Random graphs containing few disjoint excluded minors

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Abstract

The Erdős-Pósa theorem (1965) states that in each graph G which contains at most k disjoint cycles, there is a 'blocking' set B of at most $f(k)$ vertices such that the graph $G - B$ is acyclic. Robertson and Seymour (1986) give an extension concerning any minor-closed class $\mathcal{A}$ of graphs, as long as $\mathcal{A}$ does not contain all planar graphs: in each graph G which contains at most k disjoint excluded minors for $\mathcal{A}$, there is a set B of at most $g(k)$ vertices such that $G - B$ is in $\mathcal{A}$.

In an earlier paper [**?**], we showed that, amongst all graphs on vertex set $[n] = \{1, \ldots, n\}$ which contain at most k disjoint cycles, all but an exponentially small proportion contain a blocking set of just k vertices.

In the present paper we build on the previous work, and give an extension concerning any minor-closed graph class $\mathcal{A}$ with 2-connected excluded minors, as long as $\mathcal{A}$ does not contain all fans (here a 'fan' is a graph consisting of a path together with a vertex joined to each vertex on the path). We show that amongst all graphs G on $[n]$ which contain at most k disjoint excluded minors for $\mathcal{A}$, all but an exponentially small proportion contain a set B of k vertices such that $G - B$ is in $\mathcal{A}$. (This is not the case when $\mathcal{A}$ contains all fans.) For a random graph R_n sampled uniformly from the graphs on $[n]$ with at most k disjoint excluded minors for $\mathcal{A}$, we consider also vertex degrees and the uniqueness of small blockers, the clique number and chromatic number, and the probability of being connected.

Keywords: disjoint excluded minors, blocker, fan.

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1 Introduction

The classical theorem of Erdős and Pósa from 1965 [?] states that for each positive integer k there is a value $f(k)$ such that the following holds: for each graph G which contains at most k disjoint cycles, there is a set B of at most $f(k)$ vertices such that the graph $G-B$ obtained by deleting the vertices in B is acyclic. (In this paper, 'disjoint' always means pairwise vertex-disjoint.) It is also shown in [?] that the least value we may take for $f(k)$ is of order $k \ln k$.

For a class $\mathcal{A}$ of graphs (always taken to be closed under isomorphism), let $\mathcal{A}_n$ denote the set of graphs in $\mathcal{A}$ on the vertex set $[n]$ and let $\mathrm{apex}^k \mathcal{A}$ denote the class of all graphs such that by deleting at most k vertices we may obtain a graph in $\mathcal{A}$. Given a class $\mathcal{B}$ of graphs we let $\mathrm{Ex}\,\mathcal{B}$ denote the class of graphs with no minor in $\mathcal{B}$. Also, given a positive integer j, we let $j\mathcal{B}$ denote the class of graphs consisting of j vertex disjoint copies of graphs in $\mathcal{B}$ (with repetitions allowed). For notational convenience, we write C for C_3, and we omit brackets where the meaning stays clear. Thus $\mathrm{Ex}\,C$ and $\mathrm{Ex}\,C_3$ both denote the class $\mathcal{F}$ of forests, and $\mathrm{Ex}\,jC$ is the class of graphs which do not have j disjoint cycles.

In our notation, the Erdős-Pósa theorem says that

$$\mathrm{Ex}\,(k+1)C \subseteq \mathrm{apex}^{f(k)} \mathcal{F}.$$

Also, clearly

$$\mathrm{Ex}\,(k+1)C \supseteq \mathrm{apex}^{k} \mathcal{F}, \tag{1}$$

since if $G-B$ is a forest then G can have at most $|B|$ disjoint cycles. The inclusion (**??**) is 'nearly an equality'. To be more precise, it was shown in [?] that as $n \to \infty$

$$|(\mathrm{Ex}\,(k{+}1)C)_n| = \left(1 + e^{-\Omega(n)}\right)\; |(\mathrm{apex}^{k} \mathcal{F})_n|. \tag{2}$$

Thus $\mathrm{Ex}\,(k{+}1)C$ consists of $\mathrm{apex}^k \mathcal{F}$ together with an exponentially smaller class of 'exceptional' graphs. A similar result holds for unlabelled graphs [?]; we consider only labelled graphs in this paper.

The Erdős-Pósa theorem was generalised in 1986 by Robertson and Seymour [?]. Let $\mathcal{A}$ be a *minor-closed* class of graphs; that is, if $G \in \mathcal{A}$ and H is a minor of G then $H \in \mathcal{A}$. Then $\mathcal{A}$ may be written as $\mathrm{Ex}\,\mathcal{B}$, where $\mathcal{B}$ consists of the minor-minimal graphs not in $\mathcal{A}$, the *excluded minors of* $\mathcal{A}$; and $\mathcal{B}$ is finite by the fundamental result in 2004 by the same authors [?]. A $\mathcal{B}$*-minor-blocker* for a graph G is a set B of vertices such that $G-B$ is in $\mathrm{Ex}\,\mathcal{B}$.

The generalisation of the Erdős-Pósa theorem is as follows. Let $\mathcal{A}$ be any minor-closed class of graphs which does not include some planar graph, and let $\mathcal{B}$ be its set of excluded minors, so that $\mathcal{A}$ is $\mathrm{Ex}\,\mathcal{B}$. Then for each positive integer k there is a value $g(k)$ such that the following holds: each graph G which does not have as a minor a graph formed from $k+1$ vertex disjoint members of $\mathcal{B}$ contains a set B of at most $g(k)$ vertices such that $G-B$ is in $\mathcal{A}$ (that is, B is a $\mathcal{B}$-minor-blocker)[1]. In symbols we have

$$\mathrm{Ex}\,(k+1)\mathcal{B} \subseteq \mathrm{apex}^{g(k)}\,\mathcal{A}.$$

The assumption that some excluded minor is planar cannot be dropped [**?**].

Of course, there is an obvious containment result corresponding to (**??**), namely

$$\mathrm{Ex}\,(k+1)\mathcal{B} \supseteq \mathrm{apex}^{k}\,\mathcal{A}. \tag{3}$$

How much bigger is the left hand side than the right in this case? For which classes $\mathcal{B}$ is there an 'almost equality' result corresponding to (**??**)? Our main result provides a sufficient and essentially best possible condition for the class $\mathrm{Ex}\,(k+1)\mathcal{B}$ to be 'almost apex'.

A class of graphs is *proper* if it is not the class of all graphs. Following [**?**] we call a minor-closed class $\mathcal{A}$ *addable* if each excluded minor is 2-connected. (Thus each such class contains all forests.) The *fan* F_j is the graph consisting of a path P_{j-1} of $j-1$ vertices together with a vertex joined to each vertex on this path. Observe that the addable class $\mathcal{F} = \mathrm{Ex}\,C$ of forests does not contain the fan F_3: in contrast, the addable class $\mathrm{Ex}\,K_4$ of series-parallel graphs contains all fans. The following theorem is our central result.

Theorem 1.1 *Let $\mathcal{A}$ be a proper minor-closed class of graphs, with set $\mathcal{B}$ of excluded minors. If $\mathcal{A}$ is addable and does not contain all fans, then for each positive integer k, as $n \to \infty$*

$$|(\mathrm{Ex}\,(k{+}1)\mathcal{B})_n| = (1 + e^{-\Theta(n)})|(apex^k\,\mathcal{A})_n|. \tag{4}$$

On the other hand, if $\mathcal{A}$ contains all fans then this result fails; and indeed there is a constant c such that for all positive integers k and n

$$|(\mathrm{Ex}\,(k{+}1)\mathcal{B})_n| \geq 2^{(k-c)n}\ |(apex^k\,\mathcal{A})_n|.$$

Let us consider a few examples illustrating this result. Recall that the number $f(k)$ in the Erdős-Pósa theorem [**?**] must be of order $k \ln k$. From (**??**) and Theorem **??** it follows that by removing just k vertices we can obtain:

[1] Only the case of a single excluded minor is considered in [**?**] but the extension is straightforward, see also Proposition **??** below.

- a forest from almost every graph with at most k disjoint cycles [**?**];
- more generally, a graph without any cycles of length at least ℓ from almost every graph with at most k disjoint cycles of length at least ℓ (see also [**?**]);
- a collection of cacti (that is, a graph with each edge in at most one cycle) from almost every graph with at most k disjoint subdivisions of the diamond graph $D = K_4 - e$.

In contrast, by Remark **??** below, almost none of the graphs in $\mathrm{Ex}\, 2K_4$ can be turned into a series-parallel graph by removing one vertex.

A natural partner for this theorem is an asymptotic estimate for sizes of apex classes. Recall that a class $\mathcal{A}$ of graphs has *growth constant* $\gamma > 0$ if

$$|\mathcal{A}_n| = (\gamma + o(1))^n n! \quad \text{as } n \to \infty.$$

It is easy to see that if $\mathcal{A}$ has growth constant γ then $\mathrm{apex}^j\, \mathcal{A}$ has growth constant $2^j\gamma$, see for example [**?**, **?**], but we want a more precise result. Every proper addable minor-closed class of graphs has a growth constant $\gamma > 0$, see [**?**, **?**]. For two sequences of reals (a_n) and (b_n) which are positive for n sufficiently large, we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. The next result extends Theorem 1.2 of [**?**] on forests.

Theorem 1.2 *Let $\mathcal{A}$ be a proper addable minor-closed class of graphs, with growth constant γ; and let k be a fixed positive integer. Then as $n \to \infty$*

$$|(apex^k\, \mathcal{A})_n| \;\sim\; c_k 2^{kn}\; |\mathcal{A}_n|$$

where $c_k = \left(2^{\binom{k+1}{2}} \gamma^k k!\right)^{-1}$.

The above results yield asymptotic properties of typical graphs with at most k disjoint excluded minors. We state three theorems. First we note that with high probability k vertices really stand out – they each have degree about $n/2$ whereas each other vertex has much smaller degree – and they form the only minimal blocker of sublinear size. We write $R_n \in_u \mathcal{A}$ to mean that the random graph R_n is sampled uniformly from the graphs in $\mathcal{A}_n$. Thus for $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$, equation (**??**) in Theorem **??** says that R_n has a blocker of size k with probability $1 - e^{-\Theta(n)}$: the next theorem refines this result.

Theorem 1.3 *Let $\mathcal{A}$ be an addable minor-closed class of graphs which does not contain all fans, and let $\mathcal{B}$ be the set of excluded minors for $\mathcal{A}$. There is a constant $\delta > 0$ such that the following holds. Let k be a positive integer and let $0 < \varepsilon < \frac{1}{2}$. For $n = 1, 2, \ldots$ let $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$, and let S_n be the set of vertices in R_n with degree $> \varepsilon n$. Then with probability $1 - e^{-\Omega(n)}$ we have:*

(i) $|S_n| = k$ and S_n is a $\mathcal{B}$-minor-blocker in R_n;

(ii) each vertex in S_n has degree between $(\frac{1}{2} - \varepsilon)n$ and $(\frac{1}{2} + \varepsilon)n$; and

(iii) each $\mathcal{B}$-minor-blocker in R_n not containing S_n has size $> \delta n$.

Our second theorem on random graphs R_n concerns the clique number $\omega(R_n)$ and the chromatic number $\chi(R_n)$. Given a class $\mathcal{A}$ of graphs let

$$\omega(\mathcal{A}) = \sup\{\omega(G) : G \in \mathcal{A}\} \ \text{ and } \ \chi(\mathcal{A}) = \sup\{\chi(G) : G \in \mathcal{A}\}.$$

If $\mathcal{A}$ is a proper minor-closed class then these quantities are finite, since the average degree of the graphs in $\mathcal{A}$ is bounded, by a result of Mader [**?**], see also Theorem 7.22 and Corollary 5.23 in [**?**]. For example, if $\mathcal{A}$ is $\mathrm{Ex}\, C_4$ then $\omega(\mathcal{A}) = \chi(\mathcal{A}) = 3$ (since each block of each graph in $\mathcal{A}$ is an edge or a triangle). If also $\mathcal{A}$ is addable then we may use the 'pendant appearances theorem' of [**?**], restated as Lemma **??** below, to show that for $R_n \in_u \mathcal{A}$

$$\omega(R_n) = \omega(\mathcal{A}) \quad \text{and} \quad \chi(R_n) = \chi(\mathcal{A}) \quad \text{with probability } 1 - e^{-\Omega(n)}.$$

Recall that the total variation distance $d_{TV}(X, Y)$ between two random variables X and Y is the supremum over all (measurable) sets A of $|\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|$. The next result shows that for $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$ as defined below, the random pair consisting of $\omega(R_n)$ and $\chi(R_n)$ is very close in total variation distance to a certain simply defined pair of random variables.

Theorem 1.4 *Let $\mathcal{A}$ be an addable minor-closed class of graphs which does not contain all fans, and let $\mathcal{B}$ be the set of excluded minors for $\mathcal{A}$. Let k be a positive integer; let the random graph R be picked uniformly from the set of all graphs on $\{1, \ldots, k\}$; and let $X = \omega(R) + \omega(\mathcal{A})$ and $Y = \chi(R) + \chi(\mathcal{A})$. For each n let $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$. Then*

$$d_{TV}\left((\omega(R_n), \chi(R_n)), (X, Y)\right) = e^{-\Omega(n)}.$$

For example, for $R_n \in_u \mathrm{Ex}\, 3C_4$, both $\mathbb{P}(\chi(R_n) = 4)$ and $\mathbb{P}(\chi(R_n) = 5)$ are $\frac{1}{2} + e^{-\Omega(n)}$. Since there is only a finite range of relevant values, the result above is saying essentially that

$$\mathbb{P}(\omega(R_n) = i \text{ and } \chi(R_n) = j) = \mathbb{P}(X = i \text{ and } Y = j) + e^{-\Omega(n)}$$

for each $1 + \omega(\mathcal{A}) \leq i \leq j \leq k + \chi(\mathcal{A})$, and the probability that the pair $(\omega(R_n), \chi(R_n))$ does not take values in this range is $e^{-\Omega(n)}$.

