24th International Meeting on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms

On the number of graphs with few disjoint excluded minors

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The problem

$\mathcal{A}$: a class of labelled graphs, closed under isomorphism.

- What is the size of $\mathcal{A}_n$ = the set of graphs in $\mathcal{A}$ on the vertex set $\{1, \dots, n\}$?
- What is the structure of a typical graph in $\mathcal{A}_n$?

Note: in general, lots of dependence in the edges of a graph drawn uniformly at random from $\mathcal{A}_n$!

$\mathcal{B}$ = fixed finite set of graphs. A graph is bad, if it has a minor (↗) in $\mathcal{B}$. $\mathrm{Ex}\,\mathcal{B}$ = graphs that are not bad (no minor in $\mathcal{B}$). We study classes $\mathrm{Ex}\,(k+1)\mathcal{B}$ of graphs that do not have $k+1$ vertex-disjoint bad subgraphs (k fixed).

Example. $\mathrm{Ex}\,(k+1)\{K_3\}$ is the class of all graphs that do not have $k+1$ vertex disjoint cycles. This and similar classes were studied before (↓).

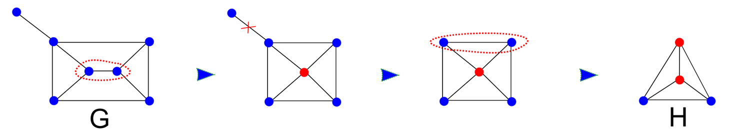
Classes in the current work include the case $\mathcal{B} = \{K_4\}$, the graphs which do not contain

$$\underbrace{\boxtimes\ \boxtimes \dots \boxtimes}_{k+1 \text{ times}}$$

as a minor.

Graph minors

H is a minor of G if it can be obtained from G by a series of edge contractions/deletions.



Minor-closed classes are subject to fundamental graph theory results. For example, Kuratowski's theorem states that planar graphs are exactly those graphs that do not have $K_{3,3}$ or K_5 as a minor. Robertson's and Seymour's theory of graph minors proves that every minor-closed class of graphs has a finite list of forbidden minors.

For the class of planar graphs the two main questions (↖) have been studied quite intensively in the past decade (Bender, Bernardi, Bodirsky, Drmota, Fusy, Gao, Gimenez, Kang, McDiarmid, Noy, Steger, Panagiotou, Welsh, Wormald…). For this class, and for classes with 2-connected excluded minors, the asymptotic growth and the properties of random graphs are now quite well understood.

Growth constants

$\mathcal{A}$ has a growth constant γ if

$$(|\mathcal{A}_n|/n!)^{1/n} \to \gamma.$$

$\overline{\gamma}(\mathcal{A})$=upper limit of the above $= \rho(\mathcal{A})^{-1}$= inverse radius of convergence of the EGF of $\mathcal{A}$.

Conjecture *(Bernardi, Noy and Welsh, 2010). Every minor-closed class of graphs has a growth constant.*

Redundant $\mathcal{B}$-blocker Q of G = every bad subgraph of G meets Q in at least two points. $\mathrm{rd}_l\,\mathcal{B}$ = the class of graphs that have a redundant $\mathcal{B}$-blocker of size at most l.

$\mathrm{rd}_{2k+1}\,\mathcal{B} \subseteq \mathrm{Ex}\,(k+1)\mathcal{B}$: each bad subgraph in G uses at least two vertices from the blocker, so we can form no more than k disjoint such subgraphs.

Theorem 1. *If $\mathrm{Ex}\,\mathcal{B}$ contains all fans and condition A holds then there is a constant $k_0 = k_0(\mathcal{B})$ such that for all $k \geq k_0$*

$$\overline{\gamma}(\mathrm{Ex}\,(k+1)\mathcal{B}) = \overline{\gamma}(\mathrm{rd}_{2k+1}\,\mathcal{B}).$$

Examples of suitable $\mathcal{B}$: $\{K_{2,3}\}$, $\{K_4\}$, $\{W_5\}$ and $\{K_{3,3}, P_0\}$. W_5 = wheel on 5 vertices; P_0 = arbitrary planar graph.

For $\mathcal{B}$ and k as in Theorem 1, there are arbitrarily large n such that in almost all graphs on $\{1, \dots, n\}$ which do not have $k+1$ bad subgraphs, we need at least $2k$ vertices to block all bad subgraphs. This is twice more than for $\mathcal{B}$ considered earlier (↙).

Theorem 2. *If $\mathrm{Ex}\,\mathcal{B}$ contains all fans and conditions A and B hold then there is a constant $k_0 = k_0(\mathcal{B})$ such that for all $k \geq k_0$ the class $\mathrm{Ex}\,(k+1)\mathcal{B}$ has a growth constant.*

Examples of suitable $\mathcal{B}$: $\{K_4\}$, $\{W_5\}$.

