ON THE NUMBER OF GRAPHS WITH FEW DISJOINT EXCLUDED MINORS

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The problem

\[ A: \] a class of labelled graphs, closed under isomorphism.

- What is the size of \( A_n \) = the set of graphs in \( A \) on the vertex set \( \{1, \ldots, n\} \)?
- What is the structure of a typical graph in \( A_n \)?

Note: in general, lots of dependence in the edges of a graph drawn uniformly at random from \( A \).

\[ B = \text{fixed finite set of graphs. A graph is bad if it has a minor (\( \uparrow \)) in } B. \text{ Ex: } B = \text{graphs that are not bad (no minor in } B). \text{ We study cases } \text{Ex}(k+1)B \text{ of graphs that do not have } k+1 \text{ vertex-disjoint bad subgraphs (} k \text{ fixed).}

\textbf{Example.}\hspace{1em} \text{Ex}(k+1)\{K_4\} \text{ is the class of all graphs that do not have } k+1 \text{ vertex disjoint cycles. This and similar classes were studied before (\( \downarrow \)).}

Classes in the current work include the case \( B = \{K_4\} \), the graphs which do not contain \( \begin{array}{ccc} & & \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ 
\end{array} \) k +1 times as a minor.

\textbf{Small blockers}

\( B \)-blocker \( Q \) of \( G \) = every bad subgraph of \( G \) meets \( Q \) in at least one point = \( G - Q \) is not bad.

\textbf{Fans} = graphs consisting of a path and an apex vertex connected to each vertex on the path.

\textbf{Theorem. (V.K. and C. McDiarmid, 2011).} If the graphs in \( B \) are 2-connected and \( A = \text{Ex} B \) does not contain all fans, then all but an exponentially small proportion of graphs without \( k+1 \) disjoint bad subgraphs have a \( B \)-blocker of size \( k \).

The number of such graphs on the vertex set \( \{1, \ldots, n\} \) is \( c(k, B)2^{kn}A_n(1 - e^{-Bn(n)}) \).

\textbf{Graph minors}

\( H \) is a minor of \( G \) if it can be obtained from \( G \) by a series of edge contractions/deletions.

\textbf{Small blockers}

\begin{itemize}
  \item A morphism.
  \item \( \text{if it has a minor (\( \uparrow \)) in } B \).
  \item \( \text{Ex}(k+1)B \text{ is the class of all graphs without } k+1 \text{ vertex-disjoint bad subgraphs (} k \text{ fixed).}
\end{itemize}

\textbf{Example.}\hspace{1em} \text{Ex}(k+1)\{K_4\} \text{ is the class of all graphs that do not have } k+1 \text{ vertex disjoint cycles. This and similar classes were studied before (\( \downarrow \)).}

\textbf{Classes in the current work include the case } B = \{K_4\}, \text{ the graphs which do not contain } \begin{array}{ccc} & & \\
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\circ & \circ & \circ \\
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\end{array} \text{ k +1 times as a minor.}

\textbf{Growth constants}

\( A \) has a growth constant \( \gamma \) if

\[ \left(\frac{|A_n|}{n!}\right)^{1/n} \rightarrow \gamma. \]

\( \tau(A) = \text{upper limit of the above} = \rho(A)^{-1} \text{ is the inverse radius of convergence of the EGF of } A. \)

\textbf{Conjecture (Bernardi, Noy and Welsh, 2010).} Every minor-closed class of graphs has a growth constant. Redundant \( B \)-blocker \( Q \) of \( G \) = every bad subgraph of \( G \) meets \( Q \) in at least two points. \( rd_kB = \text{the class of graphs that have a redundant } B \)-blocker of size at most \( k \).

\[ rd_{k+1}B \subseteq \text{Ex}(k+1)B \text{ each bad subgraph in } G \text{ uses at least two vertices from the blocker, so we can form more than } k \text{ disjoint such subgraphs.}
\]

\textbf{Theorem 1.} \text{If } ExB \text{ contains all fans and condition A holds then there is a constant } k_0 = k_0(B) \text{ such that for all } k \geq k_0

\[ \tau(Ex(k+1)B) = \tau(rd_{k+1}B). \]