The third and final theorem on random graphs R_n presented here concerns connectivity. We let $\mathrm{frag}(G)$ denote $|V(G)|$ minus the maximum number of vertices in a component of G.

Theorem 1.5 *Let $\mathcal{A}$ be an addable minor-closed class of graphs which does not contain all fans; let $\mathcal{B}$ be the set of excluded minors for $\mathcal{A}$; and let $C(z) = \sum_{n \geq 1} |\mathcal{C}_n| z^n/n!$ be the exponential generating function for the class $\mathcal{C}$ of connected graphs in $\mathcal{A}$, with radius of convergence ρ. Given a positive integer k, for $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$ we have as $n \to \infty$*

$$\mathbb{P}(R_n \textit{ is connected}) \to e^{-C(\rho/2^k)}$$

and

$$\mathbb{E}\left[\mathrm{frag}(R_n)\right] \to (\rho/2^k)\ C'(\rho/2^k) < \infty.$$

See [?] for numerical values for these limiting probabilities in the case when $\mathcal{A}$ is the class of forests. We shall actually prove a detailed extension of this result, Theorem **??** below, concerning the limiting distribution of the unlabelled 'fragment' graph formed from the vertices not in the 'giant' component.

We now state two key intermediate results needed to prove our main theorem, Theorem **??**. The first one extends a case of the Robertson-Seymour generalisation of the Erdős-Pósa theorem. Our extension asserts that, under suitable conditions, in graphs with few disjoint excluded minors there are small blockers with an additional 'redundancy' property. We write $R \cup v$ and $R \setminus v$ to denote $R \cup \{v\}$ and $R \setminus \{v\}$ respectively.

Lemma 1.6 *Let $\mathcal{B}$ be a set of 2-connected graphs containing at least one planar graph. Then for each integer $k \geq 1$ there is an integer $f(k)$ (depending on $\mathcal{B}$) such that the following holds. Each graph G in $\mathrm{Ex}\,(k+1)\mathcal{B}$ has a $\mathcal{B}$-minor-blocker R with $|R| \leq f(k)$ such that for all but at most k vertices v in R, the set $R \setminus v$ is still a $\mathcal{B}$-minor-blocker.*

The second result concerns the existence of vertex degrees of linear order. A class $\mathcal{A}$ of graphs has the *no-linear-degrees property* if, with $R_n \in_u \mathcal{A}$ as usual, for each $\delta > 0$ and each α we have $\mathbb{P}[\Delta(R_n) \geq \delta n] = O(e^{-\alpha n})$. (Here $\Delta(G)$ denotes the maximum vertex degree in G.) Observe that if $|\mathcal{A}_n| = O(\gamma^n n!)$ for some finite γ (as holds for every proper minor-closed class of graphs [**?**, **?**]), and if $\mathcal{A}$ contains all fans, then $\mathcal{A}$ does not have this property: for there are $\frac{1}{2}(n-1)!$ fans on $[n]$ with vertex 1 as the centre vertex (for $n \geq 3$), and so

$$\mathbb{P}(\text{vertex 1 has degree } n-1 \text{ in } R_n) \geq \frac{\frac{1}{2}(n-1)!}{|\mathcal{A}_n|} = \Omega(n^{-1}\gamma^{-n}).$$

The next lemma shows that, as long as $\mathcal{A}$ is not too small, the absence of some fan yields the no-linear-degrees property.

Lemma 1.7 *Let the class $\mathcal{A}$ of graphs satisfy* $\liminf \left(\frac{|\mathcal{A}_n|}{n!}\right)^{1/n} > 0$*, and suppose that for some positive integer j, no graph in $\mathcal{A}$ contains the fan F_{j+2} as a minor. Then $\mathcal{A}$ has the no-linear-degrees property.*

The plan of the rest of the paper is as follows. In Section **??** we count apex graphs and prove Theorem **??**: this work needs no preliminaries.

Section **??** concerns redundant blockers. First we introduce a useful theorem of Kloks which relates normal trees and tree decompositions (and we provide a proof). Then we give two structural lemmas on normal trees and small 'splitting sets'. We use those lemmas to prove a result related to the Robertson-Seymour generalisation of the Erdős-Pósa theorem and then we prove Lemma **??**.

In the next section, Section **??**, we consider the no-linear-degrees property and prove Lemma **??**. Following that, in Section **??**, we complete the proof of Theorem **??**. In Section **??** we use our main results to prove the theorems on properties of the random graph $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$. Finally, we make some concluding remarks.

2 Counting apex classes

We shall use the 'pendant appearances theorem', Theorem 4.1 from [**?**], several times, so for convenience we state here a suitable special case as a lemma.

Let H be a connected graph on the vertex set $\{1, \ldots, h\}$ which we consider to be rooted at vertex 1, and let G be a graph on the vertex set

$\{1,\ldots,n\}$, where $n > h$. Then an induced subgraph $\tilde{H}$ of G is a *pendant appearance* of H if (a) the increasing bijection from $\{1,\ldots,h\}$ to $V(\tilde{H})$ gives an isomorphism between H and $\tilde{H}$; and (b) there is exactly one edge in G between $V(\tilde{H})$ and the rest of G, and this edge is incident with the vertex of $\tilde{H}$ with smallest label.

Lemma 2.1 *([?]) Let $\mathcal{A}$ be a proper addable minor-closed class of graphs, and let H be a connected graph in $\mathcal{A}$. There is a constant $a > 0$ such that the following holds. For $R_n \in_u \mathcal{A}$, with probability $1 - e^{-\Omega(n)}$ R_n has at least $a \cdot n$ disjoint pendant appearances of H.*

Proof of Theorem ?? Since $\mathcal{A}$ is proper minor-closed and addable, by Theorem 1.2 of [?] $\mathcal{A}$ is *smooth* with some growth constant $\gamma > 0$, that is $|\mathcal{A}_n|/\left(n|\mathcal{A}_{n-1}|\right) \to \gamma$ as $n \to \infty$. Hence

$$|\mathcal{A}_n| \sim (n)_k \; \gamma^k \; |\mathcal{A}_{n-k}|.$$

Let $n > k$, let $V = \{1,\ldots,n\}$, and consider the following constructions of graphs on V:

(1) Choose a k-set $S \subseteq V$, and put any graph on S ($\binom{n}{k}2^{\binom{k}{2}}$ choices).

(2) Put any graph $F \in \mathcal{A}$ on $V \setminus S$ ($|\mathcal{A}_{n-k}|$ choices).

(3) Add the edges of any bipartite graph B with parts S and $V \setminus S$ ($2^{k(n-k)}$ choices).

Clearly each graph constructed is in $(\text{apex}^k\,\mathcal{A})_n$, and each graph in $(\text{apex}^k\,\mathcal{A})_n$ is constructed at least once. The number of constructions is

$$\binom{n}{k} 2^{\binom{k}{2}} 2^{k(n-k)} |\mathcal{A}_{n-k}| \sim c_k 2^{kn} |\mathcal{A}_n|$$

so $|(\text{apex}^k\,\mathcal{A})_n|$ is at most this number.

Let us bound $|(\text{apex}^k\,\mathcal{A})_n|$ from below by showing that almost all of the constructions yield distinct graphs. Observe that $G \in (\text{apex}^k\,\mathcal{A})_n$ appears just once if and only if G has a unique k-set S of vertices such that $G - S$ is in $\mathcal{A}$.

Let $\mathcal{B}$ be the set of excluded minors for $\mathcal{A}$ and fix a graph $H \in \mathcal{B}$ (which must be 2-connected). Let $h = |V(H)|$. Fix a vertex v in H, and let H^- be the connected graph $H - v$. Let us say that a graph $G \in (\text{apex}^k\,\mathcal{A})_n$ is *good* if for some k-set S_0 it satisfies the following: (a)

$G - S_0 \in \mathcal{A}$; and (b) for each vertex $s \in S_0$ there are $k+1$ pairwise disjoint sets $X_1(s), X_2(s), \ldots, X_{k+1}(s) \subseteq V(G) \setminus S_0$ such that each induced subgraph $G[X_i(s) \cup s]$ has a minor H. If G is good then S_0 must be the unique k-set S such that $G - S$ is in $\mathcal{A}$. For if S' is another set such that $G - S'$ is in $\mathcal{A}$, and $w \in S_0 \setminus S'$, then S' must contain a vertex from each of the sets $X_1(w), X_2(w), \ldots, X_{k+1}(w)$, and so $|S'| \geq k+1$.

Now by Lemma **??** (the 'pendant appearances theorem') there exist constants $a > 0$ and $b > 0$ such that the following holds for a random graph $R_n \in_u \mathcal{A}$: R_n contains at least $a \cdot n$ pairwise vertex-disjoint copies of H^- with probability at least $1 - e^{-bn}$ for n sufficiently large. If F has at least $a(n-k)$ such copies of H^- then there are at most

$$2^{k(n-k)} k \, \mathbb{P}\left(\mathrm{Bin}(\lceil a(n-k) \rceil, 2^{-h+1}) \leq k\right)$$

ways to choose the bipartite graph B with parts S and $V \setminus S$ so that the resulting graph is not good. So the number of ways to choose the graphs F and B so that the resulting graph is constructed just once is at least

$$\begin{aligned} &2^{k(n-k)} |\mathcal{A}_{n-k}| \left(1 - e^{-b(n-k)} - k\mathbb{P}\left(\mathrm{Bin}(\lceil a(n-k) \rceil, 2^{-h+1}) \leq k\right)\right) \\ &= 2^{k(n-k)} |\mathcal{A}_{n-k}| \left(1 - e^{-\Omega(n)}\right), \end{aligned}$$

by a Chernoff bound. Summing over all sets S and all graphs on S we obtain

$$|\mathrm{apex}^k\,(\mathcal{A})_n| \geq c_k 2^{kn} |\mathcal{A}_n| \left(1 - e^{-\Omega(n)}\right),$$

as required. $\square$

3 Redundant blockers

In this section, after some preliminary results we prove the 'redundant blockers lemma', Lemma **??**.

3.1 Treewidth and normal trees

We start with a theorem by Kloks stated in [**?**]. We give a proof here, as we were unable to find a proof elsewhere.

Let G be a graph, and let T be a rooted tree on the same vertex set $V(G)$, with root vertex r. (We do not insist that T is a subgraph of G.) The tree T induces a *tree-ordering* $\leq_T$ on $V(G)$, where $u \leq_T v$ if and only if u is on the path from r to v in T. T is a *normal* tree for G if u and v are comparable for every edge uv of G.

We think of the tree T as hanging down from its root. We will say that u is above v (and v is below u) in T if $u <_T v$. Think of the graph G as fixed. Given a normal tree T for G, for each vertex v of G we define its set $AA_T(v)$ of *active ancestors* by

$$AA_T(v) = \{u <_T v : \exists z \geq_T v \text{ with } uz \in E(G)\}.$$

For brevity we write $a_T(v) = |AA_T(v)|$.

Theorem 3.1 (Kloks) *The treewidth $tw(G)$ of a graph G satisfies*

$$tw(G) = \min_{T \in \mathcal{T}} \max_{v \in V(G)} a_T(v) \tag{5}$$

where $\mathcal{T}$ is the set of all normal trees for G.

We refer to [**?**] for the definition of treewidth and basics of tree decompositions (though note that in that book a normal tree in a graph is required to be a subgraph).

Proof ($\geq$) Let G have treewidth k. We shall prove that for any given vertex $s_0 \in V(G)$ there is a normal tree T for G, rooted at s_0, such that

$$\max_{v \in V(G)} a_T(v) \leq k. \tag{6}$$

Let $(T_0, (V_t : t \in T_0))$ be a tree decomposition for G with $|V_t| = k+1$ for each node t in T_0, and with $|V_s \setminus V_t| = 1$ for each edge st in T_0 (it is easy to see that such a tree decomposition always exists, see for example [**?**]). We call V_t the *bag* for t.

For convenience we shall consider the following small modification of T_0. Pick a node u of T_0 with $s_0 \in V_u$. Suppose that $V_u = \{s_0, s_1, \ldots, s_k\}$. Let $u_0, u_1, \ldots, u_k$ be a path on $k+1$ new nodes, and identify u_k with u. Let $V_{u_j} = \{s_0, \ldots, s_j\}$ for each $j = 0, 1, \ldots, k$. Let T_1 be the tree we have formed from T_0 by adjoining the path, and note that $(T_1, (V_t : t \in T_1))$ is also a tree decomposition for G. Root T_1 at u_0.

