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Correspondence between stochastic and agent based models of financial markets

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Foreword

All social sciences attempt to tackle rather complex, as is deduced from their behavior, systems. Looking from the bird's eye view one could make an assumption that this complexity arises from the huge size of analyzed system. On the other hand some huge systems (i.e. containing lots of interacting particles) were well simplified by physicists - for example thermodynamical laws describe behavior of unlimited amount of moles of matter, each of which contains $6 \cdot 10^{23}$ molecules. Therefore size by itself, be it number of particles or number of their options, should not be the cause of aforementioned complex behavior.

Economics take different and more sophisticated approach towards economics as whole. Namely human nature of, as we called, particles is noted, though neoclassical economical theories tend to overlook whole complexity of humans being human. Major simplification is done by introducing the concept of *Homo economicus* [1, 2], who by definition is rational and self-interested human. Earliest, though it might not be first, mention of this concept is found [2] to be by Vilfredo Pareto in *Manual of political economy*, which was published in Italian language back in 1906 (for English translation see [1]). In 1939, starting ongoing heated debate, Peter Drucker predicted *The End of Economic Man* [3] and later even more critique followed (in context of this work [4, 5] are most relevant, while proper historical overview is given in [2]), but this major simplification is still present within economics. *Homo economicus* is one most widely known oversimplification in economics, but it's not the only one (further in this work we will discuss similar concept of *representative agent*). Commenting on persistence of flawed and highly ideas within economics J. P. Bouchaud, renowned econophysicist, has said that *it is all easier said than done, and the task looks so formidable that some economists argue that it is better to stick with the implausible but well corseted theory* [6].

As neoclassical economics was making first steps facing complexity, physics faced it's natural counterpart - Poincare solved three body problem [7], Ising proposed famous model for ferromagnetic interaction within lattice of magnetic spins [8], Lorenz shown that chaos (i.e. complex behavior) can arise from simple deterministic equations [9]. And those are just few examples of works by physicists, where quantitative modeling of complex systems is done in order to gain qualitative understanding of phenomena. In the last decades of XX century various contributions concerned with complexity were gathered into so-called complexity physics, or some may call it synergetics [10], which few years later seen an expansion into non-traditional, i.e. non-natural, fields for physics. One of those fields was social research field, which has become concern of physics of risk [11]. Physics of risk has it's own branches, one of them being known as econophysics [6, 12, 13] - interdisciplinary science concerned with the complexity of financial markets.

Econophysics takes different approach towards financial markets than economics. The main difference of the approaches is that econophysics is driven by empirical data, amounts of which are increasing dramatically with ongoing world-wide computerization, while economics is infatuated by prevalent ideas, or so called *a priori* knowledge, [14]. Physicists and some, non-mainstream, economists rely on empirical analysis, through which they have established some statistical regularities, known as stylized facts [15, 16, 17, 18, 19, 20, 21, 22], while mainstream economists still, sometimes even zealously [23], rely on established ideas [2, 6, 12, 23, 24]. Econophysicist approach also differs from economist approach in another dimension - some econophysicist works, which are concerned with macroscopical, usually stochastic, modeling [25, 26, 27, 28, 29], doesn't take into account human nature of particles within financial market, while other works, which are concerned with microscopical, usually agent based, modeling [4, 5, 30, 31, 32, 33], give it minor, though genuinely important, account, while economics, as social science should, give it major account. To cut long story short the basic idea of difference between economics and econophysics is simple - economics is social *a priori* science, while econophysics is fundamental *a posteriori* science (for broader overview and discussion see [14]).

This difference in approaches is the reason for ongoing debate among econophysicists and mainstream economists. Some economists, for example [24], claim that economics is more complex system than any physical system could ever be. In [24] this point of view is based on the difference between the concepts of *uncertainty* (concept is introduced in [34]), which according to the authors can't be understood using mathematical tools (ex. probability of peace in the Middle East), and *risk*, which, in contrast, can be understood using mathematical tools (ex. odds at the roulette table). In some sense the authors are right, but econophysics doesn't deal with precise events or their direct influence on financial markets. Furthermore our recent research has shown that behavior of financial market during bubble (steady economic growth) phase and crash (rapid economic fall) phase is statistically the same [25, 26, 35, 36, 37].

Having achieved great results in stochastic modeling [25, 26, 27, 35, 36, 37], we move on to agent based modeling as within this framework we could further foster gained scientific insight into financial markets. Another incentive is that in modern econophysicist research [30, 31, 38] it is popular to seek connections between established stochastic differential equation framework [39, 40, 41, 42, 43] and comparatively young, thus still developing, agent based framework [32, 33]. Thus in this work we will:

- discuss Kirman ant model [5], which plays important role in agent based frameworks used in modern day research [30, 31],
- look for similarities in behavior between proposed long-range memory stochastic model of return in financial markets [25, 26, 27, 35, 36, 37], it's generalization [29] and agent based models [30, 31] inspired by [5],
- in process derive stochastic differential equations (SDE) analogous to the analyzed agent based models [30, 31].

1 Agent based model of ant colony

In economics there is a concept of *representative agent*, which is, by definition, typical decision-maker of a certain type (ex. consumer) [44]. This concept, despite of active critique over last half of a century by the non-mainstream economists [45, 46, 47], is still widely used in modern day macroeconomic models. As of today many macroeconomic problems are reduced to the optimization problem of the representative agent.

In contrast in econophysics there is a consensus that the nature of interesting phenomena observed in financial markets lies in market heterogeneity. As the result heterogeneous agent based models are being proposed, modified and extended [30, 31, 32, 33, 48, 49]. Some of those models are based on their original concept [48], while some use the ideas and frameworks proposed by other scientists, either econophysicists [49] or non-mainstream economists [30, 31]. As we have previously (see [50]) analyzed models [48, 49], this time we will discuss framework behind the [30, 31], which is also known as Kirman ant model.

Originally Kirman proposed to model the case of ant colony with two identical sources of food nearby (see [5]). Empirical entomological research have shown that at any given time most of the ants used single food source and at certain switches occurred (majority of the ants started to exploit other food source). Kirman proposed that herding behavior should be responsible for the former and stochastic search should be responsible for the latter.

1.1 Symmetric behavior case

Let us assume that ant colony of N ants has two identical food sources. For very small time intervals, Δt, one can write probabilities for small changes in population using first of the two food sources, X, as

$$p(X \to X+1) = (N-X)(a+bX)\Delta t, \tag{1}$$
$$p(X \to X-1) = X\left[a+b(N-X)\right]\Delta t, \tag{2}$$
$$p(X \to X) = p_0, \tag{3}$$

here a is parameter controlling incentive to change current group on personal decision, i.e. stochastic search, b is parameter responsible for incentive to change group due to herd behavior, while p_0 we introduce for convenience in order to simplify selection of appropriate time interval Δt. Note that probability transition equations are symmetrical - they have same form under transformation $X \to N-X$.

By assuming that Δt is small enough, only one interaction between two ants (interaction of many ants at once is forbidden) should be possible during time period, we can say that only three events, which probabilities we described above, are possible. Thus for any small time interval, $[t, t + \Delta t]$,

$$p(X \to X + 1) + p(X \to X - 1) + p(X \to X) = 1 \tag{4}$$

is true. From the former equation we can extract expression of the appropriate Δt value

$$\Delta t = \frac{1 - p_0}{(N - X)(a + bX) + X\left[a + b(N - X)\right]} = \frac{1 - p_0}{Na + 2bX(N - X)}. \tag{5}$$

As seen in Eq. (5) small Δt values are achieved with $0 \ll p_0 < 1$, thus p_0 can hold a meaning of model precision parameter.

