Bursting dynamics of the high-frequency empirical return and nonlinear stochastic model

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Abstract

Recently we have proposed a stochastic model of absolute return reproducing the main statistical features of empirical data [1, 2]. The proposed model is based on the general form SDE [3], which is known to exhibit power law bursting behavior [4, 5]. In this contribution we compare bursting behavior observed in the empirical data and in the proposed model. Explicit form of burst duration PDF for the general form SDE is derived. As this contribution is based on [6], see it for more details.

Burst statistics of the time series

Intermittent, or bursty, behavior is observed in many complex systems ranging from astronomy and geology to biology and finance [7]. As most of these complex systems have discrete observables (ex., number of the earthquakes, their aftershocks and etc.) the most direct way to understand the intermittency is to use the point process formalism [8]. In this contribution we analyze certain continuous stochastic models, which were actually derived from the same point process models [1], thus enabling us to use hitting time formalism. Bellow, in Fig. 1, we introduce different observables we consider to be related to the bursty behavior of continuous time series.

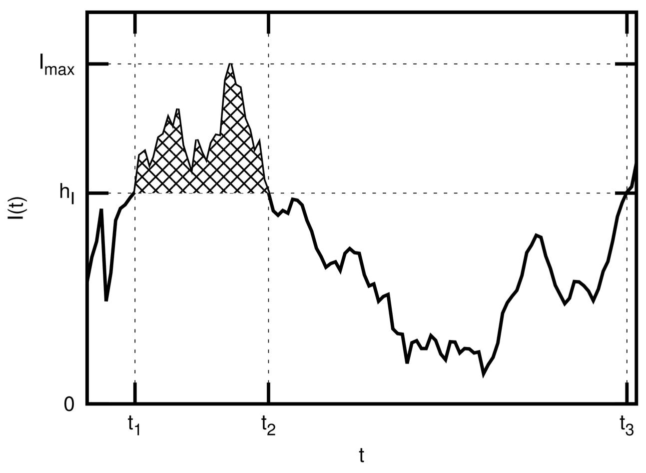

Figure 1: Time series exhibiting bursty behavior, $I(t)$. Here h_I is threshold value, above which bursts are detected. t_i are the three visible threshold passage events. I_{max} is the highlighted burst's peak value. The highlighted area is called burst size, S. The other relevant statistical properties are defined as: $T = t_2 - t_1$ (burst duration), $\theta = t_3 - t_2$ (inter-burst time) and $\tau = T + \theta = t_3 - t_1$ (waiting time).

The nonlinear stochastic model

Previously [1, 2] we have proposed a stochastic model model driven by the nonlinear SDE,

$$\mathrm{d}x = \left[\eta - \frac{\lambda}{2} - \left(\frac{x}{x_{max}}\right)^2\right]\frac{(1+x^2)^{\eta-1}}{(\epsilon\sqrt{1+x^2}+1)^2}x\mathrm{d}t_s + \frac{(1+x^2)^{\frac{\eta}{2}}}{\epsilon\sqrt{1+x^2}+1}\mathrm{d}W_s. \tag{1}$$

The resulting time series, are obtained by using the q-Gaussian noise with $\lambda = 5$ and $r_0 = 1 + \frac{\bar{r}_0}{\tau_s}\left|\int_{t_s}^{t_s+\tau_s} x(s)\mathrm{d}s\right|$. The obtained time series posses power law probability density function and fractured spectral density.

The above stochastic model appears to be too complex to tackle analytically. Yet to understand the bursty behavior of the financial markets, and the model itself, we can study a stochastic model driven by a simpler SDE

$$\mathrm{d}x = \left(\eta - \frac{\lambda}{2}\right)x^{2\eta-1}\mathrm{d}t_s + x^\eta \mathrm{d}W_s. \tag{2}$$

This SDE posses very similar statistical features - it reproduces time series with the power law probability density, $p(x) \sim x^{-\lambda}$, and single power law spectral density, $S(f) \sim f^{-\beta}$ ($\beta = 1 + \frac{\lambda-3}{2(\eta-1)}$).