The set V_{u_0} consists of the single vertex s_0: define $v(u_0) = s_0$. For each node t in T_1 other than u_0 denote its parent in T_1 by t'; and let $v(t)$ be the unique vertex in $V_t \setminus V_{t'}$. It is a property of tree decompositions that the nodes corresponding to the bags that contain $w \in V(G)$ form a subtree of T_1, which we call the tree for w. It follows that $v(s) \neq v(t)$ for distinct nodes

s and t in T_1, since $v(s) = v(t)$ would imply that both V_s and V_t contain $v(s)$ but there is a node on the path from s to t in T_1 whose bag does not contain $v(s)$. Also, for each vertex v in G there is a node t in T_1 such that $v(t) = v$ (the node t with $v \in V_t$ which is nearest to the root). Thus the map $t \to v(t)$ gives a bijection between the nodes of T_1 and the vertices of G. Let T be the tree on $V(G)$ which corresponds to T_1 under this map; that is, let T be the tree on $V(G)$ with an edge $v(t)v(t')$ for each edge tt' in T_1, rooted at s_0. We claim that T is a normal tree for G and (**??**) holds.

To see that T is a normal tree for G, consider two vertices x and y which are incomparable in T. Let t_x and t_y be the nodes of T_1 with $v(t_x) = x$ and $v(t_y) = y$ respectively. Then t_x and t_y are incomparable in T_1, so the trees for x and y do not meet in T_1 and thus x and y are not adjacent in G.

It remains to prove (**??**). Fix a vertex x in G, and let t_x be the node of T_1 with $v(t_x) = x$. If $y \in AA_T(x)$ then $y \in V_x \setminus x$; and so $a_T(x) = |AA_T(x)| \le |V_x| - 1 \le k$, as required.

($\le$) Let T be a normal tree for G, and let $\max_{v \in V} a_T(v) = k$. For each vertex x define the bag V_x as $AA_T(x) \cup x$. We claim that $(T, (V_x : x \in V(T))$ is a tree decomposition of G, of width at most k. Certainly each bag V_x satisfies $|V_x| \le k + 1$.

Let uv be an edge of G. Then u and v are comparable in T: without loss of generality suppose that $u <_T v$. So $u \in AA_T(v)$, and thus both u and v are in the bag V_v. It remains to check that for each vertex w of G the nodes t of T whose bag V_t contains w form a subtree of T. But if $w \in AA_T(v)$ then $w \in AA_T(u)$ for each vertex u (other than w) on the path in T between w and v. This completes the proof. $\square$

Note that the above proof yields a normal tree with the property (**??**) rooted at an arbitrary vertex in $V(G)$. Hence we can take the set $\mathcal{T}$ in (**??**) to be the set of all normal trees on $V(G)$ rooted at any chosen vertex $r \in V(G)$.

Note also that Theorem **??** fails if we additionally require the normal trees in $\mathcal{T}$ to be subgraphs of G. For example, consider a complete bipartite graph $K_{n,n}$ with $n \ge 3$. The treewidth of $K_{n,n}$ is n, but each normal tree T for $K_{n,n}$ which is its subgraph must be a path where the vertex w at distance $2n - 2$ from the root has $a_T(w) = 2(n - 1) > n$.

3.2 Disjoint subgraphs, splitting sets, treewidth and blockers

In this part we work with classes $\mathcal{H}$ of graphs which are closed under isomorphism but not necessarily minor-closed. Given a class $\mathcal{H}$ of graphs, denote by $\mathrm{Forb}\,(\mathcal{H})$ the class $\mathcal{A}$ of all graphs G such that no subgraph of G is in $\mathcal{H}$. Also, call a set B of vertices such that $G - B \in \mathrm{Forb}\,(\mathcal{H})$ an *$\mathcal{H}$-subgraph-blocker*.

Let T be a normal tree and let H be a connected graph on a subset of $V(T)$. For a set A of vertices we say that A *splits* H in T if there is a vertex $v \in A$ such that either $v \in V(H)$ or H contains vertices above v and below v in T.

We state and prove two general lemmas regarding graphs without k disjoint subgraphs belonging to some class $\mathcal{H}$ of connected graphs. Their proofs have a similar structure and the proof of the former is a good warm-up for the proof of the latter. At the end of this section we also present an application of Lemma **??**.

Lemma 3.2 *Let $\mathcal{H}$ be a non-empty class of connected graphs, let $k \geq 0$ be an integer, let $G \in \mathrm{Forb}\,(k+1)\mathcal{H}$, and let T be a normal tree for G. Then there is a set A of at most k vertices such that, for each subgraph $H \in \mathcal{H}$ of G, the set A splits H in T.*

Proof We will use induction on k. The statement is trivially true for the case $k = 0$, with $A = \emptyset$. Let $j \geq 1$ and suppose that the statement holds for $k = j - 1$. Let $G \in \mathrm{Forb}\,(j+1)\mathcal{H}$, and let T be a normal tree for G. Denote the subtree of T rooted at v by T_v. Let

$$B := \{v \in V(G) : G[V(T_v)] \text{ has a subgraph in } \mathcal{H}\}.$$

If $B = \emptyset$ then $G \in \mathrm{Forb}\,(\mathcal{H})$ so we may take $A = \emptyset$: thus we may assume that $B \neq \emptyset$. Consider a vertex $u \in B$ at maximum distance in T from the root r. If $u = r$ then $B = \{r\}$ so every subgraph of G in $\mathcal{H}$ must contain r. In this case we may take $A = \{r\}$: thus we may assume that $u \neq r$.

Let $G' = G - V(T_u)$ and let $T' = T - V(T_u)$. Since $G[V(T_u)]$ has a subgraph in $\mathcal{H}$, we have $G' \in \mathrm{Forb}\,j\mathcal{H}$. Clearly, T' is a normal tree for G'. Apply induction for G' and T' to obtain a set of at most $j - 1$ vertices $A' \subseteq V(G')$ such that A' splits in T' each subgraph H of G' such that $H \in \mathcal{H}$.

Now let $A = A' \cup u$. We will show that A has the required property for G and T. Suppose H is a subgraph of G and $H \in \mathcal{H}$. If $V(H) \cap V(T_u)$ is empty then H is a subgraph of G', so there is a vertex $v \in A' \subseteq A$ such that H either contains v or contains vertices both above and below v in T' and so in T. Thus it suffices to consider the case when $V(H) \cap V(T_u)$ is not empty.

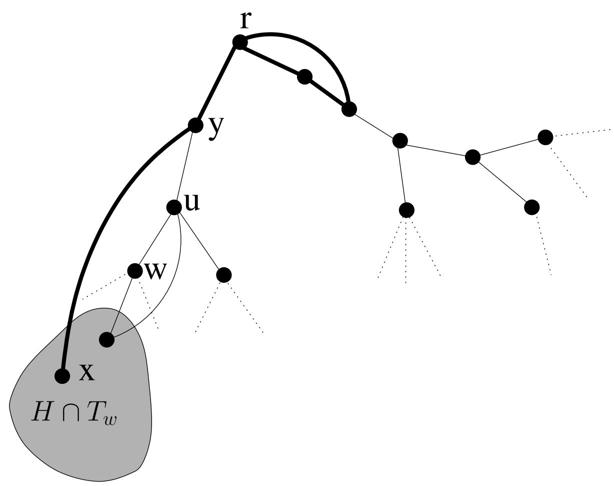

Figure 1: If $V(H) \cap [V(T_u) \setminus u] \neq \emptyset$ then H must have an edge between this set and some vertex $y \in AA_T(u)$. In this illustration H consists of a subgraph strictly below u together with the bold edges. In our terminology, $\{u\}$ splits H in T.

We may assume that H does not contain u, as otherwise we are done. Let w be a child of u in T such that T_w contains a vertex in H. Then since $w \notin B$, T_w does not contain all of $V(H)$. Since H is connected, there must be an edge between a vertex x of H in T_w and a vertex y of H not in T_w; and since T is normal, y must be in $AA_T(u)$. But now x is below u and y is above u in T, and the proof is complete. $\square$

We now assume that the class $\mathcal{H}$ consists of 2-connected graphs. The following lemma will be crucial in the proof of Lemma **??**, and thus in the proof of Lemma **??**. It asserts that any $\mathcal{H}$-subgraph-blocker can be transformed into one with a specific 'redundant' structure by adding a few extra vertices.

Lemma 3.3 *Let $\mathcal{H}$ be a non-empty set of 2-connected graphs. Let $k \geq 0$, let $G \in \mathrm{Forb}\,(k+1)\mathcal{H}$, let $Q \subseteq V(G)$ be an $\mathcal{H}$-subgraph-blocker in G, and let T be a normal tree for $G - Q$. Then there are sets $S \subseteq Q$ with $|S| \leq k$ and $A \subseteq V(G - Q)$ with $|A| \leq k$, such that for each vertex $x \in Q \setminus S$ and each subgraph H of $G - (Q \setminus x)$ in $\mathcal{H}$ the set A splits $H - x$ in T.*

To read the last sentence, it may help to observe that, given $x \in Q \setminus S$ and a subgraph H of $G - (Q \setminus x)$ in $\mathcal{H}$, we must have $V(H) \cap Q = \{x\}$.

Proof The proof is similar to the proof of Lemma **??**, except that in this case we use induction on k to obtain the set S together with the set A.

Clearly the case $k = 0$ holds, as we may take $A = S = \emptyset$. Let $j \geq 1$ and suppose that the result holds for the case $k = j - 1$. Let $G \in \text{Forb}\,(j+1)\mathcal{H}$, let $Q \subset V(G)$ be an $\mathcal{H}$-subgraph-blocker in G, and let T be a normal tree for $G - Q$. Let

$$B = \{v \in V(G) : \ G[V(T_v) \cup x] \notin \text{Forb}\,\mathcal{H} \text{ for some } x \in Q\}.$$

If B is empty then we are done (again take $A = S = \emptyset$, and note that there are no relevant graphs H); so assume that B is non-empty. Choose a vertex $u \in B$ with maximum distance from the root r in T.

Consider first the case $u = r$. Let $H \in \mathcal{H}$ be a subgraph of G with $V(H) \cap Q = \{x\}$. Since $u = r$, the vertices of the connected graph $H - x$ are not contained in $V(T_v)$ for any subtree T_v where v is a child of r. Also, since T is normal, there are no edges between subtrees T_v and $T_{v'}$ for distinct children of r. Hence H must contain r. Thus each subgraph in $\mathcal{H}$ of G which meets Q in just one vertex must contain r. Hence we may take $S = \emptyset$ and $A = \{r\}$.

We may now assume that $u \neq r$. Let $z \in Q$ be such that $G[V(T_u) \cup z] \notin \text{Forb}\,\mathcal{H}$. Let $G' = G - (V(T_u) \cup z)$, let $Q' = Q \setminus z$, and let $T' = T - V(T_u)$. Then clearly $G' \in \text{Forb}\,j\mathcal{H}$, Q' is an $\mathcal{H}$-subgraph-blocker in G', and T' is a normal tree for G'. Hence we can apply the induction hypothesis to G', Q' and T'. We obtain sets $S' \subseteq Q'$ with $|S'| \leq j - 1$ and $A' \subseteq V(G' - Q')$ with $|A'| \leq j - 1$, such that for each vertex $x \in Q' \setminus S'$ if $H \in \mathcal{H}$ is a subgraph of $G' - (Q' \setminus x))$ then A' splits $H - x$ in T'.

Now let $S = S' \cup z$ and $A = A' \cup u$. Let $x \in Q \setminus S$, and suppose that the subgraph H of $G - ((Q \setminus x) \cup A)$ is in $\mathcal{H}$. Note that $u \in A$ so $u \notin V(H)$. If $V(H) \cap V(T_u) = \emptyset$ then H is a subgraph of $G' - ((Q' \setminus x) \cup A')$: hence there is a vertex $v \in A' \subseteq A$ for which H has vertices above and below v in T' and so in T. This leaves the case that $V(H) \cap V(T_u) \neq \emptyset$. Suppose that H has no vertex above u in T: we want to find a contradiction.

Now $V(H - x)$ cannot be contained in $V(T_v)$ for any subtree T_v where v is a child of u, as this would imply that $v \in B$ which would contradict our choice of u. But as in the case $u = r$, since T is normal the connected graph $H - x$ cannot have vertices in subtrees T_v and $T_{v'}$ of T_u where v and v' are distinct children of u. Thus we have a contradiction, and the proof is complete. $\square$

A variant of the following lemma was first proved by Robertson and Seymour [**?**] in order to prove the generalised Erdős-Pósa theorem. We state it in a more general form (proved by Thomassen [**?**]) and give a simple proof using Theorem **??** and Lemma **??**.

Lemma 3.4 *Let $\mathcal{H}$ be a class of connected graphs and let k and w be non-negative integers. If $G \in \text{Forb}\,(k+1)\mathcal{H}$ and $tw(G) \leq w$ then G has an $\mathcal{H}$-subgraph-blocker of size at most $k(w+1)$.*

Proof of Lemma ?? Suppose $G \in \text{Forb}\,(k+1)\mathcal{H}$ and $tw(G) \leq w$. By Theorem **??**, there is a normal tree T for G such that $\max_{v \in V} a_T(v) \leq w$. Let A be a set of at most k vertices as in Lemma **??**, and let

$$B = A \cup \left(\cup_{v \in A} AA_T(v)\right).$$

Observe that $|B| \leq k(w+1)$. We claim that B is an $\mathcal{H}$-subgraph-blocker in G. For suppose it is not, and let $H \in \mathcal{H}$ be a subgraph of $G - B$. By Lemma **??** there is a vertex $v \in A$ such that H contains vertices both above v and below v in T (note that $v \notin V(H)$ since $A \cap V(H) = \emptyset$). Since H is connected, it has an edge xy with x above v and y below v. But then $x \in AA_T(v) \subseteq B$, a contradiction. □

3.3 Treewidth and blockers: a more general case

Lemma **??** is what we need in this paper, to prove Lemma **??**; but it does not apply to disconnected excluded subgraphs. We include for completeness a treatment of this case, and give a more general version of Lemma **??**.

