Nonlinear stochastic differential equations and $1/f$ noise

Julius Ruseckas and Bronsilovas Kaulakys

Institute of Theoretical Physics and Astronomy, Vilnius University, Lithuania

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1. Introduction: $1/f$ noise

2. Stochastic differential equations giving $1/f$ noise

3. Some models resulting in proposed SDE
   - Point processes
   - Simple model of herding behavior

4. Summary
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What is $1/f$ noise?

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a type of noise whose power spectral density $S(f)$ behaves like

$$S(f) \sim 1/f^\beta, \quad \beta \text{ is close to } 1$$

- occasionally called “flicker noise”
- or “pink noise”
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Fig. 6. Frequency variation for tube No. 2, coated filament; same data as in Fig. 4 plotted to a frequency scale; curves E and F give Hartmann's results for 2 m-a. and 20 m-a.; points G were obtained with less steady measuring circuit.

First observed (in 1925) by Johnson in vacuum tubes.
Fluctuations of signals exhibiting $1/f$ behavior of the power spectral density at low frequencies have been observed in a wide variety of physical, geophysical, biological, financial, traffic, Internet, astrophysical and other systems.
Many mathematical models:

- Superposition of relaxation processes

\[ S(f) = \int_{\gamma_1}^{\gamma_2} \frac{N}{\gamma^2 + \omega^2} \, d\gamma \approx \frac{\pi N}{2\omega}, \quad \gamma_1 \ll \omega \ll \gamma_2 \]

- Dynamical systems at the edge of chaos

\[ x_{n+1} = x_n + x_n^z \mod 1 \]

- Alternating fractal renewal process

- Self-Organized Criticality
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![Diagram of alternating fractal renewal process]

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A bibliography on $1/f$ noise is vast

Published items in each year. Topic: $1/f$ noise, $1/f$ fluctuations, flicker noise, pink noise (Web of Science)
1/f noise

- 1/f noise is intermediate between white noise, $S(f) \sim 1/f^0$ and Brownian motion $S(f) \sim 1/f^2$

- In contrast to the Brownian motion generated by the linear stochastic equations, the signals and processes with 1/f spectrum cannot be understood and modeled in such a way.

Goal

to find a simple nonlinear stochastic differential equation (SDE) generating signals exhibiting 1/f noise
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Often $1/f$ noise is defined by a long-memory process, characterized by $S(f) \sim 1/f^\beta$ as $f \to 0$.

A pure $1/f$ power spectrum is physically impossible because the total power would be infinity.

We search for a model where the spectrum of signal has $1/f^\beta$ behavior only in some intermediate region of frequencies, $f_{\text{min}} \ll f \ll f_{\text{max}}$, whereas for small frequencies $f \ll f_{\text{min}}$ the spectrum is bounded.
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Heuristic derivation of SDE

- If $S(f) \sim f^{-\beta}$ then power spectral density has a scaling property
  \[ S(af) = a^{-\beta} S(f) \]

- Wiener-Khintchine theorem
  \[ C(t) = \int_{0}^{+\infty} S(f) \cos(2\pi ft) \, df \]

- Autocorrelation function $C(t)$ has scaling property
  \[ C(at) \sim a^{\beta-1} C(t) \]
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\[ C(t) = \int dx \int dx' \, xx' P_0(x) P_x(x', t|x, 0) \]

- \( P_0(x) \) is the steady state PDF
- \( P_x(x', t|x, 0) \) is the transition probability
- The transition probability can be obtained from the solution of the Fokker-Planck equation with the initial condition

\[ P_x(x', 0|x, 0) = \delta(x' - x). \]
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  \[ P(ax', t|ax, 0) = a^{-1} P(x', a^{2(\eta - 1)} t|x, 0) \]

Then the autocorrelation function will have the required scaling with

\[ \beta = 1 + \frac{\nu - 3}{2(\eta - 1)} \]
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To get the required scaling of transition probability:

- SDE should contain only powers of $x$
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Proposed SDE

$$dx = \sigma^2 (\eta - \nu/2) x^{2\eta-1} dt + \sigma x^{\eta} dW_t$$


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Introducing

\[ z = x^\alpha \]

we get the equation of the same type

\[ dz = \sigma'^2 (\eta' - \nu' / 2) z^{2\eta'-1} dt + \sigma' z^{\eta'} dW_t \]

only with different parameters

\[ \sigma' = \alpha \sigma, \quad \eta' = (\eta - 1) / \alpha + 1, \quad \nu' = (\nu - 1) / \alpha + 1 \]
Because of the divergence of the power-law distribution and the requirement of the stationarity of the process, the SDE should be analyzed together with the appropriate restrictions of the diffusion in some finite interval.

When diffusion is restricted, scaling properties are only approximate, but $1/f$ spectrum remains in a wide interval of frequencies.
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When diffusion is restricted, scaling properties are only approximate, but $1/f$ spectrum remains in a wide interval of frequencies.
Possible forms of restriction:

- Reflective boundary conditions at $x = x_{\text{min}}$ and $x = x_{\text{max}}$
- Exponential restriction of the diffusion

$$\begin{align*}
dx &= \sigma^2 \left( \eta - \frac{\nu}{2} + \frac{m}{2} \left( \frac{x_{\text{min}}}{x} \right)^m - \frac{m}{2} \left( \frac{x}{x_{\text{max}}} \right)^m \right) x^{2\eta-1} dt + \sigma x^\eta dW_t 
\end{align*}$$

Steady state PDF:

$$P_0(x) \sim x^{-\nu} \exp \left( - \left( \frac{x_{\text{min}}}{x} \right)^m - \left( \frac{x}{x_{\text{max}}} \right)^m \right)$$
Restriction of diffusion

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Restriction of diffusion

- $q$-exponential steady-state PDF

\[
\frac{dx}{dt} = \sigma^2 (\eta - \nu/2) (x + x_0)^{2\eta-1} dt + \sigma (x + x_0)^\eta dW_t
\]

\[P_0(x) \sim \exp_{1+1/\nu}(-\nu x/x_0)\]

Reflective boundary condition at $x = 0$

- $q$-Gaussian steady-state PDF

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\frac{dx}{dt} = \sigma^2 (\eta - \nu/2) (x^2 + x_0^2)^{\eta-1} x dt + \sigma (x^2 + x_0^2)^{\eta/2} dW_t
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\[P_0(x) \sim \exp_{1+2/\nu}(-\nu x^2/2x_0^2)\]

- $q$-exponential function: $\exp_q(x) \equiv (1 + (1 - q)x)^{1/(1-q)}$

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For some choices of parameters our SDE takes the form of well-known equations.

- \( \eta = 0 \) and \( \sigma = 1 \) corresponds to the **Bessel process**

\[
dx = \frac{\delta - 1}{2} \frac{1}{x} dt + dW_t
\]

of dimension \( \delta = 1 - \nu \)

- \( \eta = 1/2, \sigma = 2 \) corresponds to the **squared Bessel process**

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dx = \delta dt + 2\sqrt{x} \, dW_t
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of dimension \( \delta = 2(1 - \nu) \)
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$$\frac{d\eta}{\delta - \frac{1}{2}} - \frac{1}{\delta} \frac{1}{\eta} dt + dW_t$$

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of dimension $\delta = 2(1 - \nu)$
Connection with other equations

- SDE with exponential restriction with $\eta = 1/2$, $x_{\text{min}} = 0$ and $m = 1$ gives **Cox-Ingersoll-Ross (CIR) process**

  \[ dx = k(\theta - x)dt + \sigma \sqrt{x} \, dW_t \]

  where $k = \sigma^2 / 2x_{\text{max}}$, $\theta = x_{\text{max}}(1 - \nu)$

- When $\nu = 2\eta$, $x_{\text{max}} = \infty$ and $m = 2\eta - 2$ then we get the **Constant Elasticity of Variance (CEV) process**

  \[ dx = \mu x dt + \sigma x^\eta \, dW_t \]

  where $\mu = \sigma^2 (\eta - 1) x_{\text{min}}^{2(\eta - 1)}$
SDE with exponential restriction with \( \eta = 1/2 \), \( x_{\min} = 0 \) and \( m = 1 \) gives Cox-Ingersoll-Ross (CIR) process