The understanding of burst dynamics of the (2) also proves useful as in certain cases it can be reduced to widely known and used stochastic processes, namely Bessel process ($\eta = 0$, $t_s = t$), squared Bessel process ($\eta = 1/2$, $t_s = 2t$), CIR (add linear restriction from the top, $\eta = 1/2$, $t_s = \sigma^2 t$) and CEV processes (add restriction from bottom with $m = 2\eta - 2$, $\lambda = 2\eta$, $t_s = \sigma^2 t$).

Obtaining burst duration

In order to obtain the analytical expression of the burst duration probability density function we assume that the burst duration is the same as the first hitting time of the stochastic process starting infinitesimally near the threshold. This approach can be backed by the recalling that the main financial market observables are actually discrete. As we are willing to use known results of the first hitting times, we have to transform (2) into the other known stochastic process. The Bessel process appears to be the best choice:

$$x^\eta \partial_x y(x) = \pm 1, \quad \Rightarrow \quad y(x) = \frac{1}{(\eta-1)x^{\eta-1}}, \quad \Rightarrow \quad \mathrm{d}y = \left(\nu + \frac{1}{2}\right)\frac{\mathrm{d}t_s}{y} + \mathrm{d}W_s,\ \nu = \frac{\lambda - 2\eta + 1}{2\eta - 2}. \tag{3}$$

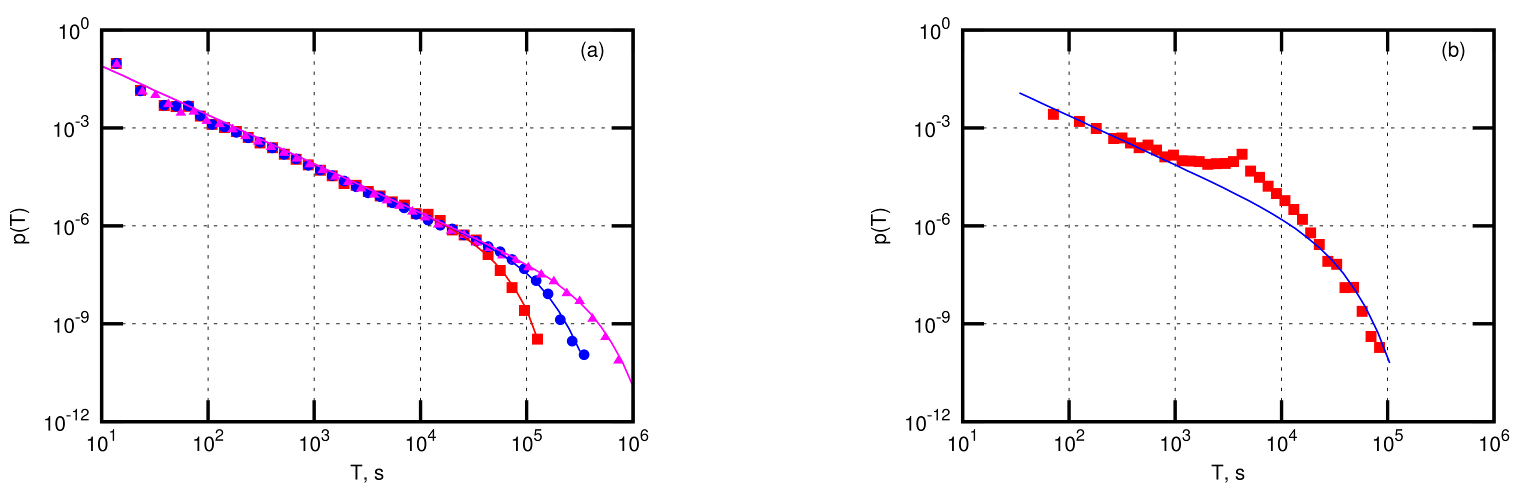

Figure 2: Fitting model (a) and empirical (b) data with (5). Model parameters were set as follows: $\lambda = 4$ (all three cases), $\eta = 2.5$ (red squares), 2 (blue circles) and 1.5 (magenta triangles). Model data fitted using $\nu = 0$ (red curve), 0.5 (blue curve), 2 (magenta curve). Empirical data fitted assuming $\nu = -0.2$.