We give two preliminary lemmas. The proof of the first one uses induction much as in [**?**], but as in the proof of Lemma **??** we use normal trees. For any graph H we let $0H$ denote the graph with no vertices: thus for any graph H we have $G \cup 0H = G$.

Lemma 3.5 *For $t \geq 1$ let $H_1, H_2, \ldots, H_t$ be connected graphs. Let $k_1, k_2, \ldots, k_t$ be nonnegative integers, not all zero. If*

$$G \in \text{Ex}\,(k_1H_1 \cup k_2H_2 \cup \cdots \cup k_tH_t)$$

and T is a normal tree for G, then there is a set $A \subseteq V(G)$ and an integer j with $k_j \geq 1$, such that $|A| \leq (\sum_i k_i) - 1$ and A splits in T each connected subgraph of G with a minor H_j.

Proof We use induction on $\sum_i k_i$. If $\sum_i k_i = 1$, then $G \in \text{Ex}\,H_j$ for some j: so we may take $A = \emptyset$ and we are done.

Let $s \geq 2$, suppose we have proved the hypothesis for each sequence $k'_1, k'_2, \ldots, k'_t$ with $\sum_i k'_i < s$, and let $\sum_i k_i = s$. We proceed as in the proof of Lemma **??**. Let B be the set of all vertices v of T such that $G[V(T_v)] \notin \mathrm{Ex}\, H_i$ for some i with $k_i \geq 1$, and let u be a vertex in B with maximum distance from the root.

Suppose first that u is the root of T. Then we may take $A = \{u\}$ as u must be in every subgraph of G with a minor in $\{H_1, \ldots, H_t\}$. So we may assume that u is not the root of T.

Let j be such that $k_j \geq 1$ and $G[V(T_u)] \notin \mathrm{Ex}\, H_j$. Write $T' = T - T_u$ and let $k'_i = k_i$, for $i \neq j$ and $k'_j = k_j - 1$. Since

$$G' := G - T_u \in \mathrm{Ex}\, \left(k'_1 H_1 \cup \cdots \cup k'_t H_t\right),$$

T' is a normal tree for G' and $\sum_i k'_i = s - 1$, we may apply induction to find some l with $k'_l \geq 1$ and a set A' of at most $s - 2$ vertices that splits in T' each connected subgraph H of G' with a minor H_l. We claim that $A = A' \cup u$ splits in T every connected subgraph of G with a minor H_l. As in the proof of Lemma **??**, every connected subgraph H of G with a minor H_l such that H is not a subgraph of G' and $u \notin V(H)$ must have vertices both in $V(T_u)$ and $V(G')$ (or otherwise we get a contradiction to the choice of u). But then $\{u\}$ splits in T each such subgraph H. □

Lemma 3.6 *Let $\mathcal{A}$ be a minor-closed class of graphs (perhaps the class of all graphs), and let $\mathcal{H}$ consist of the graphs not in $\mathcal{A}$ together with an arbitrary class of connected graphs. Let s be the sum over the disconnected excluded minors M for $\mathcal{A}$ of the number $\kappa(M)$ of components (so $0 \leq s < \infty$). Let $k \geq 0$ be an integer, let $G \in \mathrm{Forb}\,(k+1)\mathcal{H}$, and let T be a normal tree for G. Then there is a set A of at most $s(k+1) + k$ vertices such that for each subgraph $H \in \mathcal{H}$ of G the set A splits in T some component of H.*

Proof Let $\tilde{\mathcal{H}}$ consist of the connected graphs in $\mathcal{H}$. By Lemma **??**, there is a set $A_0 \subseteq V(G)$ with $|A_0| \leq k$ such that A_0 splits in T each subgraph $H \in \tilde{\mathcal{H}}$ of G.

Suppose that $\mathcal{A}$ has $j \geq 0$ disconnected excluded minors G_i, $i = 1, \ldots, j$. Let $i \in \{1, \ldots, j\}$. Since $G \in \mathrm{Ex}\,(k+1)G_i$, by Lemma **??**, there is a set $A_i \subseteq V(G)$ with $|A_i| \leq \kappa(G_i)(k+1) - 1$ such that A_i splits in T some component of each subgraph G' of G which has a minor G_i, that is, of each G' in the complement $(\mathrm{Ex}\, G_i)^c$ of $\mathrm{Ex}\, G_i$.

Finally, observe that $\mathcal{H}$ is the union of $\tilde{\mathcal{H}}$ and $\cup_i (\mathrm{Ex}\, G_i)^c$, and so we may form a set A as required from the union of the $j+1$ sets A_i. □

Now from the above lemma and Theorem **??** following the lines of the proof of Lemma **??** we have:

Proposition 3.7 *Let $\mathcal{A}$ be a minor-closed class of graphs and let $\mathcal{H}$ consist of the graphs not in $\mathcal{A}$ together with an arbitrary class of connected graphs. Then for each pair of non-negative integers k and w there is an integer $f(k,w)$ such that if $G \in \mathrm{Forb}\,(k+1)\mathcal{H}$ and $tw(G) \leq w$ then G has an $\mathcal{H}$-subgraph-blocker of size at most $f(k,w)$.*

3.4 Proof of Lemma ??

We introduce the following fundamental result, Theorem (2.1) of Robertson and Seymour [**?**].

Lemma 3.8 *For every planar graph H, there is a number $\alpha(H)$ such that every graph with no minor H has treewidth at most $\alpha(H)$.*

We now prove the redundant blockers lemma, Lemma **??** (see Figure 2), quickly using the above results. We give a slightly more general version first.

Lemma 3.9 *Fix a 2-connected planar graph H_0. Then for each integer $k \geq 1$ there is an integer $f(k)$ such that the following holds.*

Let $\mathcal{H}$ be a set of 2-connected graphs such that $\mathcal{H}$ contains H_0 and all 2-connected graphs contractible to H_0. Then each graph G in $\mathrm{Forb}\,(k+1)\mathcal{H}$ has an $\mathcal{H}$-subgraph-blocker R with $|R| \leq f(k)$ such that for all but at most k vertices v in R, the set $R \setminus v$ is still an $\mathcal{H}$-subgraph-blocker.

Proof of Lemma ?? If a graph contains H_0 as a minor then it contains a 2-connected subgraph contractible to H_0. Thus $\mathrm{Forb}\,k\mathcal{H} \subseteq \mathrm{Ex}\,kH_0$ for each positive integer k.

Let $k \geq 1$ and let $G \in \mathrm{Forb}\,(k+1)\mathcal{H}$. Then $G \in \mathrm{Ex}\,(k+1)H_0$ and $(k+1)H_0$ is planar; so by Lemma **??**, G has treewidth at most w for some constant $w = w(k, H_0)$. Therefore by Lemma **??** there is a positive integer $g(k)$ (depending only on k and H_0) such that each graph in $\mathrm{Forb}\,(k+1)\mathcal{H}$ has an $\mathcal{H}$-subgraph-blocker Q of size at most $g(k)$.

Now $G - Q \in \mathrm{Forb}\,\mathcal{H} \subseteq \mathrm{Ex}\,H_0$. Using Lemma **??** again, we see that $G-Q$ has treewidth at most α for some constant $\alpha = \alpha(H_0)$; and so it has a normal tree T with $a_T(v) \leq \alpha$ for each vertex $v \in V(T)$ by Theorem **??**. Let A and S be sets obtained in Lemma **??** for k, G, Q and T. Define $R \subseteq V(G)$ by

$$R := Q \cup A \cup \left(\bigcup_{v \in A} AA_T(v) \right).$$

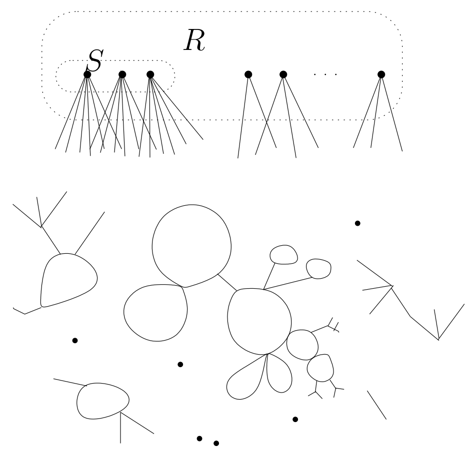

Figure 2: For example, it follows easily from Lemma **??** that we can always decompose a graph G in $\mathrm{Ex}\,4D$ into a set R of a constant size and a collection of cacti, where each vertex in R, except at most 3 of them, can have at most 2 edges to each component of $G-R$. (Here D is the diamond graph $K_4 - e$.)

Note that $|R| \leq f(k) := g(k)+(\alpha+1)k$. We want to show that $R-u$ is still an $\mathcal{H}$-subgraph-blocker for G for each vertex $u \in R \setminus S$. This is clearly true for $u \in R\setminus Q$; so assume that $u \in Q\setminus S$, and some subgraph H of $G-(R\setminus u)$ is in $\mathcal{H}$. But then H is a subgraph of $G-((Q\setminus u)\cup A)$, and so by Lemma **??**, for some vertex $w \in A$, H must have vertices both above and below w in T. Hence the connected subgraph $H-u$ of $G-R$ has vertices both above and below w in T. But by the definition of normal tree, $AA_T(w)$ is a separating set for the vertices below w and the rest of T, so $H-u$ contains a vertex in $AA_T(w) \subset R$, a contradiction. $\square$

Proof of Lemma ?? Let $\mathcal{H}$ be the class of all 2-connected graphs contractible to a graph in $\mathcal{B}$. Then $\mathrm{Forb}\,\mathcal{H} = \mathrm{Ex}\,\mathcal{B}$, and more generally $\mathrm{Forb}\,(k+1)\mathcal{H} = \mathrm{Ex}\,(k+1)\mathcal{B}$ for each $k \geq 0$. Also, a $\mathcal{B}$-minor-blocker is the same as an $\mathcal{H}$-subgraph-blocker. Now the result follows directly from Lemma **??**. $\square$

4 Graph classes not containing all fans

In this section, after a preliminary lemma on coloured forests, we prove Lemma **??** on the no-linear-degrees property.

For graphs where each vertex is coloured black or white, we define the contraction operation as usual, but colour black each vertex resulting from contracting a connected graph which contains a black vertex. A forest is *rooted* if in each component tree a vertex is distinguished as the root: often we will think of the edges as being oriented away from the root. Let us say that a class $\mathcal{A}$ of graphs is *very small* (or $\mathcal{A}$ has growth constant 0) if

$$\left(\frac{|\mathcal{A}_n|}{n!}\right)^{\frac{1}{n}} \to 0 \quad \text{as } n \to \infty.$$

For example, the class of graphs with no path of length j is very small [**?**]: we shall use this result in the proof of the next lemma. Recall that the path P_j has j vertices.

Lemma 4.1 *Fix an integer $j \geq 2$ and $0 < \alpha \leq 1$. Let $\mathcal{F}'$ be the class of black/white coloured rooted forests such that no forest in $\mathcal{F}'$ has a black path P_j as a minor, and each forest in $\mathcal{F}'_n$ has at least αn black vertices. Then $\mathcal{F}'$ is very small.*

Proof Call a colouring of a forest as above *good*, and call a good colouring (black-) *maximal* if there is no vertex coloured white such that recolouring it black gives another good colouring. Let $\hat{\mathcal{F}}$ be the set of all possible (unrooted) forests F together with a specified maximal good colouring of F. Since $|\mathcal{F}'_n| \leq 4^n |\hat{\mathcal{F}}_n|$, it suffices for us to prove that $\hat{\mathcal{F}}$ is very small. (For an n-vertex forest, there are at most 2^n choices for the colouring and at most 2^n choices for the roots.)

Let $F \in \hat{\mathcal{F}}_n$. Observe that trimming off a white leaf yields another coloured forest in $\hat{\mathcal{F}}$. The *core* of F is the coloured forest obtained by repeatedly trimming off white leaves until none remain. Thus $core(F)$ is in $\hat{\mathcal{F}}_{n_1}$ for some $n_1 \geq \alpha n$.

The key observation is that in $core(F)$ each white vertex has degree 2. For suppose that vertex v is white and has degree at least 3. Let us change the colour of v to black. By the maximality of the colouring, $core(F)$ now has a black P_j minor. There must be a vertex w of the minor to which v is contracted; and if T_v denotes the tree in $core(F)$ containing v which is contracted to w, then with the original colour of v each vertex in T_v is white (for otherwise the colour change would not have mattered). Thus there are at least 3 edges between T_v and subtrees of the rest of $core(F)$, which each contain a black vertex. At least one of these subtrees must be entirely deleted when the minor is formed (since the minor has maximum degree

2): but if instead we added such a subtree to T_v and contracted all these vertices to form w then w would be black even without the colour change. Thus $core(F)$ has a black minor P_j with the original colouring, and this contradiction shows that indeed each white vertex in $core(F)$ has degree 2.

We have now seen that each coloured forest $F \in \hat{\mathcal{F}}$ may be obtained from an all black forest with no path P_j by subdividing edges using white vertices, and then attaching pendant trees using more white vertices (to reverse the initial operation of repeatedly trimming off white leaves). Thus the following constructions yield each member of $\hat{\mathcal{F}}_n$ at least once.

1. Choose a set $V_1 \subseteq V = [n]$ of $n_1 \geq \alpha n$ vertices; colour each of these vertices black; and choose a forest F' on V_1 which does not contain a path P_j.