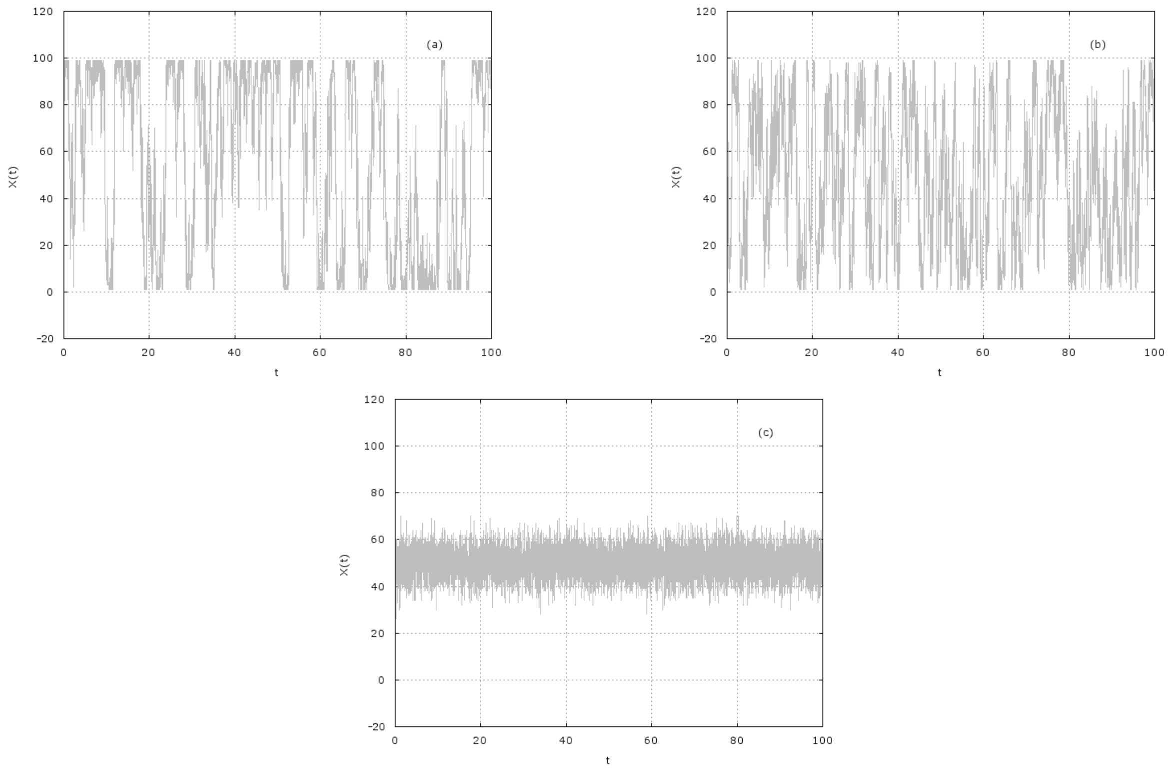

Figure 1: Time series of population using first of the two food sources, $X(t)$, if (a) herding behavior is prevalent ($a = 0.01$), (b) individual (stochastic) and herd behavior are balanced ($a = 1$), (c) individual (stochastic) behavior is dominating ($a = 100$). Other model parameters were set as follows: $b = 1$, $p_0 = 0.9$, $N = 100$.

As we see in Figure 1 and Figure 2 (a) behavior of modeled colony qualitatively depends on ratio between individual (stochastic) behavior parameter, a, and herding behavior parameter, b. In case of $\frac{a}{b} \ll 1$ agents representing ants within modeled colony demonstrate strong herding behavior - agents most of the time exploit one of the two food sources and switches between food sources are very fast (see Figure 1 (a) and black curve in Figure 2 (a)). In the opposite case, $\frac{a}{b} \gg 1$, individual (stochastic) behavior dominates - both food supplies are exploited almost evenly most of the time, thus no switches are visible (see Figure 1 (c) and light gray curve in Figure 2 (a)). Interesting behavior is observed with $\frac{a}{b} \approx 1$ - ants demonstrate both herding tendency (significant time is spent using one of the two food sources) and individual behavior (deviations are large, switches between used food supplies are slow; see Figure 1 (c)). In Figure 2 (a) (gray curve) we see that in the balanced case, $a \approx b$, modeled colony has equiprobable distribution and thus maximal entropy.

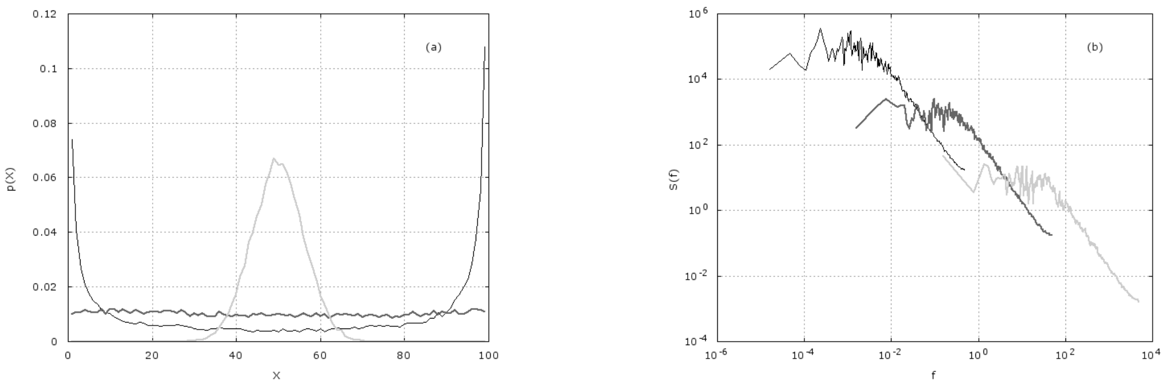

Figure 2: (a) stationary state occupation probability distributions of ant population using first food source, $X(t)$, obtained from realizations with differing $\frac{a}{b}$ values ($a = 0.01$, $b = 1$ - black curve, $a = 1$, $b = 1$ - dark gray curve, $a = 100$, $b = 1$ - light gray curve), (b) power spectral densities of ant population using first food source, $x(t)$, realizations with differing b values ($a = 0.01$, $b = 0.01$ - black curve, $a = 1$, $b = 1$ - dark gray curve, $a = 100$, $b = 100$ - light gray curve). Other model parameters were set as follows: $p_0 = 0.9$, $N = 100$.

In Figure 2 we see statistical properties - stationary state occupation probability distribution (a) and power spectral density of time series (b). We have chosen those parameter sets in order to demonstrate how model qualitative and quantitative properties change with differing parameters. As we see in Figure 2 (a) (those features are also visible in Figure 1) stationary distribution depends only on ratio between two behavioral parameters, $\frac{a}{b}$, and two different qualitative behavior phases are visible - for $\frac{a}{b} < 0$ dominating opinion exist at almost any given time, while for $\frac{a}{b} > 0$ opinions become distributed almost evenly. Quantitatively important, as we see in Figure 2 (b), is parameter b, which adjusts time scale of events in modeled system (the smaller b values is the slower processes in modeled colony become). Total power remains constant for same $\frac{a}{b}$ values.

1.2 Asymmetric behavior case

Previously discussed simple model for symmetric ant behavior can be generalized for asymmetric scenario. This approach is reported to yield statistical features comparable with statistical features obtained from empirical return time series [30].

As we have shown before, model parameter b controls event time scale. Therefore if we want to have events on the same time scale while introducing qualitatively different behavior, which is described by ratio $\frac{a}{b}$, we ought to keep b same for all transition probabilities. Furthermore one might argue that herding behavior is feature of population itself rather than being dependent on the options. By taking all that into account it is clear that the best choice is to use different

a values in order to introduce asymmetry in transition probabilities. Therefore

$$p(X \to X+1) = (N-X)(a_1+bX)\Delta t, \tag{6}$$
$$p(X \to X-1) = X\left[a_2+b(N-X)\right]\Delta t, \tag{7}$$
$$p(X \to X) = p_0. \tag{8}$$

Note that probability transition equations now are asymmetrical - they slightly change, though not in essential sense, under transformation $X \to N-X$.

