A special example of $(\max, +)$ polynomial*

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In the present supplementary note we provide an example of tropical polynomial P illustrating a very specific feature of computations over $(\max, +)$ semiring. Any minimal $(\max, +, -x)$ circuit computing P has to produce terms with negative coefficients. Nothing similar cannot be observed in the cases of $(+, *)$ and $(\min, +)$ semirings.

Consider the polynomial

$$P(x,y,z) = \max(2x+y+2z,\, 3x+z,\, 2y).$$

We will prove below that its $(\max, +, -x)$ complexity is 6, and any minimal circuit produces terms with negative coefficients. Indeed, it follows that $(\max, +, -x)$ complexity of P is smaller than its $(\max, +)$ complexity.

The polynomial P may be computed by a circuit (actually, a formula)

$$\max(2(x+z), y) + \max(y, x-z) \tag{1}$$

of size 6. Note that this formula produces a redundant term $x+y-z$.

Let $t(P)$ and $d(P)$ denote the set of (defining) terms of P, and the set of their pairwise differences (up to the sign), respectively:

$$t(P) = \{2x+y+2z,\, 3x+z,\, 2y\}, \quad d(P) = \{x-y-z,\, 2x-y+2z,\, 3x-2y+z\}.$$

In what follows, under the *weight* (*absolute weight*) of a linear term we mean the sum of its coefficients (the sum of absolute values of coefficients, respectively). For example, the term $3x-y+z$ has weight 3 and absolute weight 5. The weight $w(p)$, and absolute weight $W(p)$ of a polynomial p are defined as the maximum of weight (absolute weight, respectively) over all terms of p. The following claim is quite obvious.

Claim 1.

(i) If a circuit produces some polynomial p with either $w(p) \geq s$ or $W(p) \geq s$, then the circuit contains at least $\lceil \log_2 s \rceil$ addition gates.

(ii) Moreover, in any circuit producing p with exactly $\lceil \log_2 s \rceil$ addition gates, there is a path from an input to an output running through all addition gates.

(iii) If a term t has weight (absolute weight) 2^s, and it can be produced by a circuit of s addition gates, then t has a form $2^{s-1}(x+y)$ (or $2^{s-1}(\pm x \pm y)$, respectively).

*Supplementary note for the main paper "Reciprocal inputs in arithmetic and tropical circuits".

Proof. For $p = \max(p_1, p_2)$, we have $w(p) = \max(w(p_1), w(p_2))$, and $W(p) = \max(W(p_1), W(p_2))$. For $p = p_1 + p_2$, we have $w(p) = w(p_1) + w(p_2)$, and $W(p) \leq W(p_1) + W(p_2)$. For inputs, $|w(\pm x)| = W(\pm x) = 1$. Now it can be verified easily, that to produce a polynomial p with $w(p) \geq s$ or $W(p) \geq s$, a tree of addition depth at least $\lceil \log_2 s \rceil$ is required. Hence, the former two assertions of the claim follow. To produce the weight 2^s by s additions, all these additions, except possibly the first one, should be doublings. This leads to (iii). □

We call a polynomial p *prime*, if it contains the only term of maximal weight $w(p)$. For a prime polynomial, we also call its maximal weight term *prime*. So, the polynomial P is prime (with prime term being $2x + y + 2z$), and the polynomial $\{3x - y, 2z, x + y - z\}$ is not.

In what follows, for a circuit C producing some term t, we call a gate $g \in C$ *essential* for t if after replacing this gate with some term t', the circuit no longer produces t.

Claim 2. If a circuit produces some prime polynomial p, then at any gate essential for the prime term of p, a prime polynomial is produced. Moreover, the prime term produced at this gate contributes to the prime term of p.

Proof. Note that at any gate (either max or $+$) maximal weight terms are produced exactly from the maximal weight terms of inputs (the property of higher envelopes). Let t denote the prime term of p. Assume that in some gate essential for p, besides a term t_1 contributing to t, another term t_2, $w(t_2) \geq w(t_1)$, is produced. Then $t + t_2 - t_1$ is a term of p (just take all terms contributing to t, except that t_2 instead of t_1). But $w(t + t_2 - t_1) \geq w(t)$, which makes term t not prime. A contradiction. □

Now we are ready to prove what promised.

Lemma 1. *Any minimal* $(\max, +, -x)$ *circuit computing* $P(X)$ *produces terms with negative coefficients.*

Proof. We have already presented in (1) a $(\max, +, -x)$ circuit of size 6 for P. First, we show that the above circuit is indeed optimal, i.e. there are no circuits of size 5 computing P.