2. Choose a set $V_2 \subseteq V \setminus V_1$ of $n_2 \geq 0$ vertices; colour each of these vertices white; and use them to subdivide edges in F'.

3. Take the remaining set $V_3 = V \setminus (V_1 \cup V_2)$ of vertices; colour each of these vertices white; and use them to form $n_1 + n_2$ trees T_x rooted at the vertices in $V_1 \cup V_2$.

4. The edges of the coloured forest F are those of the subdivided forest F' together with those of the trees T_x.

Let us bound the number of constructions. For step 1, let $\tilde{f}_{n_1}$ denote the number of forests on $[n_1]$ which do not contain a path P_j. Now consider step 2, after we have chosen the set V_2. List the $t \leq n_1 - 1$ edges of F' in some fixed order, with a fixed orientation: then we see that the number of ways to use the n_2 vertices in V_2 to subdivide the edges of F' is $(n_2+t-1)!/(t-1)! = (n_2 + t - 1)_{n_2} \leq n^{n_2}$. For step 3, recall that the number of forests on the vertex set V containing exactly $n_1 + n_2$ rooted trees with given roots is $(n_1 + n_2)n^{n-n_1-n_2-1} \leq n^{n-n_1-n_2}$.

From the above we see that in total the number of constructions is at most

$$\begin{aligned} & \sum_{n_1=\lceil \alpha n \rceil}^{n} \binom{n}{n_1} \tilde{f}_{n_1} \sum_{n_2=0}^{n-n_1} \binom{n-n_1}{n_2} n^{n_2} n^{n-n_1-n_2} \\ = \quad & (2n)^n \sum_{n_1=\lceil \alpha n \rceil}^{n} \binom{n}{n_1} \tilde{f}_{n_1} (2n)^{-n_1}. \end{aligned}$$

Now let $\varepsilon > 0$, and suppose that $\varepsilon \leq \frac{1}{e}$ so that $\sum_{m \geq 0} (e\varepsilon/2)^m \leq 2$. From [?], there is a m_0 such that for each $m \geq m_0$ we have $\tilde{f}_m \leq (\varepsilon m)^m$. Hence for all n sufficiently large that $\alpha n \geq m_0$, the number of constructions is at most

$$\begin{aligned} & (2n)^n \sum_{n_1 = \lceil \alpha n \rceil}^{n} \binom{n}{n_1} (\varepsilon n_1)^{n_1} (2n)^{-n_1} \\ \leq \; & (2n)^n \sum_{n_1 = \lceil \alpha n \rceil}^{n} \left(\frac{ne}{n_1} \cdot \frac{\varepsilon n_1}{2n} \right)^{n_1} \\ \leq \; & (2n)^n \sum_{m \geq \lceil \alpha n \rceil} (e\varepsilon/2)^m \leq 2 \cdot (2^{1-\alpha} (e\varepsilon)^{\alpha} n)^n. \end{aligned}$$

This completes the proof, since we may make $2^{1-\alpha}(e\varepsilon)^{\alpha}$ arbitrarily small by choice of ε. □

We have already seen that each proper minor-closed class of graphs that contains arbitrarily large fans fails to have the no-linear-degrees property. We now use Lemma **??** to prove Lemma **??**, which says that excluding some fan ensures that a suitable class has the no-linear-degrees property.

Proof of Lemma ?? For a class $\mathcal{A}$ and a random graph $R_n \in_u \mathcal{A}$ the no-linear-degrees property is equivalent to the requirement that

$$\mathbb{P}(d(v_0) \geq cn) = o(e^{-an}) \quad \text{for every } c, a > 0$$

where v_0 is vertex 1 and $d(v_0)$ is its degree in the graph R_n.

Let j be a positive integer, and suppose that no graph in $\mathcal{A}$ contains as a minor the fan F_{j+2}. Fix $c \in (0, 1]$. We are going to bound the number of graphs in $\mathcal{A}_n$ such that $d(v_0) \geq cn$.

Consider a graph $G \in \mathcal{A}_n$. Let the least vertex in each component be the root vertex of that component (so v_0 is a root). Perform a depth-first search starting from each root vertex in G. Recall that the DFS procedure produces a spanning forest F of G where we consider each tree as rooted as above; also each tree is a normal subtree of its component.

Let T_r be a tree component of F, with root r; let v be any vertex in T_r; and let T_v be the subtree of T rooted at v. Observe that such a subtree T_v can have edges (in G) to at most j ancestors of v (from T_r), since otherwise we could form a minor F_{j+2} by contracting T_v into a single vertex and considering the path from v to the root of the tree. (Note also that by Theorem **??** this establishes that $\mathcal{A}$ has treewidth at most j, though we will not use this fact.)

Now colour black each vertex of F which is adjacent in G to v_0, and colour each other vertex (including v_0) white. We have described G as a forest of rooted trees on n vertices coloured black or white, with some additional edges (the 'back edges' from the DFS).

Consider the rooted forest $\bar{F} = F - v_0$ (set the nodes that were adjacent to v_0 in F as roots of the newly created trees). Note that $\bar{F}$ can have no black path P_{j+1} as a minor, and if $d(v_0) \geq cn$ then there are at least cn black vertices in $\bar{F}$. Thus by Lemma **??** the number of all possible rooted forests $\bar{F}$ is $o(\varepsilon^n n^n)$ for every $\varepsilon > 0$. But since there are at most 2^{n-1} ways to add the node v_0 back to the forest $\bar{F}$ to obtain a valid forest F, the number of different forests F we can get is also $o(\varepsilon^n n^n)$ for all $\varepsilon > 0$.

From the rooted forests F on $[n]$ we can construct the graphs in $\mathcal{A}_n$ by adding some DFS back edges. Let us show that there are not too many ways to do that. We shall see that, given $G \in \mathcal{A}_n$ and a DFS spanning forest F for G, we can record a small amount of information at the vertices and edges of F such that from F and this information we can reconstruct G. The total amount of information recorded is at most $(2j+1)n$ bits.

Assuming that we are able to do that, then the number of constructions on n vertices which yield a graph with $d(v_0) \geq cn$ is at most $|\mathcal{F}'_n| \cdot 2^n \cdot 2^{(2j+1)n} = o(\varepsilon^n n!)$ for any $\varepsilon > 0$. But $|\mathcal{A}_n| = \Omega\,(\gamma^n n!)$ for some $\gamma > 0$. Therefore, for any $\varepsilon > 0$

$$\mathbb{P}(d(v_0) \geq cn) \leq \varepsilon^n$$

for n sufficiently large. Thus the following lemma will complete the proof of Lemma **??**. □

Lemma 4.2 *Let j be a positive integer and suppose that no graph in $\mathcal{A}$ contains as a minor the fan F_{j+2}. Given a graph $G \in \mathcal{A}$ and a DFS spanning forest F for G, we can uniquely describe G using F together with j bits for each vertex of F other than a root, and $j+1$ bits for each arc of F other than those leaving a root.*

Proof of Lemma ?? Consider a rooted tree T in F. For each vertex x other than the root, let L_x be the list of the ancestors of x (in F) other than the parent of x which have an edge (in G) to T_x, listed in order of increasing distance from the root. Thus L_x has length between 0 and j. Let B_x be the binary j-tuple $b_x(1), \ldots, b_x(j)$, where $b_x(i) = 1$ if the list L_x has length at least i and its ith member is adjacent to x, and otherwise $b_x(i) = 0$.

Also, for each edge xy of T oriented away from the root where x is not the root, let C_{xy} be the binary $(j+1)$-tuple $c_{xy}(1), \ldots, c_{xy}(j+1)$ defined as follows. Let L be the list L_x with the parent of x appended at the end; and for $i = 1, \ldots, j+1$ let $c_{xy}(i) = 1$ if the list L has length at least i and its ith member has a non-tree edge to T_y, and otherwise let $c_{xy}(i) = 0$.

Given G and F we can of course construct all the lists L_x and the tuples B_x and C_{xy}. But conversely, given F and the tuples C_{xy} we can construct the lists L_x and then we can use the tuples B_x to recover G.

We do this as follows. For each tree T in F, we construct the lists L_x by moving one step at a time away from the root. If x is a child of the root then L_x is empty. Now let xy be an edge in T oriented away from the root where x is not the root, and suppose that we know L_x. We form L_y as follows. Let L be L_x with the parent of x appended at the end. For $i = 1, \ldots, j+1$, if $c_{xy}(i) = 1$ we keep the ith member of L, otherwise we delete it, maintaining the same order: this gives L_y.

Thus we can determine each list L_x for x not the root; and now we can use F to determine the parent of x, and L_x and B_x to determine all its other ancestors to which it is adjacent. Thus we can determine G. $\square$

5 Proof of Theorem ??

In the last two sections we proved Lemma **??** and Lemma **??**. In this section, after a further few preliminary lemmas we use the earlier results to prove Theorem **??**.

5.1 Minors, paths and pendant subgraphs

The following lemma is 'nearly obvious' but we spell out a proof.

Lemma 5.1 *Let the graph G have H as a minor. Then G has a subgraph $\tilde{H}$ which contracts to a graph isomorphic to H, and a family $(Q(xy) : xy \in E(H))$ of paths in $\tilde{H}$, which partition the edges of $\tilde{H}$ and have no internal vertices in common.*

Proof Since G has H as a minor, there is a family $(T_x^0 : x \in V(H))$ of disjoint (that is, pairwise vertex-disjoint) subtrees of G, such that for each edge $xy \in E(H)$ there is at least one edge uv in G between the vertices of T_x^0 and T_y^0. Form a set $D \subseteq E(G)$ by picking exactly one such edge uv for each edge $xy \in E(H)$. Call these the 'cross edges'.

Let $x \in V(H)$ and consider the tree T_x^0. Repeatedly remove leaves that are not incident with any cross edge, until no such leaves are left or the tree has just one vertex. The resulting tree T_x has the property that if it has at least two vertices then each leaf of T_x is incident with a cross edge. Do this for each vertex $x \in V(H)$, and let the subgraph $\tilde{H}$ of G be the graph consisting of all the trees T_x together with the edges in D. If we contract each of these trees T_x to a single vertex we obtain a copy of H. Thus it will suffice to show that there is an appropriate family of paths for $\tilde{H}$; and so the following claim will complete the proof.

Claim There is a family $(Q(uv) : uv \in D)$ of pairwise internally vertex-disjoint paths such that (a) for each edge uv in D, $Q(uv)$ contains the edge uv and if uv has one end vertex in T_x and one in T_y then so does $Q(uv)$; and (b) each edge in the trees T_x is contained in one of the paths.

We will prove the claim by induction on $|D|$. It is trivial if $|D| = 0$, so suppose that $|D| \geq 1$ and we know the result for smaller values. Let $uv \in D$ and suppose that u is in T_x and v is in T_y (and so $x \neq y$). We form the path $Q(uv)$ as follows.

If T_x consists just of u, or u is not a leaf of T_x, or u is incident with another edge in D, then let $P(u)$ be the trivial path consisting just of u. Otherwise u is a leaf of T_x and T_x has at least two vertices, and u is not incident with any edge in $D \setminus uv$: in this case we let the path $P(u)$ be the shortest path in T_x between u and a vertex u' such that either u' is incident with an edge in $D \setminus uv$ or u' has degree > 2 in T_x. Similarly we form a corresponding path $P(v)$ in T_y. Let $Q(uv)$ be the path formed by concatenating $P(u)$, uv and $P(v)$.

Now we remove uv from D, and from T_x and T_y we remove the edges and internal vertices of $Q(uv)$. It is easy to see that may use the induction hypothesis to obtain a family of paths for the new configuration, and then add the path $Q(uv)$ to complete the proof of the claim, and thus of the lemma. □

The next lemma follows quickly from the last one.

Lemma 5.2 *Let the graph G have H as a minor and let $W \subseteq V(G)$. Then G has a subgraph $\tilde{H}$ which contracts to a graph isomorphic to H, and a family $\mathcal{F}$ of paths in $\tilde{H}$ such that*

(a) the paths in $\mathcal{F}$ partition the edges of $\tilde{H}$,

(b) no path in $\mathcal{F}$ has an internal vertex in W, and

(c) $|\mathcal{F}| \leq |E(H)| + |W|$.

Proof By the last lemma, G has a subgraph $\tilde{H}$ which contracts to a graph isomorphic to H, and a family $(Q(xy) : xy \in E(H))$ of paths in $\tilde{H}$, which partition the edges of $\tilde{H}$ and have no internal vertices in common.

If a vertex $w \in W$ is internal for a path P in this family then P is the only such path for w, and we form two new paths P' and P'' by 'cutting' P at w (so that both of the new paths have w as an endpoint and not as an internal vertex). We do this for each vertex $w \in W$, and thus obtain a family $\mathcal{F}$ of paths as required. □

We call a connected subgraph H of G a *pendant subgraph* if there is exactly one edge of G between $V(H)$ and $V(G) \setminus V(H)$. From the last lemma we deduce:

Lemma 5.3 *Let $G = (V, E)$ be a graph, let $W \subset V$, let $G' = G - W$ and let $\mathcal{P}$ be a collection of pairwise vertex-disjoint pendant subgraphs of G'. Suppose the graph H has no isolated vertices, and let G have H as a minor. Then G has a subgraph $\tilde{H}$ which contracts to a graph isomorphic to H, and which has vertices in at most $2(|E(H)| + |W|)$ of the subgraphs in $\mathcal{P}$.*

Proof Invoke Lemma **??** to obtain a family $\mathcal{F}$ of at most $|E(H)| + |W|$ paths where no path has an internal vertex in W and $\bigcup_{P \in \mathcal{F}} P$ yields a graph which contracts to a graph isomorphic to H.