Appropriate Δt values once again can be chosen using same ideas as in subsection before, though this time we can't make mathematical simplifications made in Eq. (5) and the must use full expression

$$\Delta t = \frac{1-p_0}{(N-X)(a_1+bX)+X\left[a_2+b(N-X)\right]}. \tag{9}$$

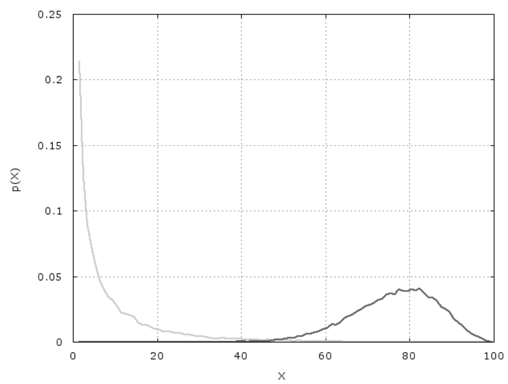

Figure 3: Influence of stochastic (individual) behavioral asymmetry on to the stationary state occupation probability distribution of ant population using first food source, $X(t)$. Dark gray line was obtained numerically for parameter set $a_1 = 16$, $a_2 = 5$, while light gray line was obtained for $a_1 = 0.2$, $a_2 = 5$. Other model parameters were set as follows: $b = 1$, $N = 100$, $p_0 = 0.9$

As one could have expected from previous discussion of symmetric case, slightly different a_i values cause movement of stationary distribution maxima (see Figure 3). In asymmetric case it would natural to expect that ants will more frequently use food resource which seems more attractive. Thus we see distribution maxima in Figure 3 shifted according to the relations between a_1 and a_2 - if $a_1 > a_2$ (dark gray curve) distribution maxima shifts to the right (i.e. first food source is more frequently used), while in inverse case, $a_1 < a_2$, (light gray curve) distribution maxima shift to the left (i.e. second food source is more frequently used).

Having said all that, though it was possible earlier, we can rewrite model transition probabilities by dropping model parameter a_i for more convenient parameter ε_i which is defined by

$$\varepsilon_i = \frac{a_i}{b}. \tag{10}$$

Thus transition probability equations become

$$p(X \to X+1) = (N-X)(\varepsilon_1 + X)b\Delta t, \quad (11)$$

$$p(X \to X-1) = X\left[\varepsilon_2 + (N-X)\right]b\Delta t, \quad (12)$$

$$p(X \to X) = p_0. \quad (13)$$

Equations above are more transparent - it is easier to determine for what each parameter is responsible for.

1.3 Conclusions

We have discussed agent based framework proposed by Kirman in [5] and later used by Alfarano and Lux in [30, 31]. During the discussion we have determined that:

- ratio $\frac{a}{b}$, later noted as ε_i, controls qualitative behavior of the modeled system, or in other words stationary probability distribution,
- herding parameter b is responsible for speed of the processes, in Figure 2 we see that features of spectral density is shifted by changing aforementioned model parameter b,
- asymmetric generalization should be very useful then modeling financial markets as one might see financial market as minority game with constantly changing rules (ex. differential between market and real price).

We have also created interactive Java applet, for symmetric case, and short model description in Lithuanian language, which both are available at *Physics of Risk* website ([51]). In future we plan to improve this applet by implementing equations of the asymmetric case.

2 Derivation of SDE analogous to Kirman agent based model

In the previous section we have discussed agent based model of ant colony proposed by Kirman in [5] and later generalized by Alfarano and Lux in [30]. Within the latter work there was derived SDE for population using one of the food sources, noted as X. In this section we will follow derivation of that SDE, while in the second half of this section we will use Ito formula for variable substitution [41] in order to obtain corresponding SDE for return (the way it was defined in [30, 31]).

2.1 SDE for ant population using one of the two food sources

Alfarano and Lux start from the idea of *birth-death processes* or *one-step processes* (overview of the ideas is given in [40, 41, 43]). The authors start by simplifying notation of Eq. (11) and Eq. (12) (in [30] Eq. (13) is not present and was introduced by us for numerical convenience)

$$p(X \to X+1) = \pi_+(X)\Delta t, \tag{14}$$

$$p(X \to X-1) = \pi_-(X)\Delta t. \tag{15}$$

In such case Master equation, for very short times Δt, can be expressed as

$$\frac{\Delta \bar{\omega}_X}{\Delta t} = \bar{\omega}_{X+1}\pi_-(X+1) + \bar{\omega}_{X-1}\pi_+(X-1) - \bar{\omega}_X \pi_-(X) - \bar{\omega}_X \pi_+(X), \tag{16}$$

here $\bar{\omega}_X$ is probability for system to be in the state described by agent number X, or in the other words probability of X ants at given time (we drop notation as it should be evident that on the left hand side we have current time moment, t, and on the right hand side we have past time moment, $t - \Delta t$, and also to conserve space) to be using one of the two food sources.

Eq. (16) can be expressed as so-called *discrete continuity equation*

$$\frac{\Delta \bar{\omega}_X}{\Delta t} + \bar{j}_{X+1} - \bar{j}_X = 0, \tag{17}$$

here $\bar{j}_i$ is probability flux from state described by agent number $i-1$ to state described by agent number i,

$$\bar{j}_i = \bar{\omega}_{i-1}\pi_+(i-1) - \bar{\omega}_i \pi_-(i). \tag{18}$$

If probability flux vanishes at boundaries, $\bar{j}_0 = \bar{j}_{N+1} = 0$, one can easily show that $\sum_i \bar{\omega}_i = 1$ for every time moment, t, (provided that it is true for $t = 0$). This actually stands behind the idea of *continuity*.

For $N \gg 1$ authors introduce continuous variable $x = \frac{X}{N}$. Probability density ω for x can be expressed through $\bar{\omega}_n$ as

$$\omega(x) = N\bar{\omega}_X. \tag{19}$$

Discrete probability flux function was also re-expressed in continuous terms

$$j\left(x - \frac{1}{2N}\right) = \bar{j}_X. \tag{20}$$

The reasoning behind the offset lies within the fact that flux noted by $\bar{j}_n$ connects two discrete states $n-1$ and n, thus it should be located in the middle of that interval. This offset also helps to avoid tedious mathematics in further derivation. Alfarano and Lux also mention that this offset in flux is widely used in discretization of Maxwell's equations and in gauge theories on a discrete lattices (see the list of references in [30]).

Eq. (17) should also be re-expressed in continuous terms. In order to do so one must plug Eq. (20) in to the difference $\bar{j}_{X+1} - \bar{j}_X$,

$$\bar{j}_{X+1} - \bar{j}_X = j\left(x + \frac{1}{2N}\right) - j\left(x - \frac{1}{2N}\right). \tag{21}$$

Using Taylor series expansion of Eq. (21) up to third order we confirm that even order terms cancel out, while odd orders remain

$$\bar{j}_{X+1} - \bar{j}_X = \frac{1}{N}\left[\partial_x j(x) + \frac{1}{24N^2}\partial_x^3 j(x)\right]. \tag{22}$$

One can simplify even further by recalling that the demanded that $N \gg 1$, then moving onto continuous description. In this case one can simply neglect N^{-2} term

$$\bar{j}_{X+1} - \bar{j}_X = \frac{1}{N}\partial_x j(x). \tag{23}$$

Thus we arrive at *discrete continuity equation* for the continuous case

$$\frac{\Delta\omega(x)}{\Delta t} + \frac{\partial j(x)}{\partial x} = 0. \tag{24}$$

Putting Eq. (18) into Eq. (20) yields

$$j\left(x - \frac{1}{2N}\right) = \bar{\omega}_{X-1}\pi_+(X-1) - \bar{\omega}_X\pi_-(X). \tag{25}$$

Lets apply Eq. (19) and shift variable x by $\frac{1}{2N}$. Latter operation yields continuous description of probability flux

$$j(x) = \frac{1}{N}\left[\omega\left(x - \frac{1}{2N}\right)\pi_+\left(xN - \frac{1}{2}\right) - \omega\left(x + \frac{1}{2N}\right)\pi_-\left(xN + \frac{1}{2}\right)\right]. \tag{26}$$

