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Monotone Circuits and Local Computations

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Let $CLIQUE_{n,k}$ denote the monotone Boolean function of $n = \binom{m}{2}$ variables which, given (a binary code of) an m-vertex graph G, computes 1 iff G contains a clique (i.e. a complete subgraph) on k vertices.

A. Razborov [1,2] has proved that, for some k, this function has no polynomial-size circuit over the basis $\{\wedge, \vee, 0, 1\}$, i.e. $CLIQUE$ can not be effectively computed using fan-in 2 monotone gates. A. Yao has recently announced in [3] that this remains true even if one uses all the monotone gates of fan-in at most $n^{1/100}$, i.e. as long as the gates perform all but sufficiently "local" monotone operations, one still needs superpolynomial-size circuits to compute the clique function.

The aim of this note is to show that the clique function can not be effectively computed even allowing gates to perform much more complicated monotone operations.

A *minterm(maxterm)* of a Boolean function $f(x_1, \ldots, x_n)$ is a minimal subset of its variables such that at least one assignment of constants to them determines f to be equal 1 (0), regardless of the values of remaining variables. Denote

$$\mathcal{M}_{n,d} = \left\{ f \in \mathcal{M}_n \; : \; \mu(f) \; \leq \; d \right\}$$

where $\mu(f)$ stands for the size of the largest minterm of f and $\mathcal{M}_n$ is the set of all monotone Boolean functions on n variables.

Definition: A sequence of functions $[f_n] = \{f_1, f_2, \ldots\}$ is called to be (m,d)-*local* (with respect to polynomial-size monotone circuits) if there exists a sequence of circuits $[C_n] = \{C_1, C_2, \ldots\}$ such that, for each n

(i) C_n is over the basis $\mathcal{M}_{m,d}$,

(ii) C_n computes f, and

(iii) $size(C_n) \; \leq \; n^{O(1)}$.

Theorem 1 (Razborov [1,2]): *Let* $k = k(n) \to \infty$ *and* $k \leq n^{1/4}$. *Then* $[CLIQUE_{n,k}]$ *is not (2,2)-local.*

Theorem 2 (Yao [3]): *Let* $\epsilon = 1/100$ *and* $k = \lceil \log\log n \rceil$. *Then* $[CLIQUE_{n,k}]$ *is not* (n^ϵ, n^ϵ)*-local.*

Notice that $CLIQUE_{n,k}$ belongs to $\mathcal{M}_{n,k}$ and hence *is* (n,k)-local. So, one might believe that this function is $(n, k^{1-\epsilon})$-local for some small $\epsilon > 0$. The following theorem refutes this belief.

Theorem 3 : *Let* $k = \lceil (m/\ln m)^{2/3} \rceil$. *Then* $[CLIQUE_{n,k}]$ *is* (n,k)*-local but is not* (n,d)*-local if* $d = o(k)$.

The theorem follows fairly easily from the following symmetric version of Razborov's argument [1,2] for unbounded fan-in monotone circuits.

Every monotone Boolean function $f(x_1, \ldots, x_n)$ is uniquely determined by any of the two hypergraphs $min(f)$ and $Max(f)$ over $X = \{x_1, \ldots, x_n\}$ where $min(f)$ ($Max(f)$) is the set of all minterms (maxterms) of f. A hypergraph $\mathcal{S} \subseteq 2^X$ is an *i-star* ($i \geq 0$) if

$$| \bigcap_{E \in \mathcal{S}} E | \geq i.$$

For a hypergraph $\mathcal{G}$, let $\mathtt{Star}(\mathcal{G}, i)$ denote the minimum number of i-stars covering $\mathcal{G}$. (Put $\mathtt{Star}(\mathcal{G}, i) = 0$ if $\mathcal{G}$ cannot be covered by i-stars, i.e. if it contains an edge of cardinality less than i.

Theorem 4: *Let* C *be a monotone circuit over the basis* $\mathcal{M}_{n,d}$ ($d \geq 1$) *computing a function* f. *Then for any hypergraphs* $\mathcal{G} \subseteq min(f)$ *and* $\mathcal{H} \subset Max(f)$ *the following bound holds:*

$$size(C) \geq \max_{r,s \geq 1} \min \left\{ d^{-1} r^{-u}\, \mathtt{Star}(\mathcal{G}, u), \qquad (r+1)^{-s} s^{-r}\, \mathtt{Star}(\mathcal{H}, r) \right\}$$

where

$$u = \lceil (s+1)/2d \rceil.$$

Proof (sketch): We need some auxilary notations concerning hypergraphs. For a hypergraph $\mathcal{F}$, set $co - \mathcal{F} = \{ X - A \ : \ A \in \mathcal{F} \}$. We say that a hypergraph $\mathcal{E}$ *covers* a hypergraph $\mathcal{F}$ iff each edge of $\mathcal{F}$ contains an edge of $\mathcal{E}$. An *interval* generated by a hypergraph $\mathcal{F}$ is the set of all subsets $E \subseteq X$ such that for any $A, B \in \mathcal{F}$,

$$E \supseteq A \cap B \iff A \neq B.$$

A *co-interval* generated by $\mathcal{F}$ is the set of all subsets $E \subseteq X$ such that for any $A \in \mathcal{F}$,

$$E \cap \left(A - \bigcup_{B \in \mathcal{F} - \{A\}} B \right) \neq \emptyset.$$

In what follows we will consider intervals and co-intervals generated by hypergraphs $\mathcal{F}$ consisting of r edges each of cardinality at most s.

Now we turn to the proof of the theorem. Theorem 3 from [4] directly yields the following lower bound

$$size(C) \ \geq \ \max_{r,s\geq 1} \min \left\{ \ d^{-1} r^{-u} \, \mathtt{Star}(\mathcal{G}, u), \qquad t \ (r+1)^{-s} \ \right\}$$

where t is the minimum number of intervals covering the hypergraph

$$\mathcal{M}(f) = \{ \ E \subseteq X \ \ : \ \ A - E \neq \emptyset \qquad \text{for all} \qquad A \in min(f) \ \}.$$

So, it is enough to prove that

$$t \ \geq \ s^{-r} \mathtt{Star}(\mathcal{H}, r). \tag{1}$$

One may easily verify the following three facts:

(i) $Max(f) \subseteq co - \mathcal{M}(f)$;

(ii) *if* $\mathcal{I}$ *is an interval then* $co - \mathcal{I}$ *is a co-interval*;

(iii) *any co-interval* $\mathcal{J}$ *can be covered by at most* s^r *r-stars.*
(Moreover, $\mathtt{Star}(\mathcal{J}, r) = s^r$ if the underlying hypergraph $\mathcal{F}$ consists of r pairwise disjoint subsets each of cardinality s.)

Now, if $\mathcal{M}(f)$ can be covered by t intervals then by (i) and (ii), $Max(f)$ can be also covered by at most t co-intervals, each of which, by (iii), has no more than s^r r-stars. Hence, $\mathtt{Star}(\mathcal{H}, r) \leq ts^r$ for any $\mathcal{H} \subseteq Max(f)$. This completes the proof of (1), and thus, the proof of Theorem 4.

Thus, Theorem 4 gives non-trivial lower bounds on the unbounded fan-in monotone circuit complexity of f if neither $min(f)$ nor $Max(f)$ can be covered by a small number of stars, i.e. if f has large "star complexity". To obtain the theorem 3 it is sufficient to verify that $CLIQUE_{n,k}$ has super-polynomial star complexity for some $r, s \leq k$ with $s/d \to \infty$.

References

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