We claim that any path $P \in \mathcal{F}$ can touch at most 2 distinct pendant subgraphs in $\mathcal{P}$. Clearly the lemma will follow from this claim.

To establish the claim, assume for a contradiction that some path P in $\mathcal{F}$ shares vertices with each of 3 distinct subgraphs C_1, C_2 and C_3 in $\mathcal{P}$. Since P does not have internal vertices in W, we may assume it is entirely contained in G' (otherwise consider P less any vertices in W). Travel along P and without loss of generality suppose that C_1 is visited first and C_2 is visited second. But in order to visit C_2 after C_1 we must cross the bridge e connecting C_2 to the rest of G', and there is no path in $G' - e$ from $V(C_2)$ to $V(C_3)$, so P cannot reach C_3. □

5.2 Completing the proof of Theorem ??

Lemma 5.4 *Let $\mathcal{A}$ be an addable minor-closed class of graphs such that its family $\mathcal{B}$ of excluded minors includes at least one planar graph. Assume that $\mathcal{A}$ has the no-linear-degrees property. Then for each positive integer k, we have*

$$|(\mathrm{Ex}\,(k+1)\mathcal{B})_n| = (1 + e^{-\Omega(n)})|(apex^k\,\mathcal{A})_n|.$$

The idea of the proof is similar to that of the proof of Theorem 1.1 in [**?**]. We first describe constructions which yield every graph in $(\mathrm{Ex}\,(k{+}1)\mathcal{B})_n$ at least once (as well as other graphs); we then show that there are few 'unrealistic' constructions; and finally we show that few 'realistic' constructions yield a graph not in $\mathrm{apex}^k\,\mathcal{A}$.

Proof of Lemma ?? Fix a positive integer k. By Lemma **??**, there is a positive integer r such that the following holds. For each graph G in $\mathrm{Ex}\,(k+1)\mathcal{B}$ with at least r vertices, there is a $\mathcal{B}$-minor-blocker R of size r and a subset S of R of size k such that $R \setminus v$ is still a $\mathcal{B}$-minor-blocker for each vertex $v \in R \setminus S$.

Let $n > r$. By the above, the following constructions yield every graph in $(\mathrm{Ex}\,(k+1)\mathcal{B})_n$ at least once (as well as other graphs).

(i) Choose an r-subset $R \subseteq V$, put any graph on R, and choose a k-subset $S \subseteq R$ ($\binom{n}{r}2^{\binom{r}{2}}\binom{r}{k} = O(n^r)$ choices)

(ii) Add the edges of any bipartite graph $H(S, V \setminus R)$ with parts S and $V \setminus R$ ($2^{k(n-r)}$ choices)

(iii) Put any graph F in $\mathcal{A}$ on $V \setminus R$ ($|\mathcal{A}_{n-r}|$ choices)

(iv) Add the edges of any bipartite graph $H(R \setminus S, V \setminus R)$ with parts $R \setminus S$ and $V \setminus R$, subject to the restriction that for each $v \in R \setminus S$ the induced subgraph on $V \setminus (R \setminus v)$ is in $\mathcal{A}$.

By the graph minors theorem of Robertson and Seymour [**?**], $\mathcal{B}$ is a finite set of $j \geq 1$ graphs $H_1, \ldots, H_j$; and by assumption each H_i is 2-connected. Let m be the maximum number of edges in these graphs H_i.

Pick distinct vertices v_1 and r_1 in H_1, and consider the connected graph $H_1 - v_1$ in $\mathcal{A}$. Write $h_1 = |V(H_1)|$. From this graph, form the graph $\tilde{H} \in \mathcal{A}$ by attaching a path of length h_1 to r_1; let vertex r the other end of the path; and let $\tilde{H}$ be rooted at r. Call the part corresponding to $H_1 - v_1$ in a pendant appearance of $\tilde{H}$ a *spike* (following the terminology in [**?**]). Our construction ensures that spikes must be disjoint.

By Lemma **??** (the 'pendant appearances' theorem of [**?**]), there exist constants $a > 0$ and $b > 0$ such that (assuming n is sufficiently large) the number of graphs $F \in \mathcal{A}_{n-r}$ with less than an spikes is at most $e^{-bn}|\mathcal{A}_{n-r}|$. We define further constants in terms of a and b. By a Chernoff bound, there is a constant $c > 0$ such that

$$\mathbb{P}\left(\mathrm{Bin}(\lceil an \rceil, 2^{-h_1+1}) < 2m + 2r\right) = O(e^{-cn}).$$

Let η, $0 < \eta < \frac{1}{2}$ be sufficiently small that

$$\left(\frac{e}{\eta}\right)^{\eta r} < e^{\min\{b,c\}}.$$

Let α be sufficiently large that $e^{\alpha} > 2^r$, and let $t = t(n) = \lceil \eta n \rceil$.

Call a construction *realistic* if there are at most t edges between each vertex $v \in R \setminus S$ and the vertices in $V \setminus R$; and *unrealistic* otherwise. Let $\mathcal{C}(n)$ denote the set of graphs in $\mathcal{A}_{n-r+1}$ such that vertex $n{-}r{+}1$ has degree $> t$. Since $\mathcal{A}$ has the no-linear-degrees property,

$$|\mathcal{C}(n)| = O(e^{-\alpha n})\ |\mathcal{A}_n|.$$

For a given choice of R and S, the number of graphs $\tilde{F}$ on $V \setminus S$ such that the induced subgraph $\tilde{F}[R \setminus S]$ is some fixed graph, and some vertex in $R \setminus S$ has $> t$ edges to the vertices in $V \setminus R$ is at most

$$(r-k)\ |\mathcal{C}(n)|\ 2^{(r-k-1)(n-r)}.$$

Hence the number of unrealistic constructions is at most

$$O(n^r) \cdot 2^{k(n-r)}\ 2^{(r-k-1)(n-r)} |\mathcal{C}(n)| \le O(n^r) \cdot 2^{rn} |\mathcal{A}_n|\ e^{-\alpha n} = |\mathcal{A}_n|\ e^{-\Omega(n)}.$$

Thus there are few unrealistic constructions so that we may ignore them further. Note that in realistic constructions, the number of choices for the bipartite graph $H(R \setminus S, V \setminus R)$ in step (iv) is

$$\left(\sum_{i=0}^{t} \binom{n-r}{i}\right)^{r-k} \le \left(n\binom{n}{t}\right)^{r} \le n^r \left(\frac{ne}{t}\right)^{tr} \le n^r \left(\frac{e}{\eta}\right)^{(\eta n+1)r}.$$

Let us bound the number of realistic constructions which yield a graph G in $\mathrm{Ex}\,(k+1)\,\mathcal{B} \setminus \mathrm{apex}^k\,\mathcal{A}$. For each such construction, the graph $G[V \setminus S]$ must contain a minimal subgraph K which contracts to an excluded minor H_i for some $i \in \{1, \ldots, j\}$; and by Lemma **??**, such a subgraph K can touch at most $2(m+r-k)$ spikes.

Now suppose that each vertex v in S is adjacent to all $h_1 - 1$ vertices of each spike in a set A_v of at least $2m+2r-k$ spikes. Since the graph K does not touch at least $2m+2r-k-2(m+r-k) = k$ spikes in A_v for each vertex $v \in S$, we can pick a spike in A_v (for example, greedily) for each $v \in S$ to form k disjoint subgraphs isomorphic to H_1 with none of them touching the graph K.

But now there are at least $k+1$ disjoint excluded minors in G, contradicting G being in $\mathrm{Ex}\,(k+1)\mathcal{B}$. Hence, for at least one vertex v in S, v must be adjacent to all $h_1 - 1$ vertices of at most $2m+2r-k-1 < 2m+2r$ spikes.

Therefore, given any choices at steps (i),(iii) and (iv), if F has z spikes then the number of choices at step (ii) to obtain a graph in $\mathrm{Ex}\,(k+1)\mathcal{B} \setminus \mathrm{apex}^k\,\mathcal{A}$ is at most

$$2^{k(n-r)}\ k\ \mathbb{P}\left(\mathrm{Bin}(z, 2^{-h_1+1}) < 2m+2r\right).$$

Hence, by considering separately the realistic constructions which yield a graph in $\mathrm{Ex}\,(k+1)\mathcal{B} \setminus \mathrm{apex}^k\,\mathcal{A}$ such that F has $< an$ spikes and those ones where F has $\geq an$ spikes, we see that the number of such constructions is at most

$$\begin{aligned} & O(n^{2r})\ 2^{kn} \left(\frac{e}{\eta}\right)^{\eta r n} |\mathcal{A}_{n-r}| \left(e^{-bn} + e^{-cn}\right) \\ = \ & e^{-\Omega(n)}\ 2^{kn}\ |\mathcal{A}_{n-k}| \ = \ e^{-\Omega(n)}\ |(\mathrm{apex}^k\,\mathcal{A})_n|. \end{aligned}$$

since $|(\mathrm{apex}^k\,\mathcal{A})_n| \geq 2^{k(n-k)}|\mathcal{A}_{n-k}|$. This completes the proof of the lemma. $\square$

The following two simple facts will be useful.

Lemma 5.5 *Let $\mathcal{A}$ be a proper minor-closed class of graphs, with set $\mathcal{B}$ of excluded minors, such that no graph in $\mathcal{B}$ has a component which is a path. Let k be a positive integer. Then there is a positive integer t such that for all $n \geq t$*

$$(\mathrm{Ex}\,(k+1)\mathcal{B} \setminus apex^k\mathcal{A})_n \geq n!/(2 \cdot t!).$$

Proof We need only show that for some $t \geq 3$, the graph K_t is in $\mathrm{Ex}\,(k+1)\mathcal{B} \setminus \mathrm{apex}^k\mathcal{A}$, since then the graphs formed from K_t together with a disjoint path would also be in this class.

Let h be the least number of vertices in a graph in $\mathcal{B}$, so $h \geq 3$. Let $t = h+k$, and note that $t < (k+1)h$, since $(k+1)h-(h+k) = k(h-1) > 0$. Then K_t is not in $\mathrm{apex}^k\,\mathcal{A}$ because removing k vertices from K_t leaves a copy of K_h; and K_t cannot contain a minor in $(k+1)\mathcal{B}$ since $t < (k+1)h$. $\square$

Lemma 5.6 *Let $\mathcal{A}$ be a proper minor-closed class of graphs with set $\mathcal{B}$ of excluded minors, and suppose that $\mathcal{A} \supseteq apex^l\,\mathcal{C}$ for some class $\mathcal{C}$ of graphs and some positive integer l. Then* $\mathrm{Ex}\,(k+1)\mathcal{B} \supseteq apex^{(k+1)(l+1)-1}\,\mathcal{C}$.

Proof Let $G \in \text{apex}^{(k+1)(l+1)-1}\,\mathcal{C}$; and suppose for a contradiction that G has $k+1$ disjoint subgraphs $H_1, \ldots, H_{k+1}$ each with a minor in $\mathcal{B}$. Let $S \subseteq V(G)$ be a set of size at most $(k+1)(l+1)-1$ such that $G-S \in \mathcal{C}$. Since $\text{apex}^{l}\,\mathcal{C} \subseteq \text{Ex}\,\mathcal{B}$, each graph H_i must have at least $l+1$ vertices in S; and since the graphs H_i are pairwise disjoint we must have that $|S| \geq (k+1)(l+1)$, contradicting our choice of S. $\square$

Now Theorem **??** will follow easily.

Proof of Theorem ?? For the first part, suppose that $\mathcal{A}$ is addable and does not contain all fans. Then directly from Lemmas **??** and **??**, for each positive integer k, as $n \to \infty$

$$|(\text{Ex}\,(k+1)\mathcal{B})_n| = (1 + e^{-\Omega(n)})|(\text{apex}^{k}\,\mathcal{A})_n|.$$

But now Lemma **??** allows us to replace the $e^{-\Omega(n)}$ by $e^{-\Theta(n)}$.

Let us prove the second part of the theorem. Suppose that $\mathcal{A}$ contains all fans. Let $\mathcal{P}$ denote the class of all paths, so $\mathcal{A} \supseteq \text{apex}\,\mathcal{P}$. Then by Lemma **??** we have $\text{Ex}\,(k+1)\mathcal{B} \supseteq \text{apex}^{2k+1}\,\mathcal{P}$. So, by counting just the graphs where the first $2k+1$ vertices form an apex set,

$$\begin{aligned}
|(\text{Ex}\,(k+1)\mathcal{B})_n| &\geq |(\text{apex}^{2k+1}\,\mathcal{P})_n| \\
&\geq 2^{(2k+1)(n-2k-1)}\frac{1}{2}(n-2k-1)! \\
&\geq n!\; 2^{(2k+1)n-o(n)}.
\end{aligned}$$

But since $\mathcal{A}$ is proper minor-closed, it is small [?] (see [?] for another proof); that is, $|\mathcal{A}_n| = O(\gamma^n n!)$ for some γ. So

$$|(\text{apex}^{k}\,\mathcal{A})_n| \leq \binom{n}{k} 2^{kn} |\mathcal{A}_{n-k}| = O((2^k\gamma)^n n!).$$

Therefore

$$|(\text{Ex}\,(k+1)\mathcal{B})_n| / |(\text{apex}^{k}\,\mathcal{A})_n| \geq \left(2^{k+1-o(1)}/\gamma\right)^n$$

and we have completed the proof. $\square$

Remark 5.7 *If the class $\mathcal{A}$ contains an apex class larger than apex $\mathcal{P}$ we can obtain a better lower bound than in the last part of the proof above. For example, the class $\text{Ex}\,K_4$ of series-parallel graphs contains apex $\mathcal{F}$, so for each fixed $k \geq 1$, by Lemma* **??**, $|(\text{Ex}\,(k+1)K_4)_n| \geq 2^{(2k+1)n} e^{n-o(n)} n!$. *Also $\text{Ex}\,K_4$ has growth constant $\gamma \approx 9.07$ [?], and $2^{k+1}e \geq 4e \approx 10.87$ so $2^{k+1}e > \gamma$; and hence the graphs on $[n]$ in apex^k $(\text{Ex}\,K_4)$ form only an exponentially small proportion of those in $\text{Ex}\,(k+1)K_4$.*

6 Properties of the random graphs R_n

In this section we use the 'counting' results Theorems **??** and **??** to prove Theorems **??** and **??**, as well as Theorem **??** which extends Theorem **??**.