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Numerical example

Power spectral density

Typical signal

Distribution of $x$

$$dx = \left( 1 + \frac{x_{\text{min}}}{2x} - \frac{x}{2x_{\text{max}}} \right) x^4 dt + x^{\frac{5}{2}} dW_t$$

$$\nu = 3, \eta = \frac{5}{2}, x_{\text{min}} = 1, x_{\text{max}} = 10^3.$$
Signals generated by proposed SDE exhibit **intermittent behavior**: there are bursts corresponding to large deviations, separated by laminar phases.

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4. Summary
Point processes

The signal of the model consists of pulses or events

\[ I(t) = a \sum_k \delta(t - t_k) \]

Point processes arise in different fields such as physics, economics, ecology, neurology, seismology, traffic flow, financial systems and the Internet.
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Let us assume that the signal $x$ is the number of pulses per unit time. How to obtain equation for inter-event time $\tau_k = t_k - t_{k-1}$:

- Transform the equation from the variable $x$ to $\tau = 1/x$
- Discretize the equation according to Euler-Maruyama approximation
- Take time step equal to $\tau_k$
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Example: equation

\[ dx = \sigma^2 x^4 \, dt + \sigma x^{5/2} \, dW \]

leads to

\[ \tau_{k+1} = \tau_k + \sigma \epsilon_k \]

We obtained a simple random walk of inter-event time

One of possible origins of $1/f$ noise

Brownian motion in time axis leads to $1/f$ noise
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General case

\[ \tau_{k+1} = \tau_k + \gamma \tau_k^{2 \mu - 1} + \sigma \tau_k^\mu \varepsilon_k \]

where \( \mu = 5/2 - \eta, \gamma = \sigma^2(1 - \eta + \nu/2) \).

Used for modeling of the internote interval sequences of the musical rhythms

Point processes

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Herding model

Simple model describing heterogeneous interacting agents:

- fixed number $N$ of agents
- each of them can be in state 1 or in state 2
- agents do not have memory, dynamics described as a Markov chain
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Herding model

- Transition probabilities per unit time:

\[ p(n \to n + 1) = (N - n)(\sigma_1 + hn) \]
\[ p(n \to n - 1) = n(\sigma_2 + h(N - n)) \]

- \( n \) is the number of agents in state 1
- \( N - n \) is the number of agents in state 2
- \( \sigma_1 \) and \( \sigma_2 \) are probabilities to change the state spontaneously
- \( h \) describes herding tendency

- Non-linear terms represent interaction between agents
- Connectivity between agents increases with the number of agents \( N \). The interactions have a global character, the range of the correlations involves a macroscopic fraction of agents.
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- Transition probabilities per unit time:

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- Ratio of the number of agents in the state 2 to the number of agents in the state 1:
  \[ y = \frac{N - n}{n} \]

- For large \( N \) we can represent the dynamics by SDE
  \[ dy = [(2h - \sigma_1)y + \sigma_2](1 + y)dt + \sqrt{2hy(1 + y)}dW \]

- When \( y \gg 1 \) we get our non-linear SDE with parameters \( \eta = 3/2, \nu = 1 + \sigma_1/h \)

If \( \sigma_1 = 2h \), we obtain \( 1/f \) spectrum

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Power spectral density of the ratio of the numbers of agents.

\( N = 10000 \)
We obtain a class of nonlinear SDEs giving the power-law behavior of the power spectral density in any desirably wide range of frequencies and power-law steady state distribution of the signal intensity.

In special cases we obtain other well-known SDEs.

One of the reasons for the appearance of the $1/f$ spectrum are scaling properties of the SDE.

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Thank you for your attention!