Previous equation can be simplified even further by assuming that $\omega(x \pm \frac{1}{2N})$ can be approximated as $\omega(x) \pm \frac{1}{2N}\partial_x\omega(x)$ (terms of second order and above are dropped in accordance with previous approximations),

$$j(x) = \frac{\pi_+\left(xN - \frac{1}{2}\right) - \pi_-\left(xN + \frac{1}{2}\right)}{N}\omega(x) - \frac{\pi_+\left(xN - \frac{1}{2}\right) + \pi_-\left(xN + \frac{1}{2}\right)}{N}\partial_x\omega(x). \tag{27}$$

Next one should put Eq. (14) and Eq. (15) into the above equation. Then by introducing two custom functions

$$D(x) = 2bx(1-x), \tag{28}$$

$$A(x) = b[\varepsilon_1(1-x) - \varepsilon_2 x], \tag{29}$$

one can re-express probability flux function $j(x)$ as (terms of second order and above are dropped once again in accordance with previous approximations)

$$j(x) = A(x)\omega(x) - \frac{1}{2}\frac{\partial}{\partial x}[D(x)\omega(x)]. \tag{30}$$

Finally by recalling Eq. (24) one obtains Fokker-Planck equation

$$\frac{\partial}{\partial t}\omega(x,t) = -\frac{\partial}{\partial x}[A(x)\omega(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[D(x)\omega(x,t)]. \tag{31}$$

Process described by above Fokker-Plank equation can be also modeled via Langevin stochastic differential equation [30, 40, 41, 43]:

$$\mathrm{d}x = A(x)\mathrm{d}t + \sqrt{D(x)}\mathrm{d}W = b\left[\varepsilon_1(1-x) - \varepsilon_2 x\right]\mathrm{d}t + \sqrt{2bx(1-x)}\mathrm{d}W, \tag{32}$$

here W is Wiener-Brownian process. It should be evident that the above SDE must be solved with vanishing boundary conditions - $\omega(x,t) = 0$ for every x, which does not lie in the interval $[0,1]$ - as there is no logical meaning in negative or larger than maximal population using one of the two food sources.

2.2 Comparison of statistical properties of population time series obtained from agent based and stochastic model definitions

In previous subsection we have followed derivation of stochastic differential, Langevin, equation, which should yield same statistical properties as in previous section discussed agent based model. As we see in Figure 4 it is indeed true - for qualitatively different parameter sets, namely $\varepsilon_1 = 0.2$, $\varepsilon_2 = 5$ (thick gray line for stochastic model, black dashed line for agent based

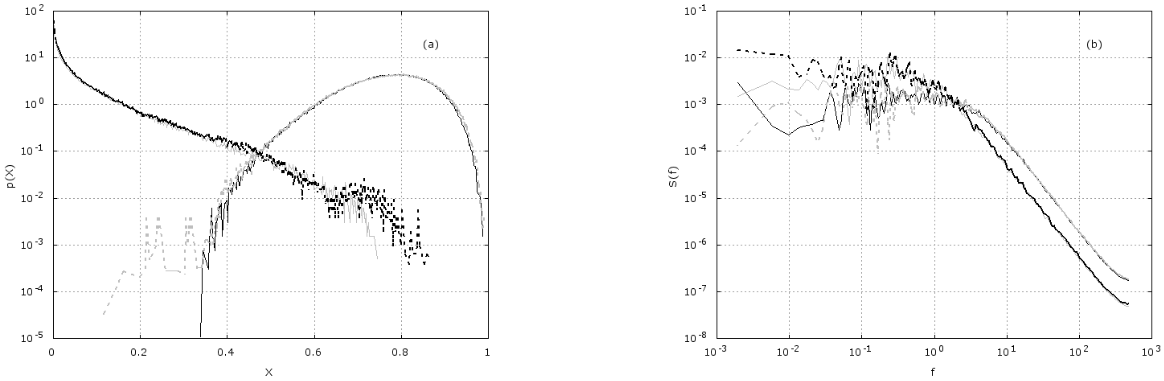

Figure 4: Correspondence between statistical properties, (a) PDF and (b) PSD, of external observable, population using one of the two food sources, x, obtained from macroscopic (Eq. (33); thick lines) and microscopic (Section 1; dashed lines) description of the same modelic system - ant colony. Gray thick line and black dashed line (colors swapped for visual convenience) correspond to the case of $\varepsilon_1 = 0.2$, $\varepsilon_2 = 5$, while black thick line and gray dashed line correspond to the case of $\varepsilon_1 = 16$, $\varepsilon_2 = 5$. Other model parameters were set as follows: $b = 1$, $\Delta t = 10^{-3}$, $N = 500$, $p_0 = 0.9$.

model) and $\varepsilon_1 = 16$, $\varepsilon_2 = 5$ (thick black line for stochastic model, gray dashed line for agent based model), we have obtained almost identical statistical behavior.

Numerical solution of Eq. (33) can be obtained by using simple Euler-Maruyama method (see [42]) as x is strictly restrained in the interval of small positive values

$$x_{i+1} = x_i + b\left[\varepsilon_1(1-x_i) - \varepsilon_2 x_i\right]\Delta t + \sqrt{2bx_i(1-x_i)\Delta t}\zeta_i, \tag{33}$$

here Δt is constant time step between consecutive iterations, ζ_i is Gaussian random variable (zero mean, unit variance). Boundary conditions should be implemented by using *min* and *max* functions available in used programing language.

2.3 Defining asset return within the ant colony framework

In [30] it is assumed that there exists two essentially different trading strategies - chartists and fundamentalist - which one can interpret as two differing food sources in previously discussed model. Thus one can model population dynamics using same model as for ants. Though if we want to compare model with empirical data, we must introduce other observable into the model, as it is impossible to directly observe popularity of trading strategies. One of the options is to introduce return, or equivalently price, through Walrasian price adjustment mechanism,

$$\frac{1}{\pi}\frac{\mathrm{d}\pi}{\mathrm{d}t} = \beta \sum_i ED_i, \tag{34}$$

here β stands for speed of adjustment, ED_i is excess demand, equivalently shortage of supply. Next we will establish excess demands of different strategies in order to obtain expression of return.

As chartists are noise traders, their excess demand is given by

$$ED_c(t) = -r_0 N_c(t) W(t), \tag{35}$$

here r_0 empirically defined scaling constant, $N_c(t)$ is number of chartists in the market, while negative sign is used for notational convenience. Fundamentalists are completely opposite type of traders - they have some information about market (fundamental price of stock or commodity), and thus try to adjust their trading according to that information

$$ED_f(t) = N_f(t) \ln \frac{\pi_f(t)}{\pi(t)}, \tag{36}$$

here $N_f(t)$ is number of fundamentalists within the system, $\pi_f(t)$ is fundamental price, $\pi(t)$ is market price.

In [30] it is required for market maker to clear the market. Thus according to Walrasian scenario it is required that $ED_c + ED_f = 0$, or alternatively in expanded form

$$\ln \frac{\pi(t)}{\pi_f(t)} = r_0 \frac{N_c(t)}{N_f(t)} W(t). \tag{37}$$

Let us now assume that $\pi_f(t)$ is constant or changes very slowly in comparison to the $\pi(t)$, if so one can easily obtain definition of return

$$r(t) = \ln \frac{\pi(t+\Delta t)}{\pi(t)} = r_0 \left[\frac{x(t+\Delta t)}{1 - x(t+\Delta t)} W(t+\Delta t) - \frac{x(t)}{1-x(t)} W(t) \right]. \tag{38}$$

Note that above we have expressed $N_c(t)$ and $N_f(t)$ using $x(t)$ - we have chosen that the first food source, in further calculations, will correspond to the chartist trading strategy, while the second one will correspond to the fundamentalist trading strategy (one may choose differently - final results, replicated statistical properties, shouldn't vary). By making additional adiabatic approximation, let $x(t)$ be much slower than $W(t)$, one can obtain final expression of return

$$r(t) = r_0 \frac{x(t)}{1-x(t)} \xi(t), \tag{39}$$

here $\xi(t)$ is simple random variable, it can follow uniform, spin-noise or any other distribution. Though in further calculation we will drop scaling and noise terms, as we expect to introduce them then there will be a need for them.