Let $\mathcal{M}^k$ be the multiset of graphs produced by the constructions in the proof of Theorem **??**. For graphs $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$ and $R'_n \in_u \mathcal{M}^k$, that proof together with Theorem **??** gives that

$$d_{TV}(R_n, R'_n) = e^{-\Omega(n)} \tag{7}$$

where d_{TV} denotes total variation distance. Therefore it is enough to prove Theorems **??** and **??** with R_n replaced by R'_n (except for the easy lower bound in the second result). Notice that the graph R'_n can be generated by choosing the set S and the graphs $G[S]$, F and B in the steps in the proof of Theorem **??** uniformly at random.

Proof of Theorem ?? Suppose $S = S_0$ was chosen for step (1). Note that for each $v \in S_0$

$$\mathbb{P}\left(\left(\frac{1}{2} - \varepsilon\right) n \le d(v) \le \left(\frac{1}{2} + \varepsilon\right) n\right) = 1 - e^{-\Omega(n)}.$$

Indeed, since the graph $G[S]$ and the bipartite graph B are chosen uniformly at random, each vertex $v \in S_0$ has $d(v) \sim \mathrm{Bin}\left(n-1, \frac{1}{2}\right)$.

To show that with probability $1 - e^{-\Omega(n)}$ no vertex $v \notin S_0$ may belong to S_n (that is, have degree $> \varepsilon n$) we can apply Lemma **??** to the class $\mathcal{A}$ (as each proper addable minor-closed class of graphs has growth constant at least e). Thus for the graph F picked in step (2) uniformly from $\mathcal{A}_{n-k}$ we get

$$\mathbb{P}\left(\Delta(F) > \frac{\varepsilon n}{2}\right) = e^{-\Omega(n)}.$$

Hence with probability $1 - e^{-\Omega(n)}$, given that $S = S_0$ was chosen in step (1), each vertex in $V(F)$ has degree not larger than $\frac{\varepsilon n}{2} + k < \varepsilon n$ in R'_n, for n sufficiently large. Considering now all k-subsets $S_0 \subset [n]$ completes the proof of (i) and (ii).

For (iii), let a, H, H^- and h be such as in the proof of Theorem **??**. We say that a construction G obtained in that proof is *very good* if for each vertex $s \in S$ there are at least $m = 2^{-h}an$ pairwise disjoint subsets $X_1(s), X_2(s), \ldots, X_m(s) \subseteq V(F)$ such that each graph $G[X_i(s) \cup s]$ has H as a minor.

If G is very good and S' is another $\mathcal{B}$-blocker of G such that $S \not\subseteq S'$, then S' must have at least δn vertices where $\delta := 2^{-h}a$ (to see this, let

$s \in S \setminus S'$ and note that S' has to contain a vertex from each of the disjoint sets $X_i(s)$). But given that $S = S_0$ and that F has at least $a(n-k)$ copies of H^-, the probability that a random construction R'_n is not very good is at most

$$k\,\mathbb{P}\left(\mathrm{Bin}\left(\lfloor a(n-k)\rfloor, 2^{-h+1}\right) < \lceil 2^{-h}an\rceil\right) = e^{-\Omega(n)}.$$

□

Proof of Theorem ?? Let $K, H \in \mathcal{A}$ be (fixed) connected graphs such that $\omega(K) = \omega(\mathcal{A})$ and $\chi(H) = \chi(\mathcal{A})$. Fix a k-set S_0 of vertices and a graph H_0 on S_0. By Lemma **??** (the pendant appearances theorem of [**?**]), there exists $a > 0$ such that, with probability $1 - e^{-\Omega(n)}$, the graph F chosen in step (2) of the construction has at least $2\lceil an\rceil$ disjoint sets of vertices such that the first $\lceil an\rceil$ induce copies of K, and the second $\lceil an\rceil$ induce copies of H.

Standard bounds for the binomial distribution now show that with probability $1-e^{-\Omega(n)}$ there is a copy of K and a copy of H such that each possible edge between S_0 and these copies is present. Thus with probability $1-e^{-\Omega(n)}$ we have $\omega(R'_n) \geq \omega(H_0)+\omega(\mathcal{A})$ and $\chi(R'_n) \geq \chi(H_0)+\chi(\mathcal{A})$. Hence, removing the conditioning on S and the graph on S, with probability $1-e^{-\Omega(n)}$ we have $\omega(R'_n) \geq \omega(R') + \omega(\mathcal{A})$ and $\chi(R'_n) \geq \chi(R') + \chi(\mathcal{A})$, where R' denotes the induced subgraph $R'_n[S]$. Also $R' \sim R$; that is, R' and R have the same distribution.

Of course, the reverse inequalities, $\omega(R'_n) \leq \omega(R') + \omega(\mathcal{A})$ and $\chi(R'_n) \leq \chi(R') + \chi(\mathcal{A})$ always hold. We have now shown that

$$d_{TV}\left((\omega(R'_n), \chi(R'_n)), (X, Y)\right) = e^{-\Omega(n)}$$

and thus, by the discussion at the start of this section, that

$$d_{TV}\left((\omega(R_n), \chi(R_n)), (X, Y)\right) = e^{-\Omega(n)}.$$

To replace $e^{-\Omega(n)}$ by $e^{-\Theta(n)}$ note that $\mathbb{P}(R_n \in \mathcal{A}) = e^{-O(n)}$. □

We shall deduce Theorem **??** from a more general result, Theorem **??**. For a graph G we let $\mathrm{Big}(G)$ denote its (lexicographically first) largest component, and let the *fragment* $\mathrm{Frag}(G)$ be G less $\mathrm{Big}(G)$. Let us use $v(G)$ to denote $|V(G)|$. Thus $\mathrm{frag}(G) = v(\mathrm{Frag}(G))$. We shall investigate the asymptotic behaviour of $\mathrm{Frag}(R_n)$, following the treatment in [**?**].

A class $\mathcal{A}$ of graphs is called *decomposable* if a graph is in $\mathcal{A}$ if and only if each component is. (It is easy to see that each addable minor-closed class is decomposable.) For any graph class $\mathcal{A}$ we let $\mathcal{UA}$ denote the corresponding unlabelled graph class, with members the equivalence classes of graphs in $\mathcal{A}$ under isomorphism.

Now let $\mathcal{A}$ be any decomposable class of (labelled) graphs, and let $A(x) = \sum_{n \geq 0} |\mathcal{A}_n| x^n / n!$ be its exponential generating function. Let $\mathcal{C}$ denote the class of connected graphs in $\mathcal{A}$, with exponential generating function $C(x)$. Recall the 'exponential formula', that $A(x) = e^{C(x)}$ (see for example [**?**]). (By convention the empty graph $\emptyset$ is in $\mathcal{A}$ and not in $\mathcal{C}$.) If $\rho > 0$ is such that $A(\rho)$ is finite, then we may obtain a natural 'Boltzmann Poisson distribution' on $\mathcal{UA}$, as follows. Let

$$\lambda(H) = \frac{\rho^{v(H)}}{\mathrm{aut}(H)} \quad \text{for each graph } H \in \mathcal{UA} \tag{8}$$

where $\mathrm{aut}(H)$ denotes the number of automorphisms of H. Then

$$\sum_{H \in \mathcal{UA}} \lambda(H) = A(\rho) = e^{C(\rho)}.$$

The *Boltzmann Poisson random graph* $R = R(\mathcal{A}, \rho)$ takes values in $\mathcal{UA}$, with

$$\mathbb{P}[R = H] = \frac{\lambda(H)}{A(\rho)} \quad \text{for each } H \in \mathcal{UA}. \tag{9}$$

It is shown in [**?**] that the number of components of R isomorphic to a given graph $H \in \mathcal{UC}$ has distribution $\mathrm{Po}(\lambda(H))$, and numbers of components corresponding to distinct graphs in $\mathcal{UC}$ are independent; and thus the random number $\kappa(R)$ of components of R satisfies $\kappa(R) \sim \mathrm{Po}(C(\rho))$. Also, $v(R)$ is the sum of independent random variables $v(H)\mathrm{Po}(\lambda(H))$ for $H \in \mathcal{UC}$; and

$$\mathbb{P}[v(R) = n] = \frac{|\mathcal{A}_n| \rho^n / n!}{A(\rho)} \quad \text{for } n = 0, 1, 2, \ldots \tag{10}$$

We are interested in the limiting behaviour of the random graph $\mathrm{Frag}(R_n)$. It is convenient to deal with the corresponding random unlabelled graph which we denote by $\mathcal{U}\mathrm{Frag}(R_n)$.

Theorem 6.1 *Let $\mathcal{A}$ be an addable minor-closed class of graphs which does not contain all fans; let ρ be the radius of convergence of its exponential generating function $A(x)$; and let $\mathcal{B}$ be its set of excluded minors. Let $\mathcal{C}$ be the class of connected graphs in $\mathcal{A}$, with exponential generating function $C(x)$. Then $A(\rho) < \infty$; and given a positive integer k, for $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$ we have*

$$d_{TV}(\mathcal{U}\mathrm{Frag}(R_n), R) \to 0 \quad \text{as } n \to \infty. \tag{11}$$

where $R = R(\mathcal{A}, \rho/2^k)$ is the Boltzmann Poisson random graph for $\mathcal{A}$ and $\rho/2^k$ as in (??) and (??) above. Further,

$$\mathbb{E}\,[\mathrm{frag}(R_n)] \to \mathbb{E}\,[v(R)] = (\rho/2^k)\ C'(\rho/2^k) < \infty. \tag{12}$$

Since $\mathbb{P}(R = \emptyset) = e^{-C(\rho/2^k)}$, Theorem **??** follows as an immediate corollary. Also, $d_{TV}(\mathrm{frag}(R_n), v(R)) \to 0$ as $n \to \infty$, where the distribution of $v(R)$ is given by (??) with ρ replaced by $\rho/2^k$.

To prove (??) in Theorem **??** we use one preliminary lemma, followed by a lemma taken from [?]. After that, to prove (??) in Theorem **??** we use another preliminary lemma.

Lemma 6.2 *Let $\mathcal{A}$ be a minor-closed class of graphs, with set $\mathcal{B}$ of excluded minors. Let j be a positive integer, suppose that* $\mathrm{Ex}\, j\mathcal{B}$ *has a positive growth constant, and let $R_n \in_u \mathrm{Ex}\, j\mathcal{B}$. Then*

$$\mathbb{P}(\mathrm{Frag}(R_n) \in \mathcal{A}) = 1 - e^{-\Omega(n)} \quad \textit{as } n \to \infty.$$

Proof The case $j = 1$ is trivial, since $\mathcal{A} = \mathrm{Ex}\,\mathcal{B}$; so assume that $j \geq 2$. For $i = 1, 2, \ldots$ let $\mathcal{A}^i$ denote $\mathrm{Ex}\, i\mathcal{B}$, and let the exponential generating function $A^i(x)$ of the graphs in $\mathcal{A}^i$ have radius of convergence ρ_i. Then $\rho_{i+1} \leq \rho_i/2$, since from each graph G in $\mathcal{A}^i_n$ we may construct at least 2^n graphs in $\mathcal{A}^{i+1}_{n+1}$ by adding any set of edges between vertex $n+1$ and $V(G)$. Thus $\rho_j \leq 2^{-(j-1)}\rho_1 \leq \rho_1/2$.

Observe that if any component of R_n is in $\mathcal{A}^j \setminus \mathcal{A}^{j-1}$ then the remaining components must be in $\mathcal{A}$. Thus

$$\begin{aligned} \mathbb{P}(\mathrm{Frag}(R_n) \notin \mathcal{A}) \ &\leq\ \mathbb{P}(R_n \text{ has all components in } \mathcal{A}^{j-1}) \\ &+\ \mathbb{P}(\mathrm{Frag}(R_n) \text{ has a component in } \mathcal{A}^j \setminus \mathcal{A}^{j-1}). \end{aligned} \tag{13}$$

The first term on the right side tends to 0, because the family $\mathcal{D}$ of graphs with all components from $\mathcal{A}^{j-1}$ has radius of convergence $\rho_{j-1} \geq 2\rho_j$. To see this, let $C(x)$ denote the exponential generating function of the connected graphs in $\mathcal{A}^{j-1}$: then, by the exponential formula, $\mathcal{D}$ has exponential generating function $e^{C(x)}$, and this converges for $0 < x < \rho_{j-1}$.