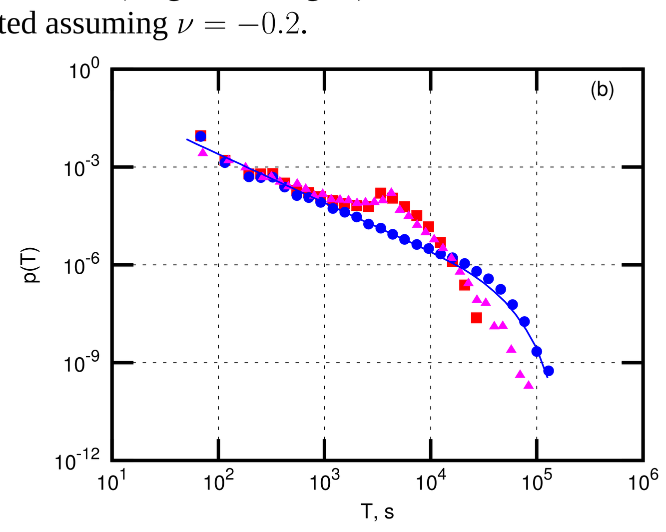

Figure 3: Comparison of the empirical data (red squares), data obtained by solving a more complex SDE (blue circles) and data obtained from the double stochastic model (magenta triangles). Model parameters were set as follows: $\eta = 2.5$, $\lambda = 3.6$, $x_{max} = 10^3$, $\epsilon = 0.017$, $\bar{r}_0 = 0.4$. Data obtained by solving a more complex SDE fitted assuming that $\nu = 0$ (blue curve).

From the transformation, $y(x)$, follows that we have to use the known results for the Bessel process, where the starting point lies bellow the certain threshold. This first hitting time is equivalent to the inter-burst duration of y, while note that we look for the burst duration of x. The needed result is given in [9]:

$$\rho^{(\nu)}_{y_0,h_y} = \frac{h_y^{\nu-2}}{y_0^\nu}\sum_{k=1}^{\infty}\frac{j_{\nu,k}J_\nu\left(\frac{y_0}{h_y}j_{\nu,k}\right)}{J_{\nu+1}(j_{\nu,k})}\exp\left(-\frac{j_{\nu,k}^2}{2}t\right). \tag{4}$$

To evaluate the above in the infinitesimal limit we have to expand $J_\nu\left(\frac{y_0}{h_y}j_{\nu,k}\right)$ near $\frac{y_0}{h_y} = 1$. And since $j_{\nu,k}$ are almost equally spaced (for k), we can replace the sum by integration, which yields:

$$p^{(\nu)}_{h_y}(t) = \lim_{y_0 \to h_y}\frac{\rho^{(\nu)}_{y_0,h_y}(t)}{h_y - y_0} \approx C\left[\frac{h_y^2 j_{\nu,1}}{t}\exp\left(-\frac{j_{\nu,1}^2 t}{2h_y^2}\right) + \sqrt{\frac{\pi}{2}}\frac{h_y^3}{t^{3/2}}\mathrm{erfc}\left(\frac{j_{\nu,1}\sqrt{t}}{\sqrt{2}h_y}\right)\right]. \tag{5}$$