2.4 Derivation of SDE for return in financial market

From the Ito calculus [41] there is known formula for variable substitution

$$\mathrm{d}f(x) = [A_x(x)\partial_x f(x) + \frac{1}{2} B_x^2(x) \partial_x^2 f(x)]\mathrm{d}t + B_x(x)\partial_x f(x) \mathrm{d}W, \tag{40}$$

here x is variable of primary SDE, $A_x(x)$ is drift term of primary SDE, $B_x(x)$ is diffusion term of primary SDE and $f(x)$ is variable substitution function. Thus by putting

$$\begin{aligned} r &= f(x) = \frac{x}{1-x}, && (41)\\ A_x(x) &= b(\varepsilon_1(1-x) - \varepsilon_2 x), && (42)\\ B_x(x) &= \sqrt{2b(1-x)x}, && (43)\\ \partial_x r &= \frac{\mathrm{d}r}{\mathrm{d}x} = \frac{1}{1-x} + \frac{z}{(1-x)^2} = \frac{1}{1-x}(1+r), && (44)\\ \partial_x^2 r &= \frac{\mathrm{d}^2 r}{\mathrm{d}x^2} = \frac{2}{(1-x)^2} + \frac{2z}{(1-x)^3} = \frac{2}{(1-x)^2}(1+r), && (45) \end{aligned}$$

into the Eq. (40) we obtain SDE for return in financial markets, r,

$$\mathrm{d}r = b(\varepsilon_1 - r[\varepsilon_2 - 2])(1+r)\mathrm{d}t + \sqrt{2br}(1+r)\mathrm{d}W. \quad (46)$$

Note that Eq. (46) is not anymore invariant, in essential sense, to transformation $x \to 1-x$ (or alternatively $\varepsilon_1 \to \varepsilon_2$, $\varepsilon_2 \to \varepsilon_1$), thus symmetry property is lost. It was lost while applying Walrasian scenario, due to derived definition of return, Eq. (39).

2.5 Comparison of statistical properties of return time series obtained from stochastic and agent based model definitions

In the Section 2.2 we have shown correspondence between statistical properties of external observable - percentage of population using one of the two food sources - obtained from two different, stochastic (macroscopic) and agent based (microscopic), definitions of the same model. In the Section 2.3 we have assumed that we can draw analogy between ant colony and financial markets and in the Section 2.4 we have derived Langevin equation for return in financial markets. In this subsection we will obtain correspondence between statistical properties of return from two possible definitions of the same model.

We solve Eq. (46) numerically by using variable time step,

$$h_i = \frac{\kappa^2}{(1+r)^2}. \quad (47)$$

In such case iterative equation equivalent for Eq. (46) becomes

$$\begin{aligned} r_{i+1} &= \kappa^2 b \frac{\varepsilon_1 - r_i[\varepsilon_2 - 2]}{1+r_i} + \kappa\sqrt{2br_i}\zeta_i, && (48)\\ t_{i+1} &= t_i + \frac{\kappa^2}{(1+r)^2}, && (49) \end{aligned}$$

here κ is precision parameter, $0 < \kappa \ll 1$.

Note that from Eq. (46), and thus from system of Eq. (48) and Eq. (49), we obtain momentary return, while actual interest lies in compounded return. To amend this discrepancy we integrate solutions of Eq. (46) in relevant time intervals, Δt wide, in order to obtain compounded return

$$\bar{r}(t) = \frac{r_0}{\Delta t} \int\limits_{t-\Delta t/2}^{t+\Delta t/2} r(\tau) \mathrm{d}\tau, \tag{50}$$

here we have reintroduce empirically defined return scaling constant, r_0. As we are solving numerically we change integration by appropriate summation in corresponding time intervals. Note that the form of Eq. (50) is similar to one of the moving average function's, thus we use overlying bar to denote compounded return.

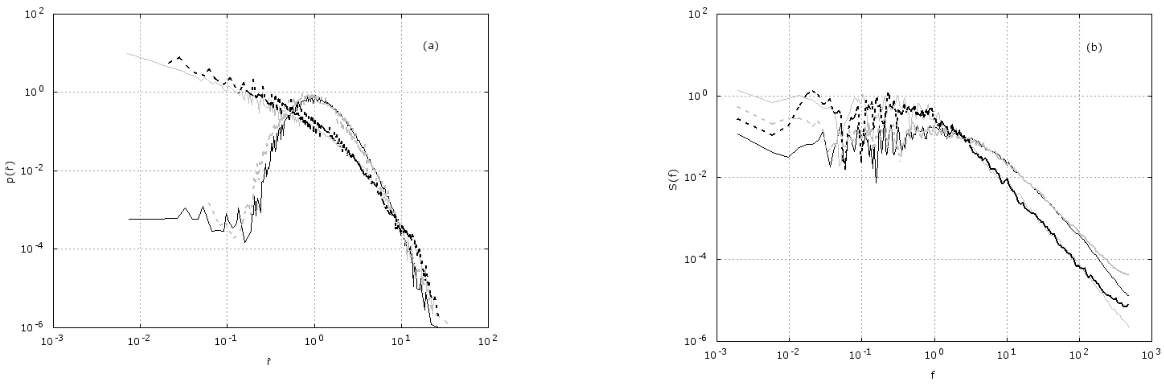

Figure 5: Correspondence between statistical properties, (a) PDF and (b) PSD, of external observable, compounded return (as defined by Eq. (50)), obtained from macroscopic (Eq. (46); thick lines) and microscopic (Eq. (39); dashed lines) description of the same modelic system - heterogeneous financial market. Gray thick line and black dashed line (colors swapped for visual convenience) correspond to the case of $\varepsilon_1 = 0.2$, $\varepsilon_2 = 5$, while black thick line and gray dashed line correspond to the case of $\varepsilon_1 = 16$, $\varepsilon_2 = 5$. Other model parameters were set as follows: $b = 1$, $r_0 = 0.1$, $\Delta t = 10^{-3}$, $\kappa = 0.3$, $N = 500$, $p_0 = 0.9$.

In Figure 5 we see that statistical properties of return time series match very well. It is to be expected as despite different definitions, they model same modelic system. Though some differences are visible (such as longer tails of PDF obtained from stochastic description), but they are easily explained by the fact that stochastic definition is continuous and agent based definition is discrete - tick sizes might have played the role.

2.6 Conclusions

In this section we have followed derivation, originally derivation was done in [30], of Langevin equation, Eq. (33), for population percentage using one of the two food sources, x. Based on aforementioned SDE we have derived Langevin equation for momentary return, Eq. (46)

In this section we have also confirmed correspondence of statistical features replicated by differing model definitions (see Figure 4 and Figure 5) - macroscopic (stochastic) and microscopic (agent based). This agreement and high power of noise multiplicativity, $\frac{3}{2}$ (though it is still smaller than $\frac{5}{2}$ in our previous works [25, 26, 35, 36, 37]), encourages us to continue to work this way.

3 Adjustment to empirical data and previous stochastic model of return

In previous section we have obtained stochastic differential equation for return in financial markets, which are assumed to be governed by ant-like behavior. In this section we will attempt reproduce signature statistical features and discuss further improvements needed to obtain agreement between empirical data, previous long range memory stochastic model of return [25, 26, 27, 35, 36, 37] and discussed model.

3.1 Achieving $1/f$ noise

$1/f$ noise is very interesting and not yet fully understood phenomenon in many physical and social systems. Though there are lots of models which strive to give an insight into it. Despite the fact that spectral density in financial markets is more sophisticated [25, 26, 35, 36, 37], in this subsection we will reproduce spectral density with $1/f$ dependency as it seems to be crucial step in developing model (we made similar approach in [35, 36]).