Consider the second term on the right side. Fix $\varepsilon > 0$ sufficiently small that

$$\frac{(\rho_1^{-1} + \varepsilon)(\rho_j^{-1} + \varepsilon)}{(\rho_j^{-1} - \varepsilon)^2} < 1.$$

There are constants $0 < a \leq b$ such that for all non-negative integers n

$$a(\rho_j^{-1} - \varepsilon)^n \leq \frac{|\mathcal{A}^j_n|}{n!} \leq b(\rho_j^{-1} + \varepsilon)^n$$

(since $\mathcal{A}^j$ has growth constant ρ_j^{-1}) and

$$\frac{|\mathcal{A}_n|}{n!} \leq b(\rho_1^{-1} + \varepsilon)^n.$$

Since each component of $\text{Frag}(R_n)$ has at most $n/2$ vertices, it now follows that the second term on the right side of (**??**) is at most

$$\begin{aligned}
\frac{1}{|\mathcal{A}_n^j|} \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{t} |\mathcal{A}_t^j| |\mathcal{A}_{n-t}| \quad &\leq \quad \frac{b^2 n!}{a n! (\rho_j^{-1} - \varepsilon)^n} \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} (\rho_j^{-1} + \varepsilon)^t (\rho_1^{-1} + \varepsilon)^{n-t} \\
&\leq \quad (b^2/a) n \left(\frac{(\rho_1^{-1} + \varepsilon)(\rho_j^{-1} + \varepsilon)}{(\rho_j^{-1} - \varepsilon)^2} \right)^{n/2} \quad = \quad e^{-\Omega(n)}.
\end{aligned}$$

□

We say that $\mathcal{A}$ is *bridge-addable* if given any graph G in $\mathcal{A}$ and vertices u and v in distinct components of G, the graph obtained from G by adding an edge joining u and v must be in $\mathcal{A}$. It is easy to see that each addable minor-closed class is bridge-addable. Given a graph H in $\mathcal{A}$, we say that H is *freely addable* to $\mathcal{A}$ if, given any graph G disjoint from H, the union of G and H is in $\mathcal{A}$ if and only if G is in $\mathcal{A}$. Recall that $\mathcal{A}$ is called smooth if $|\mathcal{A}_n| / (n|\mathcal{A}_{n-1}|)$ converges to some finite constant $\gamma > 0$ as $n \to \infty$. The following lemma is a combination of Lemmas 4.4 and 4.5 from McDiarmid [**?**].

Lemma 6.3 *Let the class $\mathcal{A}$ of graphs be minor-closed and bridge-addable; let $R_n \in_u \mathcal{A}$; let $\mathcal{B}$ denote the class of all graphs freely addable to $\mathcal{A}$; and suppose that $\mathbb{P}(\text{Frag}(R_n) \in \mathcal{B}) \to 1$ as $n \to \infty$. Suppose further that $\mathcal{A}$ is smooth, and its exponential generating function $A(x)$ has radius of convergence ρ, with $0 < \rho < \infty$.*

*Then the exponential generating function $B(x)$ of $\mathcal{B}$ satisfies $0 < B(\rho) < \infty$; and $d_{TV}(\mathcal{U}\text{Frag}(R_n), R) \to 0$ where $R = R(\mathcal{B}, \rho)$ is the Boltzmann Poisson random graph for $\mathcal{B}$ and ρ as defined in (**??**) and (**??**) above. Further, $\mathbb{E}[v(R)] = \rho C'(\rho) < \infty$, where $C(x)$ is the exponential generating function of the class $\mathcal{C}$ of connected graphs in $\mathcal{B}$.*

Proof of (??) in Theorem ?? Let $\mathcal{A}^{k+1}$ denote $\text{Ex}\,(k+1)\mathcal{B}$. Since $\mathcal{A}$ is addable, it follows that $\mathcal{A}^{k+1}$ is bridge-addable and the class of graphs freely addable to $\mathcal{A}^{k+1}$ is $\mathcal{A}$. By Theorems **??** and **??**, $\mathcal{A}^{k+1}$ is smooth and its exponential generating function has radius of convergence $\rho/2^k$. Thus by Lemma **??** we may use Lemma **??** to complete the proof. $\Box$

In order to prove (**??**) in Theorem **??** we need one more lemma. For a vertex v in a graph G, we let $\mathrm{Comp}(v, G)$ denote the component containing v and let $\mathrm{comp}(v, G)$ denote its number of vertices.

Lemma 6.4 *Let $\mathcal{A}$ be an addable minor-closed class of graphs which does not contain all fans; let $\mathcal{B}$ be its set of excluded minors; let k be a positive integer and let $R_n \in_u \mathrm{Ex}\,(k+1)\mathcal{B}$. Then for each $\varepsilon > 0$ there is a $K \geq 0$ such that*

$$\mathbb{E}\left[|\{v \in V(\mathrm{Frag}(R_n)) : \mathrm{comp}(v, R_n) \geq K\}|\right] < \varepsilon.$$

Proof By (**??**), it suffices to prove this result with R_n replaced by R'_n. Recall that R'_n specifies a k-set S and $F_{n-k} \in \mathcal{A}$. Let A_n be the event that in R'_n, each vertex in S has an edge to $\mathrm{Big}(F_{n-k})$. We first show that

$$\mathbb{P}(A_n) = 1 - e^{-\Omega(n^{\frac{1}{2}})}. \tag{14}$$

Since $\mathcal{A}$ is bridge-addable, by Theorem 2.2 of [**?**], if we let $k = \lfloor n^{\frac{1}{2}} \rfloor$ then

$$\mathbb{P}(\kappa(F_{n-k}) \geq k+1) \leq 1/k! = e^{-\Omega(n^{\frac{1}{2}} \log n)}.$$

(Recall that $\kappa(G)$ denotes the number of components of G.) Thus

$$\mathbb{P}(|\mathrm{Big}(F_{n-k})| < n^{\frac{1}{2}}) = e^{-\Omega(n^{\frac{1}{2}} \log n)}.$$

Hence, if v_i is a vertex in S then

$$\mathbb{P}(v_i \text{ has no edge to } \mathrm{Big}(F_{n-k})) \leq 2^{-n^{\frac{1}{2}}} + \mathbb{P}(|\mathrm{Big}(F_{n-k})| < n^{\frac{1}{2}}) = e^{-\Omega(n^{\frac{1}{2}})};$$

and now (**??**) follows.

Next observe that if A_n holds then S and $\mathrm{Big}(F_{n-k})$ are contained in $\mathrm{Big}(R'_n)$; and thus if $v \in V(\mathrm{Frag}(R'_n))$ then $v \in V(\mathrm{Frag}(F_{n-k}))$, and there is no edge between S and $\mathrm{Comp}(v, F_{n-k})$.

By (**??**) it suffices to upper bound $\mathbb{E}\left[|\{v \in V(\mathrm{Frag}(R'_n)) : \mathrm{comp}(v, R'_n) \geq K\}| \cdot 1_{A_n}\right]$. For each $v \in V \setminus S$ let $B_n(v)$ be the event that there is no edge between S and $\mathrm{Comp}(v, F_{n-k})$. By the above observations

$$\begin{aligned}
& \mathbb{E}\left[|\{v \in V(\mathrm{Frag}(R'_n)) : \mathrm{comp}(v, R'_n) \geq K\}| \cdot 1_{A_n}\right] \\
\leq\ & \mathbb{E}\left[|\{v \in V \setminus S : v \in V(\mathrm{Frag}(F_{n-k})), \mathrm{comp}(v, F_{n-k}) \geq K, B_n(v)\}|\right] \\
\leq\ & \sum_{v \in V \setminus S} \mathbb{P}\left(B_n(v) \mid \mathrm{comp}(v, F_{n-k}) \geq K, v \in V(\mathrm{Frag}(F_{n-k}))\right) \cdot \mathbb{P}(v \in V(\mathrm{Frag}(F_{n-k}))) \\
\leq\ & \sum_{v \in V \setminus S} 2^{-kK} \mathbb{P}(v \in V(\mathrm{Frag}(F_{n-k}))) \\
\leq\ & 2^{-K} \mathbb{E}\left[\mathrm{frag}(F_{n-k})\right] \ \leq\ c \cdot 2^{-K}.
\end{aligned}$$

In the last inequality here we used the result that $\mathbb{E}\left[\mathrm{frag}(F_{n-k})\right] \leq c$ for a constant c, see Lemma 2.6 of [?]. The lemma follows. □

Proof of (??) in Theorem ?? Let $\varepsilon > 0$. By Lemma **??**

$$\mathbb{E}\left[v(R)\right] = \sum_{H \in \mathcal{UC}} v(H)\lambda(H) = (\rho/2^k)\ C'(\rho/2^k) < \infty.$$

Thus for K sufficiently large

$$\mathbb{E}\left[v(R)\right] - \varepsilon \leq \sum_{H \in \mathcal{UC}, v(H) \leq K} v(H)\lambda(H) \leq \mathbb{E}\left[v(R)\right].$$

By (**??**)

$$\mathbb{E}\left[|\{v \in V(\mathrm{Frag}(R_n)) : \mathrm{comp}(v, R_n) \leq K\}|\right] \to \sum_{H \in \mathcal{UC}, v(H) \leq K} v(H)\lambda(H),$$

and so

$$|\ \mathbb{E}\left[|\{v \in V(\mathrm{Frag}(R_n)) : \mathrm{comp}(v, R_n) \leq K\}|\right] - \mathbb{E}\left[v(R)\right]\ | < \varepsilon + o(1).$$

Hence by Lemma **??** with K sufficiently large

$$|\ \mathbb{E}\left[\mathrm{frag}(R_n)\right] - \mathbb{E}\left[v(R)\right]\ | < 2\varepsilon + o(1)$$

and we are done. □

7 Concluding remarks

Consider an addable minor-closed class $\mathcal{A}$ of graphs, with set $\mathcal{B}$ of excluded minors. For such a class $\mathcal{A}$, we have learned much about the class of graphs with at most k disjoint excluded minors, and in particular about the relationship between $\mathrm{Ex}\,(k{+}1)\mathcal{B}$ and $\mathrm{apex}^k\mathcal{A}$. Here there are just two cases: if $\mathcal{A}$ does not contain all fans then the difference class $\mathrm{Ex}\,(k{+}1)\mathcal{B} \setminus (\mathrm{apex}^k\mathcal{A})$ forms an exponentially small proportion of $\mathrm{Ex}\,(k{+}1)\mathcal{B}$; and if $\mathcal{A}$ contains all fans then $\mathrm{Ex}\,(k{+}1)\mathcal{B}$ is exponentially larger than $\mathrm{apex}^k\mathcal{A}$ (at least for large k).

But what happens if the minors of $\mathcal{A}$ are not 2-connected? Consider for example the class $\mathrm{Ex}\,S_t$, where S_t denotes the star with t leaves (and thus with $t+1$ vertices). For each $t \geq 3$ almost all graphs in $\mathrm{Ex}\,(k+1)S_t$ are in $\mathrm{apex}^k\ \mathrm{Ex}\,S_t$, as in the case when $\mathcal{A}$ is addable and does not contain all fans; and for $t=3$ the difference class $\mathrm{Ex}\,(k+1)S_t \setminus (\mathrm{apex}^k\ \mathrm{Ex}\,S_t)$ forms an exponentially small proportion of $\mathrm{Ex}\,(k+1)S_t$; but this is not the case for $t \geq 4$, where the proportion is $2^{-\Theta(n^{\frac{2t-5}{2t-4}})}$, see [**?**]. There is more to be learned about disjoint excluded minors in such classes of graphs.

A second natural question concerns the behaviour of $\mathrm{Ex}\,(k+1)\mathcal{B}$ when $\mathcal{A}$ is minor-closed and contains all fans. We have learned little about this case, other than the fact that $\mathrm{apex}^k\mathcal{A}$ is irrelevantly small in comparison (at least for large k).

A good starting point is to consider the class $\mathcal{A} = \mathrm{Ex}\,K_4$ of series-parallel graphs, see Remark **??**. Clearly $\mathrm{Ex}\,2K_4$ contains $\mathrm{apex}^3\mathcal{F}$, where $\mathcal{F}$ denotes the class of forests, and so if $\mathrm{Ex}\,2K_4$ has a growth constant then it must be at least $8e$. It is not hard to see that this is not the right value, but it may give the right idea. We conjecture that almost all graphs G in $\mathrm{Ex}\,2K_4$ contain a set S of three vertices such that any two form a K_4-minor-blocker, or equivalently every non-series-parallel subgraph of G has at least 2 vertices in S.

More generally, consider any (fixed) planar graph H. Perhaps there is a positive integer j_H such that the following is true for every positive integer k: almost all graphs G with at most k disjoint subgraphs contractible to H contain a set S of $(k+1)j_H - 1$ vertices such that each subgraph of G contractible to H contains at least j_H vertices from S. Observe that this is true for $H = jC_3$, with $j_H = j$. For if we let $\mathcal{B} = \{jC_3\}$ and $\mathcal{A} = \mathrm{Ex}\,\mathcal{B}$, then $\mathrm{Ex}\,(k{+}1)\mathcal{B} = \mathrm{Ex}\,(k{+}1)j\,C_3$ which is very close to $\mathrm{apex}^{(k+1)j-1}\,\mathcal{F}$ (where $\mathcal{F}$ is the class of forests); and for a graph G in this class, the set S consisting of the $(k+1)j-1$ apex vertices is such that each subgraph of G not in $\mathcal{A}$ contains at least j vertices from S.

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