I. Since P contains 3 terms, it requires at least 2 max-gates to be computed. The term $2x + y + 2z$ alone requires at least 3 +-gates by Claim 1. Assume that some circuit C of 3 +-gates computes P. These +-gates should be arranged as a chain to make computation of the prime term $t_1 = 2x + y + 2z$ of weight 5 possible (again by Claim 1). Hence, all these +-gates are essential for t_1 and by Claim 2 should produce prime polynomials.

For computing t_1, we have only two types of circuits with respect to connections between +-gates. The first type: $s_1 = x + z, s_2 = 2s_1, s_3 = s_2 + y$, the second type: $s_1 = x + z, s_2 = s_1 + y, s_3 = s_2 + s_1$ (by s_i we denote prime terms of the polynomials produced at the corresponding gates).

Consider the computation of $t_2 = 3x + z$. This term also has additive complexity 3 (by Claim 1), so it should be produced at the last +-gate. Since $w(t_2) = 4$, a prime term of one of the inputs of this gate contributes to t_2. In a circuit of the first type the prime terms of the inputs are $2(x + z)$ and y. Then, t_2 cannot be produced, since it does not have a form of $2(x + z) + w_{0,0}$ or $y + w_{3,2}$, where $w_{i,j}$ are some terms of weight $\leq i$ and addition complexity $\leq j$.

For a circuit of the second type, input prime terms are $x + y + z$ and $x + z$. Though t_2 cannot be expressed as $x + y + z + w_{1,1}$, it has a representation $x + z + w_{2,2}$ with $w_{2,2} = 2x$. But the required term $2x$ cannot be produced at the first +-gate, since otherwise it would be another prime term there. Thus, the term $2x$ should be computed at the second +-gate. Again, notice that then some of input prime terms contributes to $2x$, so we obtain that either $2x = x+z+w_{0,0}$ or $2x = y+w_{1,1}$, which indeed is not possible.

We comes to a contradiction. Therefore, at least 4 +-gates are required to compute P.

II. Now we are going to check that any circuit of size 6 computing P produces terms with negative coefficients. We have already shown that such circuit should contain 4 +-gates and 2 max-gates. We will consider two subcases.

Subcase 1. The output gate g of the circuit is a max-gate.

Case 1.1. The output of the other max-gate is fed to the input of g.

It means that all 3 terms of P should be computed in a subcircuit containing +-gates alone. But then we need at least 3 gates to compute the term $2x+y+2z$ (by Claim 1), and additionally at least one gate for each of the terms $3x+z$, $2y$. Hence, the circuit contains at least 5 +-gates which makes it not minimal.

Case 1.2. Both inputs of g are connected to +-gates.

Let $\max(a,b)$ be the polynomial produced by another (inner) max-gate in the circuit (a,b are the terms). If this max-gate does not precede one of the +-gates entering g, then that +-gate produces exactly some term of P. This term has to be $2y$, since any other term of P requires 3 +-gates to be produced. It follows that e.g. the term $3x+z$ can be computed via 2 additions from inputs $\pm x, \pm y, \pm z, 2y$, which indeed does not hold.

Hence, the inner max-gate should precede to both +-gates entering g. Thus, $P = \max(\max(a,b)+c, \max(a,b)+d)$ for some terms c,d. By assumption, this formula produces the 3 required terms and a redundant term. Assume w.l.o.g. that the latter is $b+d$. So we have that $b+d = (a+d)+(b+c)-(a+c) = t_1+t_2-t_3$ for some permutation (t_1,t_2,t_3) of the defining terms of P. Hence,

$$b+d \in \{5x-y+3z,\ -x+3y+z,\ x+y-z\}.$$

Anyway, $b+d$ is a term with a negative coefficient.

Subcase 2. The output gate g of the circuit is a +-gate.

Then the produced polynomial has a form $p+q$, where $p = \max(a_1,[a_2,a_3,\ldots])$ and $q = \max(b_1,[b_2,b_3,\ldots])$ are polynomials at inputs of g (square brackets may include optional terms). These polynomials must have at least 4 terms in total to compute the 3 required terms of P. W.l.o.g. assume that p contains no less contributing terms than q.

Case 2.1. Both polynomials p, q contain 2 terms contributing to the computation of 3 terms of P.