Condition A

The graphs in $\mathcal{B}$ are 2-connected, there is at least one planar graph and there is a finite number m such that if we take any tree T of size m and add two apex vertices connected to each vertex of T, then we obtain a bad graph.

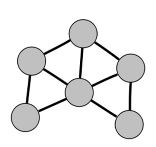
Condition B

There are finite numbers m and l such that if we take a 2-connected planar graph H of size m and a new vertex with l neighbours in $V(H)$, then we obtain a bad graph.

Small blockers

$\mathcal{B}$-blocker Q of G = every bad subgraph of G meets Q in at least one point = $G - Q$ is not bad.

Fans = graphs consisting of a path and an apex vertex connected to each vertex on the path.

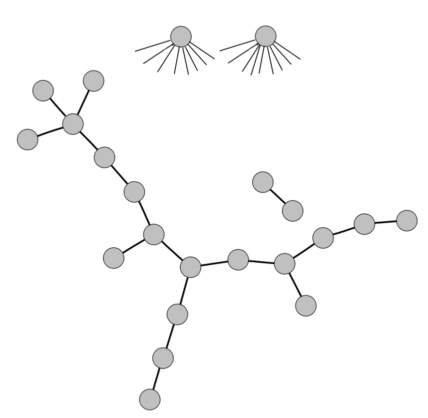


Theorem. *(V.K. and C. McDiarmid, 2011). If the graphs in $\mathcal{B}$ are 2-connected and $\mathcal{A} = \mathrm{Ex}\,\mathcal{B}$ does not contain all fans, then all but an exponentially small proportion of graphs without $k+1$ disjoint bad subgraphs have a $\mathcal{B}$-blocker of size k.*

The number of such graphs on the vertex set $\{1, \dots, n\}$ is $c(k, \mathcal{B}) 2^{kn} |\mathcal{A}_n| \left(1 - e^{-\Omega(n)}\right)$.

In other words, the simplest possible type of structures overhelmingly dominates the class of all graphs without $k+1$ disjoint minors in $\mathcal{B}$.

Example. A random graph on vertex set $\{1, \dots, n\}$ without $k+1$ disjoint cycles is "almost" a uniformly random forest F along with k "apex" vertices, where the edges between the apex vertices are independent and exist with probability 1/2. A uniformly random graph without 3 disjoint cycles:



Graphs with no $k+1$ disjoint minors K_4

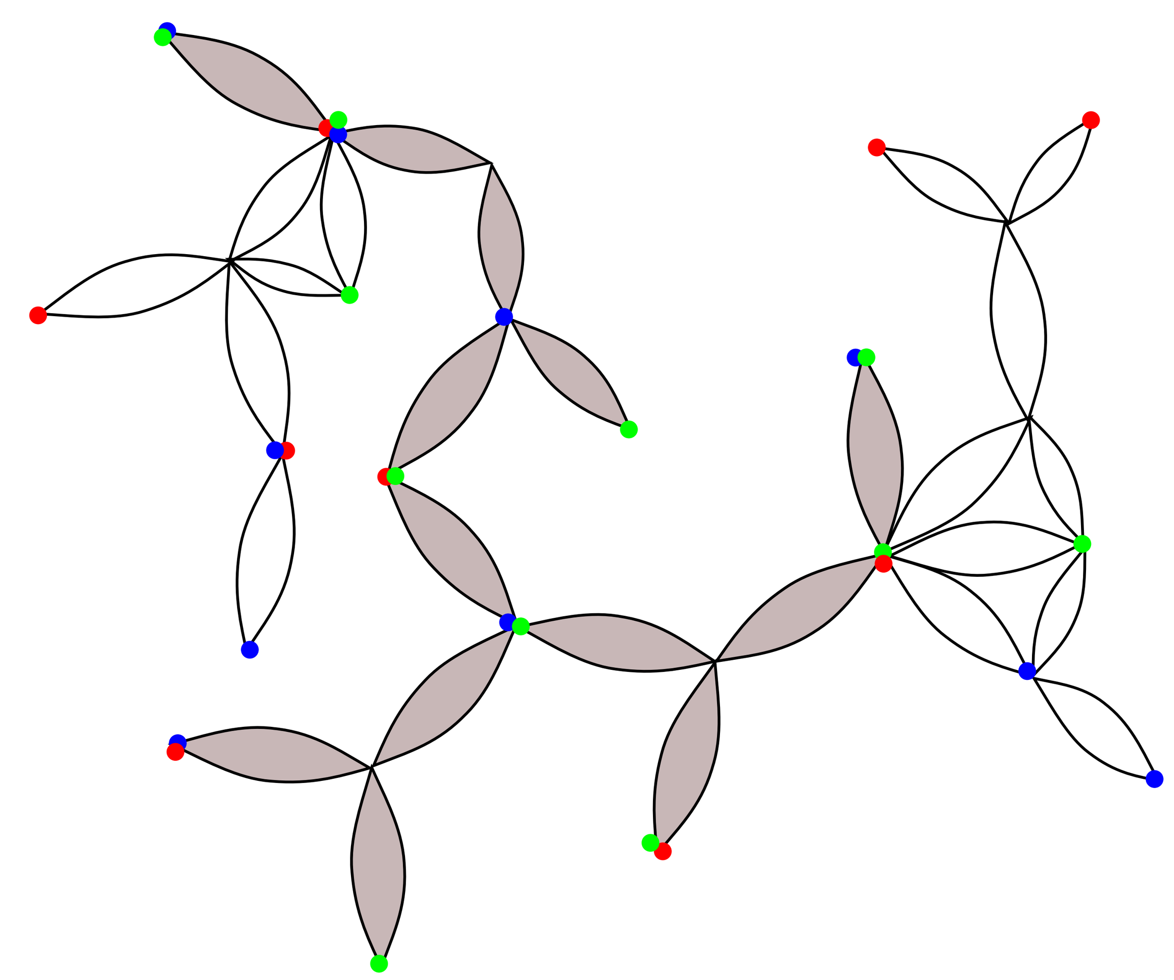
Theorem 3. *Let $\mathcal{A} = \mathrm{Ex}\,(k+1)\{K_4\}$. For any $k = 1, 2, \cdots$*