In previous work [35] we have derived Langevin equation, whose expression,

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right)(1+x^2)^{\eta-1} x \mathrm{d}t_s + (1+x^2)^{\frac{\eta}{2}} \mathrm{d}W_s, \tag{51}$$

here x is momentary return, resembles Eq. (46). Solutions of Eq. (51), as was shown in [35], yield stationary q-Gaussian distribution and $1/f^\beta$ spectral density, where β is expressed as

$$\beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}. \tag{52}$$

If we want to achieve $1/f$ noise we must set parameters in a way to obtain $\beta = 1$. From Eq. (52) it is obvious that we should choose $\lambda = 3$, while η value can be chosen freely. But η is set freely only in Eq. (51), in Eq. (46) it is set explicitly - it equals $\frac{3}{2}$. In this case $\left(\eta - \frac{\lambda}{2}\right) = 0$, thus drift term in Eq. (51) disappears and should also disappear in similar equations. Drift term can be dropped in Eq. (46) by setting model parameters to $\varepsilon_1 = 0$, $\varepsilon_2 = 2$. Thus Eq. (46) becomes

$$\mathrm{d}r = \sqrt{2br}(1+r)\mathrm{d}W. \tag{53}$$

Though above SDE will only reproduce $1/f$ of noise, but not stationary q-Gaussian distribution. From stochastic analysis [41, 43] we know that stationary distribution, $p(x)$, is related with SDE drift, $A(x)$, and diffusion, $B(x)$ functions via

$$p(x) = \frac{1}{B^2(x)} \exp\left[2 \int_{-\infty}^{x} \frac{A(u)}{B^2(u)} \mathrm{d}u\right]. \tag{54}$$

Thus Eq. (53) will have stationary distribution inversely proportional to $2br(1+r)^2$. In Figure 6

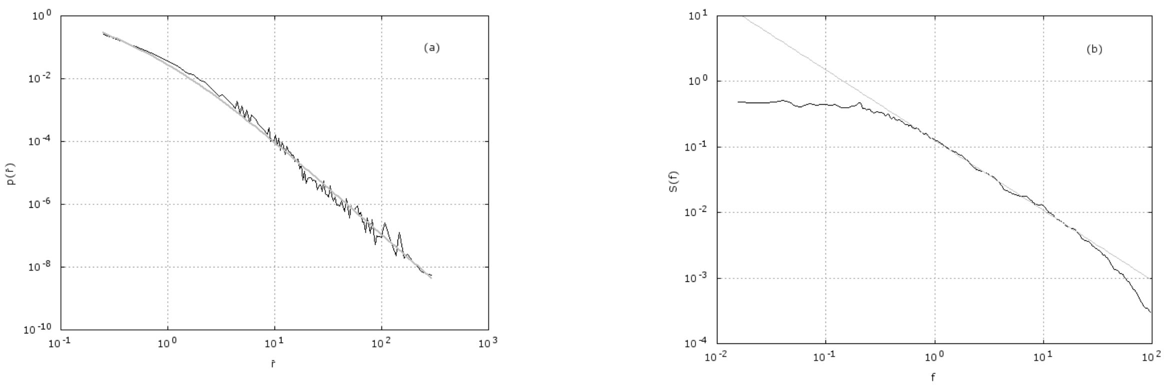

Figure 6: Statistical properties, (a) PDF and (b) PSD, obtained with reduced to diffusion term SDE. Black lines in both subfigures represent modeling results, gray curve in (a) represents theoretical PDF and in (b) is approximating power law function with power $\beta = 1.05$. Model parameters were set as follows: $\varepsilon_1 = 0$, $\varepsilon_2 = 2$, $b = 1$, $r_0 = 1$, $\Delta t = 5 \cdot 10^{-3}$, $\kappa = 0.3$.

we show that we were able to reproduce expected results - $1/f$ PSD and power law PDF (actual stocks have more sophisticated PDF, though their tail is power law) with selected parameter set.

3.2 Achieving q-Gaussian distribution

In previous works [35, 36, 37] we have shown that empirical one minute return distributions can be fitted by q-Gaussian with λ roughly being equal to 3.8 ± 0.2. Thus as this feature is important signature statistical property of varying stocks, we will attempt to recover it using discussed model.

From Eq. (54) we can derive general formula for stationary distribution of Eq. (46) solutions

$$p(r) = \frac{C r^{\varepsilon_1 - 1}}{2b(1+r)^{\varepsilon_1 + 2}} \exp\left[\frac{\varepsilon_1 + \varepsilon_2 - 2}{1+r}\right]. \quad (55)$$

We have numerically determined model parameter set ($\varepsilon_1 = 0.75$, $\varepsilon_2 = 5$, $r_0 = 2.5$) with which q-Gaussian probability density function (with $\lambda = 3.7$) and probability density function of discussed model, Eq. (55), agree well. As we see in Figure 7 gray lines overlap well, but numerical results (black line) stand out. We can explain this as Eq. (55) is true for momentary return, for which we have derived SDE, Eq. (46), while our numerical results corresponds to compounded return (defined as moving average of momentary return), thus numerical results have smaller variance and lesser probabilities of extreme values.

Though by having this in mind we can choose other parameter sets to improve agreement between numerical results and q-Gaussian. We have achieved best agreement with very similar parameter set - $\varepsilon_1 = 0.9$, $\varepsilon_2 = 3.25$, $r_0 = 1.5$. While trying various parameter sets we have determined that ε_1 controls both tail and plateau behavior of the resulting distribution, while ε_2

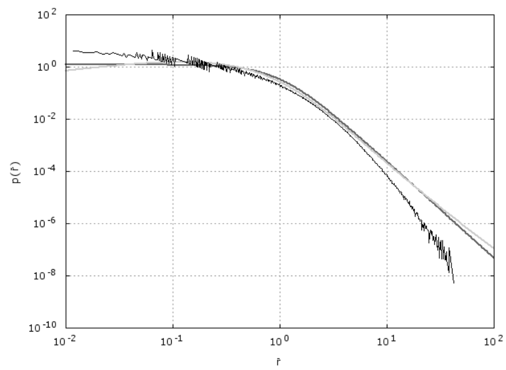

Figure 7: Comparison of q-Gaussian ($\lambda = 3.7$; dark gray line), discussed model, Eq. (55), (light gray line), numerically obtained (black line) PDFs. Model parameters were set as follows: $\varepsilon_1 = 0.75$, $\varepsilon_2 = 5$, $b = 1$, $r_0 = 2.5$, $\Delta t = 10^{-3}$, $\kappa = 0.3$.

plays a role only then establishing tail behavior. Same conclusion can be reached by analyzing Eq. (55). In Figure 8 we show that selected parameter set helps us recover return distribution

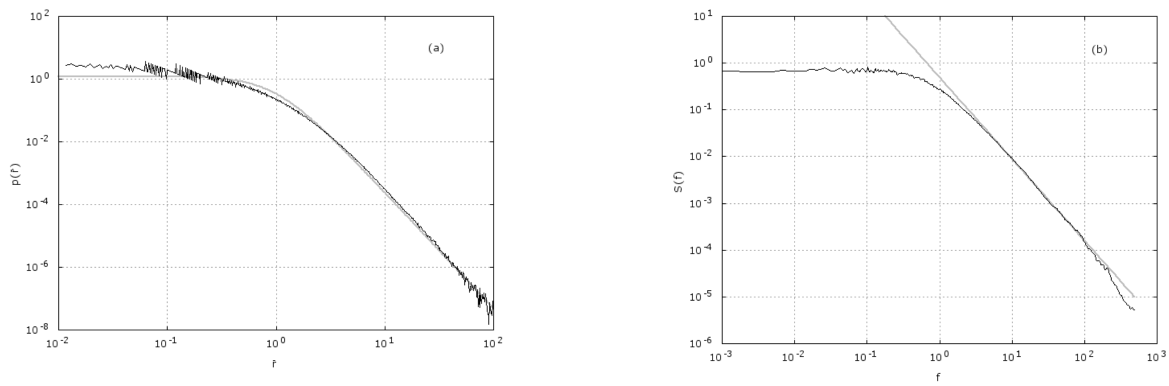

Figure 8: (a) comparison of q-Gaussian ($\lambda = 3.7$; gray line) and numerically obtained (black line) PDFs, (b) corresponding numerical PSD (black line) and fitting power law function with $\beta = 1.74$ (gray line). Model parameters were set as follows: $\varepsilon_1 = 0.9$, $\varepsilon_2 = 3.25$, $b = 1$, $r_0 = 1.5$, $\Delta t = 10^{-3}$, $\kappa = 0.3$.

nearly similar to q-Gaussian distribution, though spectral density is not even nearly similar to what one would want to see - obtained noise is almost Brownian-like ($\beta = 1.74$ is near 2).