Examples of suitable \( B : \{K_4, 3\}, \{K_4, \{W_5\}\} \text{ and } \{K_3, P_6\}. \)

\[ W_5 = \text{wheel on } 5 \text{ vertices; } P_6 = \text{any arbitrary planar graph.}
\]

\textbf{For } B \text{ and } k \text{ as in Theorem 1, there are arbitrarily large } n \text{ such that in almost all graphs on } \{1, \ldots, n\} \text{ which do not have } k+1 \text{ bad subgraphs, we need at least } 2k \text{ vertices to block all bad subgraphs. This is twice more than for } B \text{ considered earlier (\('\)').}

\textbf{Theorem 2.} \text{If } ExB \text{ contains all fans and conditions A and B hold then there is a constant } k_0 = k_0(B) \text{ such that for all } k \geq k_0 \text{ the class } Ex(k+1)B \text{ has a growth constant.}

\text{Examples of suitable } B : \{K_4, \{W_5\}\}. \]

\textbf{Graphs with no } k+1 \text{ disjoint minors } K_4

\textbf{Theorem 3.} \text{Let } A = \text{Ex}(k+1)\{K_4\}. \text{ For any } k = 1, 2, \cdots

\[ A \text{ has a growth constant } \gamma_k \text{.}
\]

\[ \gamma_1 = \gamma(\text{Ex } K_4) = 23.5241. \]

\[ A \text{ is uniformly random graph from } A_n \text{ whp has a unique redundant blocker } Q \text{ of size } 2k + 1. \]

\[ \text{Each vertex of } Q \text{ has a linear degree.}
\]

\[ \text{There is } c_k > 0 \text{ such that } |A_n| = c_kn^{-5/2}n^{\gamma_k}(1 + o(1)).\]

\textbf{Problems}

\textbf{Theory of graph minors.}

Robertson and Seymour proved that any } G \in \text{Ex} \{k+1\}B \text{ has a constant size } B \text{-blocker, as long as } B \text{ contains a planar graph.

\textbf{Earlier results on planar and other minor-closed classes.}

Ideas of McDiarmid, Steger and Welsh (2005) help to avoid generating functions in the proof of Theorem 2. A new version of a lemma of V.K. and McDiarmid (2011): Every } G \in \text{Ex} \{k+1\}B \text{ can be represented by \( (a) \) a constant-size set } Q \text{ (b) a set } S \subseteq Q \text{ of size at most } 2k \text{ and } \( c \) \text{ a graph in } ExB \text{, so that each bad subgraph that meets } Q \text{ in at most two points, must meet } S.

\textbf{New structural results}

If } G - Q \text{ is connected, we can split (= reverse contraction) a constant number of vertices in } G \text{ and represent } G \text{ as } O(1) \text{ graphs in } rd_{2k+1}B \text{ and } O(1) \text{ edges (the crucial point}).

\textbf{Analytic combinatorics.}

Easy by the “First principle” of Analytic Combinatorics: } Ex(2k+1)B \approx \text{ a class of sets of graphs composed of } O(1) \text{ elements from } rd_{2k+1}B \Rightarrow \rho(Ex(k+1)B) = \rho(rd_{2k+1}B).

\textbf{More analytic combinatorics}

Bodirsky, Gimenez, Kang and Noy (2005) obtained expressions for the EGF of series-parallel graphs = Ex \{K_4\}. Using these, we analyse the structure (\( I \)) of graphs with small redundant blockers. This includes counting trees where leaves, internal vertices and edges are substituted by objects of different types.

\textbf{Typical graphs without two disjoint minors } K_4

\[ \text{Leaf-like shapes} = \text{arbitrary series-parallel networks, most of them small. Above is just a “core” of a graph in } Ex2K_4 \text{ (is typically of linear size). Attach arbitrary rooted series-parallel graphs at arbitrary vertices of this; add a redundant blocker - (} x, y, z \text{) - and connect } x \text{ to each red point, } y \text{ to each green point and } z \text{ to each blue point. Unlike in planar graphs, there are vertices of different types: colours can occur only at the “joints”; only the joints of the grey blocks can have arbitrary colour } \).

\text{the top-right vertex cannot have colour blue, otherwise } z \text{ alone would form a minor } K_3 \text{ and } (x, y, z) \text{ would not be redundant.}