It means that $P = \max(a_1,a_2)+\max(b_1,b_2)$. This formula produces the 3 required terms and a redundant term. Let the latter be a_2+b_2. So we have that $a_2+b_2 = (a_1+b_2)+(a_2+b_1)-(a_1+b_1) = t_1+t_2-t_3$ for some permutation (t_1,t_2,t_3) of terms of P. Hence,

$$a_2+b_2 \in \{5x-y+3z,\ -x+3y+z,\ x+y-z\},$$

so a_2+b_2 is a term with a negative coefficient. Note, that this case covers the circuit (1).

Case 2.2. The polynomial p has 3 terms, and q has at least 2 terms involved in computing P.

It means that $P = \max(a_1,a_2,a_3)+\max(b_1,b_2)$. Note that some two pairs of terms of P have to be computed independently, say, as $t_1 = a_1+b_1$ and $t_2 = a_2+b_2$, and as $t_2 = a_2+b_2$ and

$t_3 = a_3 + b_1$. Assume w.l.o.g. t_1, t_2 be a pair with a larger total weight (so, it is not $3x+z$, $2y$). Consider two other terms $u_1 = a_1 + b_2$ and $u_2 = a_2 + b_1$ produced by the circuit. By construction, $u_1 + u_2 = t_1 + t_2$, and $u_1, u_2 \notin \{t_1, t_2\}$.

If $\{t_1, t_2\} = \{2x+y+2z, 3x+z\}$, then some u_i has weight at least 5, hence it cannot be a term of the prime polynomial P.

If $\{t_1, t_2\} = \{2x+y+2z, 2y\}$, and since none of the terms u_i can have weight ≥ 5 (by the above reason), then u_1, u_2 have weights 4 and 3. Choose those of them (say, u_1) whose y-coefficient is ≥ 2. But then u_1 is illegal, since in a point $(1,5,1)$ it takes larger value than any of the defining 3 terms of P.

Case 2.3. The polynomial p has 3 terms (a_1, a_2, a_3), and q has just one term contributing into computing P.

We may assume that q is simply a 1-term polynomial, $q = b_1$ (that is, max-gates are not involved in computing q). If it had another term b_2, then $b_2 < b_1$ would hold, since $p + b_2 \leq p + b_1 = P$. But then the circuit would also produce the terms of the polynomial $P + (b_2 - b_1)$. Since $b_2 - b_1$ has negative coefficients, then at least one of the produced terms $3x+z+b_2-b_1$, $2y+b_2-b_1$ also has negative coefficients.

Subcase 2.3.1. The term q is computed by a subcircuit consisting of 3 +-gates.

Then the terms a_i are either from the set $V = \{0, \pm x, \pm y, \pm z\}$ or are computed at the gates of the +-subcircuit. Only the two inner +-gates may compute some of a_i's, since $2q$ cannot be the term of P, except when $q = y$, but then q does not require gates to be computed.

Recall that some 3 differences $a_i - a_j$ form the set $d(P)$. Given that any difference in $d(P)$ has absolute weight at least 3, only one of a_i's (say, a_3) may belong to V. And it does belong, since a_1, a_2 have to be computed at the two inner gates of the +-subcircuit (so, no other possibilities for a_3 remain open). But then $W(a_1 - a_2) \leq 4$, $W(a_1), W(a_2) \leq 4$, hence $W(a_i - a_3) \leq 5$ for $i = 1, 2$. As a consequence, neither of differences $a_i - a_j$ have absolute weight 6, as required to make the set $d(P)$. A contradiction.

Subcase 2.3.2. The term q is computed by a subcircuit consisting of 2 +-gates.

Then either $q = 2s$ or $q = s + v$ for some $v \in V$, where s is the term computed at the +-gate entering the gate computing q. Assuming that the circuit is non-degenerate, we require s to have absolute weight 2, and q to have absolute weight 3 or 4. The remaining subcircuit (besides those computing q) contains 2 max-gates and a +-gate.

Subsubcase 2.3.2.1. The polynomial p is computed at a +-gate.

By checking all ways to locate two max-gates, we obtain

$$p \in \{2\max(c_1, c_2, c_3), \max(c_1, c_2, c_3) + d_1, \max(c_1, c_2) + \max(d_1, d_2)\}, \tag{2}$$

where $c_i, d_i \in V \cup \{q, s\}$. By construction, the pairwise differences between some three terms of p form $d(P)$. Recall that absolute weights of terms in $d(P)$ are 3, 5, and 6.