Statistical properties of the other burst related variables

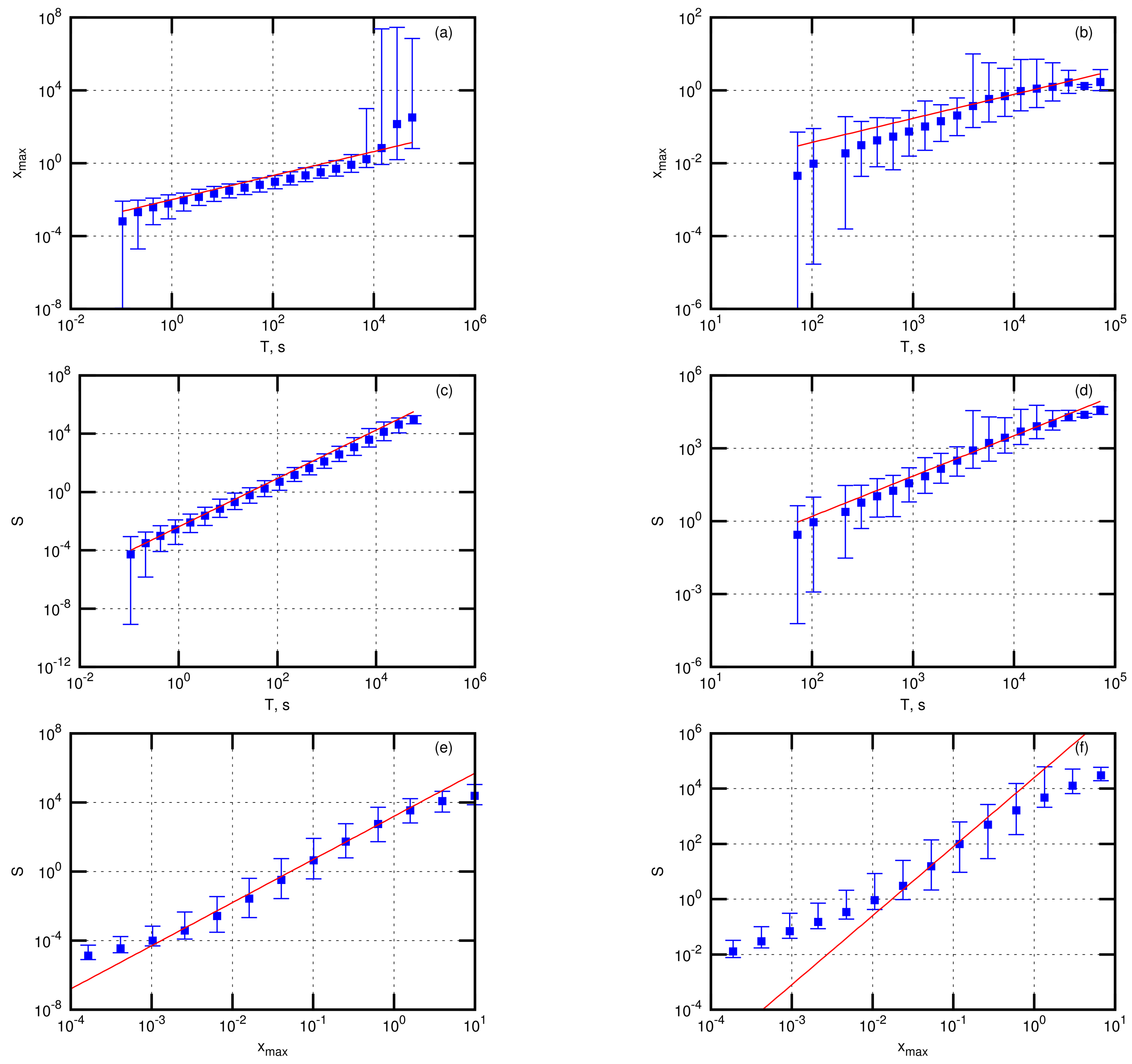

Figure 4: Scatter plots of three burst related variables, T, S and x_{max}, observed in the simple model ((a), (c), (e)) and empirical data ((b), (d), (f)). Curves provide power law fits with the exponents: $\alpha = 0.66$ ((a), (b)), 1.66 ((c), (d)), 2.5 ((e), (f))

Conclusions

- The proposed stochastic model offers analytical treatment for the bursty behavior. Which would be complicated under the point process formalism.
- Simple stochastic model, (2), is enough to reproduce most of the statistical features of bursty behavior in the financial markets.
- Double stochastic model, driven by (1) and q-Gaussian noise, may be used to recover PDF of burst durations.
- Numerical modeling suggests that the stochastic models for absolute return should be nonlinear with large powers of noise multiplicativity, $\eta > 1$.

Acknowledgments

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