- *$\mathcal{A}$ has a growth constant γ_k.*
- $\gamma_1 = \gamma(\mathrm{Ex}\ \boxtimes\ \boxtimes) = 23.5241..$
- *A uniformly random graph from $\mathcal{A}_n$ whp has a unique redundant blocker Q of size $2k+1$.*
- *Each vertex of Q whp has a linear degree.*
- *There is $c_k > 0$ such that $|\mathcal{A}_n| = c_k n^{-5/2} n! \gamma_k^n (1 + o(1))$.*

Proof elements

- **Theory of graph minors.** Robertson and Seymour proved that any $G \in \mathrm{Ex}\,(k+1)\mathcal{B}$ has a constant size $\mathcal{B}$-blocker, as long as $\mathcal{B}$ contains a planar graph.
- **Earlier results on planar and other minor-closed classes.** Ideas of McDiarmid, Steger and Welsh (2005) help to *avoid* generating functions in the proof of Theorem 2. A new version of a lemma of V.K. and McDiarmid (2011): every $G \in \mathrm{Ex}\,(k+1)\mathcal{B}$ can be represented by (a) a constant-size set Q (b) a set $S \subseteq Q$ of size at most $2k$ and (c) a graph in $\mathrm{Ex}\,\mathcal{B}$, so that each bad sugraph that meets Q in at most two points, must meet S.
- **New structural results** If $G - Q$ is connected, we can *split* (= reverse contraction) a constant number of vertices in G and represent G as $O(1)$ graphs in $\mathrm{rd}_{2k+1}\,\mathcal{B}$ and $O(1)$ edges (the crucial point).
- **Analytic combinatorics.** Easy by the "First principle" of Analytic Combinatorics: $\mathrm{Ex}\,(2k+1)\mathcal{B} \approx$ a class of sets of graphs composed of $O(1)$ elements from $\mathrm{rd}_{2k+1}\,\mathcal{B} \implies \rho(\mathrm{Ex}\,(k+1)\mathcal{B}) = \rho(\mathrm{rd}_{2k+1}\,\mathcal{B})$.
- **More analytic combinatorics** Bodirsky, Gimenez, Kang and Noy (2005) obtained expressions for the EGF of series-parallel graphs = $\mathrm{Ex}\,\{K_4\}$. Using these, we analyse the structure (↓) of graphs with small redundant blockers. This includes counting trees where leaves, internal vertices and edges are substituted by objects of different types.

Typical graphs without two disjoint minors K_4



Leaf-like shapes = arbitrary series-parallel *networks*, most of them small. Above is just a "core" of a graph in $\mathrm{Ex}\,2K_4$ (it is typically of linear size). Attach arbitrary rooted series-parallel graphs at arbitrary vertices of this; add a redundant blocker - $\{x, y, z\}$ - and connect x to each red point, y to each green point and z to each blue point. Unlike in planar graphs, there are vertices of different types: colours can occur only at the "joints"; only the joints of the grey blocks can have arbitrary colour. I.e., the top-right vertex cannot have colour blue, otherwise z alone would form a minor K_4, and $\{x, y, z\}$ would not be redundant.

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