So, the first possibility in (2) cannot occur, since (absolute) weights of all three differences in it would be even.

The second one is also impossible, since neither of terms $c_i - c_j$ can have absolute weight 6 (absolute weights of q, s, and $q - s$ are at most 4, 2, and 2, respectively).

In the remaining case $p = \max(c_1, c_2) + \max(d_1, d_2)$, one of the terms $c_i + d_j$ (say, $c_2 + d_2$) is redundant. Then, the three pairwise differences are $c_1 - c_2$, $d_1 - d_2$ and $c_1 + d_2 - c_2 - d_1$ (up to the sign). Either $c_1 - c_2$, or $d_1 - d_2$ has absolute weight at least 5 (assume that it is $c_1 - c_2$), and neither of them has absolute weight less than 3. Then $c_i = q$ for some i, and $W(q) = 4$. Hence, $q = 2(v + v')$ for some $v, v' \in V$ by Claim 1. Also $d_j \in V$ for some j (since $\{d_1, d_2\} \neq \{s, q\}$). But then the circuit produces the term $c_i + d_j + q = 2q + d_j$, which either has negative coefficients (when $w(q) < 4$), or weights at least 7 (when $w(q) = 4$) and makes the computed polynomial different from P.

Subsubcase 2.3.2.2. The polynomial p is computed at a max-gate.

Here we have

$$p \in \{\max(c_1, 2c_2, 2c_3), \max(c_1, c_2 + d_1, c_3 + d_1), \max(c_1, c_2, d_1 + d_2)\}, \tag{3}$$

where $c_i, d_i \in V \cup \{q, s\}$.

The first possibility in (3) cannot occur, since it implies the difference $2(c_2 - c_3)$ between some two terms of P with all coefficients being even, contradicting the structure of $d(P)$.

Now, assuming $p = \max(c_1, c_2, d_1 + d_2)$ we obtain that $c_1 - c_2$ (up to the sign) belongs to $d(P)$. Hence, w.l.o.g. $c_1 \in V$, and $c_2 \in \{q, s\}$. But $c_2 \neq q$, otherwise $c_2 + q = 2q$ would be a term of P (recall that $q = y$ doesn't go, since the gate computing q would be redundant). Thus, $c_2 = s$. It would follow from $q = 2s$ that $3s \in t(P)$, which is impossible. Therefore, $q = s + v$ for some $v \in V$. Then we have $c_2 + q = 2s + v$ as a term of P, leading to $v = y$, $s = x + z$, and $q = x + y + z$. But neither of the 3 terms of P have the form $c_1 + q = x + y + z + c_1$. A contradiction.

We are left to check the case $p = \max(c_1, c_2 + d_1, c_3 + d_1)$. Since $c_2 - c_3$ (up to the sign) belongs to $d(P)$, we obtain w.l.o.g. $c_2 \in V$, and $c_3 \in \{q, s\}$.

Since $3q \notin t(P)$, then $c_3 + d_1 \neq 2q$. Since neither $4s$, nor $3s + v$ for some $v \in V$, cannot be a term of P, then $c_3 + d_1 \neq 2s$. Now assume that $c_3 + d_1 = s + q$. Then $2q + s$ is a term of P, hence $q \neq 2s$. So $q = s + v$ for some $v \in V$, and $3q - v$ is a term of P, but its absolute weight is at least 8, which is not possible. Thus, we obtain $d_1 \in V$.

Look at c_1. Clearly, $c_1 \neq q$, since $2q \notin P$, and $c_1 \neq s$, since otherwise the difference $c_2 + d_1 - c_1$ would have even absolute weight ≤ 4 contradicting with the structure of $d(P)$. In the remaining case $c_1 \in V$, we need $c_3 = q = 2s$ to provide a difference (namely, $c_3 + d_1 - c_1$) with absolute weight 6. But then the circuit would produce the term $c_3 + d_1 + q = 4s + d_1 \in t(P)$, which is impossible.

Subcase 2.3.3. The term q is computed by a +-gate connected to inputs.

It means that the absolute weight of q is 2. Recall that the polynomial p is produced by a subcircuit of 2 +-gates and 2 max-gates taking inputs from $V \cup \{q\}$. The weight of q cannot be non-positive, since otherwise some of a_i's would have weight at least 5, which is impossible to produce via just 2 +-gates. So the weight of q is 2.