3.3 Looking for SDE consistent with previous long range memory model

In the previous subsections we have shown that some signature features of financial markets can indeed be reproduced. Though for experiments with model parameters above it is also obvious that model should be improved further in order to reproduce sophisticated statistical features of actual financial markets.

In [25, 35] we have derived SDE using which we were able to reproduce long range memory behavior observed at NYSE (abbr. New York Stock Exchange). In [26, 36, 37] we have extended earlier proposed stochastic model and reproduced statistical properties of two differing financial markets - NYSE and VSE (abbr. NASDAQ OMX Vilnius Stock Exchange) - at different time scales. Final SDE was expressed as

$$\mathrm{d}x = \left[\eta - \frac{\lambda_0}{2} - \left(\frac{x}{x_{max}}\right)^2\right] \frac{(1+x^2)^{\eta-1}}{(\epsilon\sqrt{1+x^2}+1)^2} x \mathrm{d}t_s + \frac{(1+x^2)^{\frac{\eta}{2}}}{\epsilon\sqrt{1+x^2}+1} \mathrm{d}W_s, \tag{56}$$

here x is momentary return, η empirically defined power of multicativity, t_s scaled time, W_s scaled Wiener-Brownian process. In all above mentioned papers model includes secondary noise modulating first one, compounded solutions of Eq. (56), but at this time details about this point are not relevant.

Let us define b as a function of momentary return, r,

$$b(r) = \frac{b_0}{2} \frac{1+r^2}{(\epsilon\sqrt{1+r^2}+1)^2}. \tag{57}$$

In that case Eq. (46) becomes

$$\mathrm{d}r = \frac{b_0}{2}(\varepsilon_1 - r[\varepsilon_2 - 2]) \frac{(1+r)(1+r^2)}{(\epsilon\sqrt{1+r^2}+1)^2} \mathrm{d}t + \sqrt{b_0 r(1+r^2)} \frac{(1+r)}{\epsilon\sqrt{1+r^2}+1} \mathrm{d}W. \tag{58}$$

As equations (56) and (58) are very similar we can assume that after fine tunning of model parameters we would be able to reproduce similar statistical properties. We started with setting $\eta = 2.5$, $\lambda_0 = 4$, $\epsilon = 0.008$ and dropping diffusion limiting term in Eq. (56). By looking into expressions of those formulas one can determine primary set of parameters for Eq. (58) - $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $b_0 = 1$.

Starting with that parameter set we started to look for more precise parameter set by looking for smallest average relative difference between corresponding terms in stochastic differential equations. As diffusion term in Eq. (58) incorporates only one parameter we started from minimizing difference between diffusion functions. In Figure 9 (a) we see the dependence of difference on model parameter b_0, and it seems that difference is minimal at $b_0 = 0.9$. Thus we improve our parameter set and move on to the optimization of drift functions. Clearly parameter non-zero parameter ε_1 would become very uncomfortable, thus we have optimized parameter ε_2 (see Figure 9 (b)), whose minima is located at $\varepsilon_2 = 0.8$. In the last subfigure of Figure 9, (c), we show that ε_1 should be indeed equal to zero - difference increase with larger ε_1 values.

We solve Eq. (58) numerically by using variable time step method. We defined variable time step as

$$h_i = \kappa^2 \frac{2}{b_0} \frac{(1+0.008\sqrt{1+r^2})^2}{(1+r)(1+r^2)}. \tag{59}$$

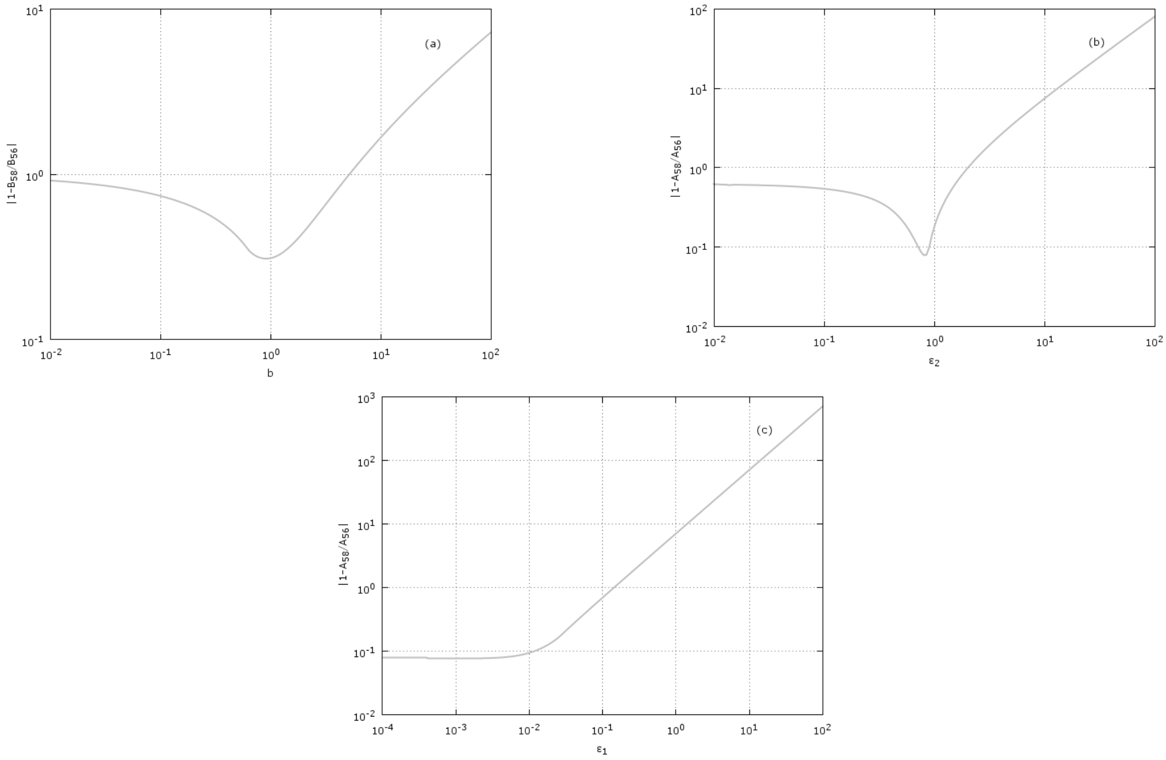

Figure 9: Relative difference between (a) diffusion functions from Eq. (56) and Eq. (58) (model parameter b is varied; $\varepsilon_1 = 0$, $\varepsilon_2 = 1$), (b) drift functions from Eq. (56) and Eq. (58) (model parameter ε_2 is varied; $\varepsilon_1 = 0$, $b = 0.9$), (c) drift functions from Eq. (56) and Eq. (58) (model parameter ε_1 is varied; $\varepsilon_2 = 0.8$, $b = 0.9$).

