On local weak limit and subgraph counts for sparse random graphs

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Valentas Kurauskas
Vilnius University

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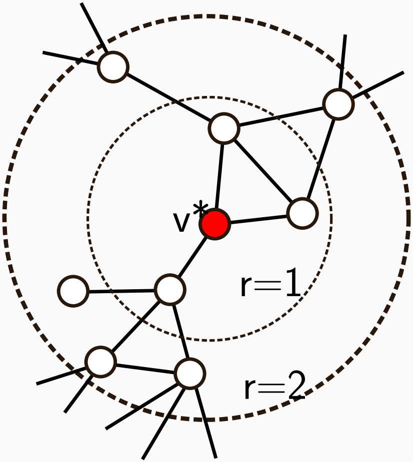
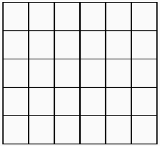
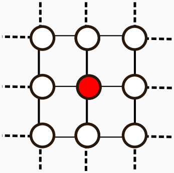
1. Sparse graphs.
2. Local weak limit.
3. Sparse random intersection graphs.
4. Why local weak limit?
5. Local and global subgraph counts.
6. Sidorenko theorem.
7. How many samples are needed?

2. Local weak limit (r-ball)



2. Local weak limit (examples)

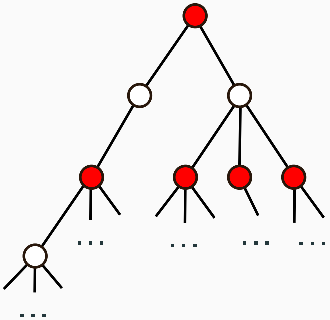
Graphs	Limit
$n \times n$ grids $n \to \infty$ 	
$G(n, \frac{c}{n})$	$Po(c)$ branching process

3. Sparse random intersection graphs (1 example)

"Passive" random intersection graph model:

- Fix a distribution μ on $\{0, 1, 2, \ldots\}$.
- For each vertex $v \in V_2$ generate random variables $Y_v \sim \mu$ independently.
- Given $\{Y_v, v \in V_2\}$ for each $v \in V_2$ select the set of its neighbours in H from V_1 independently and uniformly at random from all $\binom{|V_1|}{Y_v}$ sets of size Y_v.

3. Sparse random intersection graphs (local weak limit)



$\sim D_1$ children

$\sim D_2^* - 1$ children

$\sim D_1^* - 1$ children

$\sim D_2^* - 1$ children

...

4. Why local weak limit?

- Provides a different viewpoint to analyse random graphs: 1) prove the convergence 2) analyse the "nice" limit objects (Aldous, Asymptotic fringe distributions for general families of random trees, 1991).
- The limit implies general asymptotic results: (bi-)degree distribution, local clustering coefficient, conditional clustering coefficient, averages of bounded functionals, spectra, recurrence of random walks, spanning tree counts, etc.
- Sampling random vertices is a natural way to study empirical networks ("snowball sampling").
- Any sequence of sparse graphs with uniformly integrable degree of a random vertex has a locally weakly convergent subsequence (Benjamini, Lyons & Schramm 2015).
- Explicitly known for many random graph models.

5. Local and global subgraph counts

Theorem. (VK, 2015)

Suppose $h \geq 2$, (G_n) *has a weak local limit* $G^* = (G^*, r^*)$,

D_n *is the (random) degree of a (uniformly) random vertex in* G_n.

Then D_n^{h-1} *is uniformly integrable iff* $\frac{1}{n}\mathrm{emb}\,(H, G_n) \to \mathbb{E}\,\mathrm{emb}\,'(H', G^*)$ *for any connected graph* H *on* h *vertices and any rooted* $H' = (H, v)$, $v \in V(H)$.

$\mathrm{emb}\,(H, G)$: number of embeddings from H to G (number of copies).

$\mathrm{emb}\,'(H', G')$: number of rooted embeddings from H' to G' (the root of H' maps to the root of G').

If $X_n \geq 0$, (X_n) is *uniformly integrable* if $\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{E}\, X_n \mathbb{I}_{X_n \geq K} = 0$.

7. How many samples are needed?

Lemma. (VK, 2022)

G: any graph $|V(G)| \geq 1$.

D: degree of a uniformly random vertex in $V(G)$.

Suppose $\exists$ positive Δ and ϵ such that

$$\mathbb{E}\, D^{h-1} \mathbb{I}_{D \geq \Delta} \leq \epsilon.$$

Then the number of homomorphisms from a connected h-vertex graph H to G that "touch" a vertex of degree $\geq \Delta$ in G is at most $hn\epsilon$.

Caveat: $\mathbb{E}\, D^{h-1}$ can be huge for social networks, even for $h = 3$.