Note that if $q \neq 2x$, then the polynomial $p = P - q$ contains a term requiring 3 +-gates to be produced: either $3x + z - q$, when $q \neq x + z$, or $2y - q$, otherwise (by Claim 1). Looking over all ways to arrange 2 +-gates in the subcircuit computing p we obtain that any produced term a_i belongs to one of the two types $2(u_1 + u_2)$, $u_1 + u_2 + u_3$, where $u_i \in V \cup \{q\}$. Given that $W(t) \geq 4$, $w(t) \in \{0, 2\}$, and t contains odd coefficients, it remains the only possibility $t = q + v_1 + v_2$ for some $v_1, v_2 \in V$. In the case $q \neq x + z$, we obtain $3x + z - v_1 - v_2 = 2q$, hence $2q = (3 \pm 1)x + (1 \mp 1)z$, and it cannot be satisfied. In the case $q = x + z$, we have $2y - v_1 - v_2 = 2q = 2(x + z)$, which is obviously unsatisfiable.

In the remaining case, $q = 2x$ and $p = \max(y+2z, x+z, 2y-2x)$. There is the only way to produce the term $2y-2x$ via 2 +-gates, namely, as $2(y-x)$ (by Claim 1). It means that the inputs of the first of these two gates are polynomials $p_1 = \max(y, [\ldots])$ and $p_2 = \max(-x, [\ldots])$, where square brackets may include some additional terms. Since p is prime polynomial, then both p_1 and p_2 are prime by Claim 2. To have the prime term $y+2z$ produced for p, the term z should belong to one of p_i. It cannot belong to p_1, since p_1 is prime, so $p_2 = \max(-x, z, [\ldots])$. But then the subcircuit computing p illegally produces the term $2(y+z)$. A contradiction.

Subcase 2.3.4. $q \in V \setminus \{0\}$.

In this subcase, we are checking that no circuit with 3 +-gates can compute $P-q$. We will search through all possible values for q.

Subsubcase 2.3.4.1. $q \in \{-x, -z\}$.

We will show that the prime term $2x+y+2z-q$ of p alone requires 4 +-gates to be computed. Cases $q = -x$ and $q = -z$ are mutually symmetric, so we consider only the former. Assume that 3 additions are sufficient to compute $3x+y+2z$.

By Claim 1, these additions should be arranged into a chain s_1, s_2, s_3. The last gate cannot be doubling, since the term has odd coefficients. So to obtain the weight 6, we need $s_2 = 2s_1$ and $s_3 = s_2 + s_1$. But then the circuit would compute $3s_1$, which is not what required.

Subsubcase 2.3.4.2. $q \in \{\pm y\}$.

Again, we will show that the term $3x \pm y + z$ alone requires 4 +-gates. Assume contrariwise that 3 additions are sufficient.

By Claim 1, these additions should be arranged into a chain s_1, s_2, s_3. There cannot be doublings among the two last additions, since the term has 3 odd coefficients. So to obtain absolute weight 5, we need $s_3 = s_2 + s_1$ to hold. But then the circuit computes $2s_1 + v$ for some $v \in V$, which is clearly cannot have all coefficients odd.

Subsubcase 2.3.4.3. $q \in \{x, z\}$.

Both possibilities are treated symmetrically. For $q = x$, we have that $\{x+y+2z,\ 2y-x\} \subset t(p)$, and for $q = z$, we have $\{2x+y+z,\ 2y-z\} \subset t(p)$. We will follow the latter case.

The first term $t = 2x+y+z$ is prime and requires 3 +-gates to be computed by Claim 1. So by Claim 2, all inputs for all +-gates are prime polynomials. Next, we consider two types of interconnection between +-gates.

First, assume that +-gates are arranged as a depth-2 tree. So, the inputs for the first two gates have prime terms x, x, y, z in some order.

Second, assume that +-gates are arranged into a chain. Due to weight considerations, the first +-gate has two inputs with prime terms of weight 1 (variables). Obviously, with respect to the computation of t, the last +-gate cannot be doubling. The same goes with the second +-gate, since t cannot be represented as $2s+v$ or $3s$, where s has absolute weight 2, and $v \in V$.

In both cases, we obtain that +-gates form a tree with 4 variable inputs. Note that to produce the term $2y-z$ we should have $-z$ as a term at the input of some +-gate. Let v denote the prime term belonging to the same input polynomial. But then the subcircuit computing p produces an illegal term $2y+v$. A contradiction. □