Implementing which Eq. (58), exactly as before in this work with other SDE, turns into the set of discrete equations for momentary return and time,

$$r_{i+1} = \kappa^2(\varepsilon_1 - r_i[\varepsilon_2 - 2]) + \kappa\sqrt{2r_i(1+r_i)}\zeta_i, \tag{60}$$

$$t_{i+1} = t_i + \kappa^2 \frac{2}{b_0} \frac{(1+0.008\sqrt{1+r^2})^2}{(1+r)(1+r^2)}. \tag{61}$$

As before one can't directly use the solution of Eq. (58), in order to obtain compounded return one must integrate, or in numerical case sum, all momentary returns in the relevant time intervals (see Eq. (50)).

We have presented results of numerical calculation (black lines) versus further discussed theoretical expectations (gray lines) in Figure 10. As we see in the figure despite similarity of equations with multiplicativity power of $\frac{5}{2}$, Eq. (56) and Eq. (58), we were unable to reproduce complex behavior observed in financial markets. With selected parameter set we were able to obtain $1/f$ PSD and power-law PDF, which only in the tail yields similar probabilities as q-Gaussian probabilities. Most probably main cause for the difference is significant disagreement of diffusion functions (average relative difference is about 35% (with 7% difference for drift functions)).

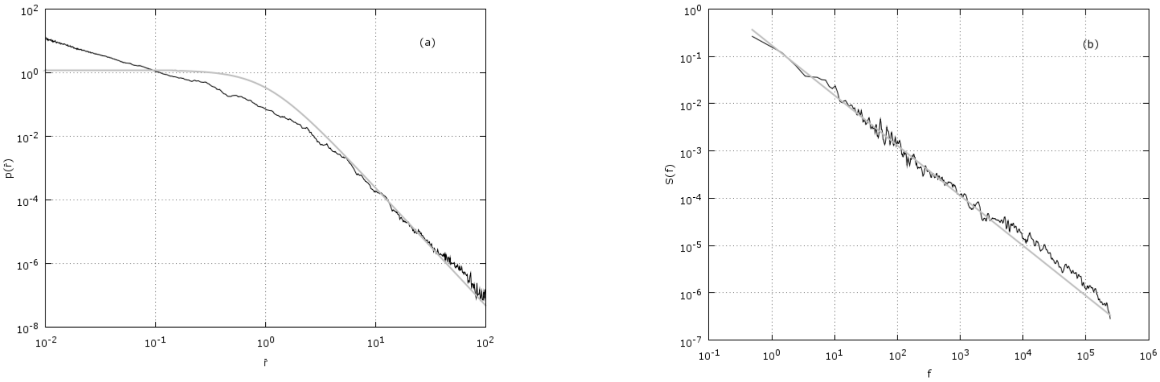

Figure 10: Statistical features, (a) PDF and (b) PSD, of SDE, given by Eq. (58), with multiplicativity power of $\frac{5}{2}$. Numerically obtained statistical features (black lines) were plotted against (a) expected q-Gaussian distribution, $\lambda = 3.7$ and (b) fitting power law function with $\beta = 1.05$ (gray lines). Model parameters were set as follows: $\varepsilon_1 = 0$, $\varepsilon_2 = 0.8$, $b_0 = 0.9$, $\Delta t = 2 \cdot 10^{-6}$, $r_0 = 2.5$, $\kappa = 0.3$

As we have seen purely mathematical approach doesn't give expected results. Though it doesn't in any way mean that this model is to rigid, bad or wrong, as we can make further attempts, from ideological point, to improve discussed model.

3.4 Conclusions

In this section we have attempted to reproduce signature statistical features of social systems and financial markets. We have shown that using our previous research [25, 26, 27] and knowledge offered by the stochastic analysis [40, 41, 42, 43] we are able to control results of stochastic implementation of agent based model discussed in previous sections. Though we were unable to obtain expected results by increasing power of multiplicativity, despite similarity of obtained equation and equation from previous research, which was successful in reproducing sophisticated statistical features of actual financial markets.

4 Conclusions

In Section 1 we have discussed agent based model proposed by Kirman [5]. We have presented it's definitions in symmetric and asymmetric cases and analyzed it's behavior for different parameter sets (see Figure 1, Figure 2 and Figure 3).

We continued our research in Section 2 by following derivation of SDE for population in asymmetric Kirman model (originally done in [30]) and showing equivalence of obtained numerical results (see Figure 4). Next by using Ito calculus [41, 43] we have derived corresponding SDE for return, defined as population ratio (see Section 2.3), - Eq. (46). To conclude this section we have shown that during transformation nothing essential was lost (see Figure 5).

Finally we successfully reproduced signature statistical features of social systems and financial markets. In Section 3 we obtained $1/f$ noise (see Figure 6) and q-Gaussian distribution (see Figure 8). Though our attempts to reproduce sophisticated statistical properties of financial markets have failed (see Section 3.3) despite agreement with successful previous research [25, 26, 35, 36, 37]. Despite the failure while utilizing purely mathematical approach in future we see a possibility to extend research presented in this paper. There are additional techniques, such as modulating signals (we have used similar technique in [25, 35, 36]), which can be applied in order to achieve best agreement between model and empirical data.

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Aleksejus Kononovičius

Atitikimas tarp finansų rinkos stochastinių ir agentų modelių

Santrauka

Šiame mokslo tiriamajame darbe mes išnagrinėjome Kirmano pasiūlytą [5] ir Alfarano su Lux apibendrintą [30, 31] agentų modelį. Originale Kirmanas pasiūlė modelį skruzdžių kolonijai greta kurios yra du identiški maisto šaltiniai. Su tam tikrais parametrais, $a \ll b$, modelio rezultatai kokybiškai sutampa su empiriniais entomologiniais tyrimais. Vėlesniuose šio autoriaus darbuose (pvz. [52]) šis entomologinis modelis yra pritaikomas finansų rinkai. Finansų rinkai jo apibendrinimą taiko ir ekonofizikai, pvz. Alfarano su Luxu [30, 31].

Mes savo darbe pagrindinį akcentą dedame ant stochastinės šio agentų modelio interpretacijos. 1 skyriuje mes trumpai aptarėme patį agentų modelį ir jo apibendrinimą nevienodiems maisto šaltiniams (asimetrinio elgesio atvejis). Vėlesniame skyriuje mes pademonstruojame Lanžaveno lygties populiacijos procentui išvedimą (originaliai atlikta [30] darbe), bei pasinaudodami Ito formulėmis [41, 43] išvedame Lanžaveno lygtį grąžai. Lygties grąžai išvedimas mums yra svarbus, nes būtent šį parametrą yra siekiama modeliuoti, bei ankstesniuose darbuose mes jau esame gavę grąžos Lanžaveno lygčių išraiškas [25, 26, 27, 29, 35, 36, 37] ir žinome kaip reikia valdyti modelio parametrus.

3 skyriuje mes pademonstruojame, kad su gauta lygtimi, kurios triukšmo multiplikatyvumo laipsnis yra $\frac{3}{2}$, yra įmanoma atkurti $1/f$ triukšmą, jis yra būdingas daugeliui socialinių sistemų, nors finansų rinkose triukšmas yra sudėtingesnės priklausomybės [35, 36], ir q-Gausinį pasiskirstymą, kuris taip pat yra būdingas socialinėms sistemoms ir finansų rinkoms [19, 35, 36]. Mes taip pat pabandėme suformuluoti sudėtingesnę Lanžaveno lygtį, kurios multiplikatyvumo laipsnis būtų panašus į anksčiau mūsų naudotų lygčių multiplikatyvumo laipsnį, bet šios lygties rezultatai nepaisant lygčių panašumo rezultatų suderinamų su empiriniais nedavė.

Nepaisant nesėkmės su $\frac{5}{2}$ multiplikatyvumo laipsnio lygtimi, mes manome, kad modelį galima bandyti patobulinti kitais keliais. Modelį galima bandyti tobulinti papildomai iš idėjinės (šiame darbe lygtis bandėme suderinti grynai matematiškai) pusės. Dar ankstesniuose tyrimuose [25, 35, 36] rašydami Lanžaveno lygtį grąžai mes esame pasinaudojame dvigubo stochastinio proceso idėja (t.y. vienas atsitiktinis procesas moduliuoja kitą).