Tropical Kirchhoff's Formula and Postoptimality in Matroid Optimization*

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Abstract

Given an assignment of real weights to the ground elements of a matroid, the min-max weight of a ground element e is the minimum, over all circuits containing e, of the maximum weight of an element in that circuit with the element e removed. We use this concept to answer the following structural questions for the minimum weight basis problem. Which elements are persistent under a given weighting (belong to all or to none of the optimal bases)? What changes of the weights are allowed while preserving optimality of optimal bases? How does the minimum weight of a basis change when the weight of a single ground element is changed, or when a ground element is contracted or deleted? Our answer to this latter question gives the tropical $(\min, +, -)$ analogue of Kirchhoff's arithmetic $(+, \times, /)$ effective conductance formula for electrical networks.

Keywords: Weighted matroid, optimization, sensitivity, persistency, postoptimality

1. Introduction

The *minimum weight basis problem* on a matroid $M = (E, \mathscr{I})$ is, given an assignment $x : E \to \mathbb{R}$ of real weights to the ground elements, to compute the minimum weight $\tau_M(x)$ of a basis, the latter being the sum of weights of the basis elements. Thanks to classical results of Rado [10], Gale [6] and Edmonds [4], the *algorithmic* aspect of this problem is well understood: the minimum weight basis problem on a downward closed set system $\mathscr{I}$ can be solved by the greedy algorithm precisely when the system $\mathscr{I}$ forms the family of independent sets of a matroid.

In this paper, we are interested in *structural* aspects of the minimum weight basis problem. Given a weighting $x : E \to \mathbb{R}$ of ground elements, the following questions naturally arise.

1. How does the optimal value $\tau_M(x)$ change when an element $e \in E$ is contracted or deleted?
2. How does the optimal value $\tau_M(x)$ change when the weight of an element $e \in E$ is changed?
3. By how much can the weight of a single element $e \in E$ be changed without changing optimal bases?
4. What simultaneous changes of the weights preserve optimality of optimal bases?
5. What ground elements belong to all, to none or to some but not to all of the optimal bases?

Question 3 is a special case of Question 4 and was already answered by Tarjan [11] (for graphic matroids) and by Libura [8] (for general matroids) in terms of fundamental circuits and cuts relative to a given optimal basis. Question 5 was answered by Cechlárova and Lacko [3] in terms of the rank function of the underlying matroid. But to our best knowledge, no answers (in any terms) to Questions 1, 2 and 4 were known so far.

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It turns out that all five questions can be answered using the concept of *min-max weight* $x[e]$ of a ground element e, which we define as the minimum, over all circuits C containing e, of the maximum weight of an element in the independent set $C-e$:

$$x[e] := \min_{\substack{C \text{ circuit} \\ e\in C}} \max_{f\in C-e} x(f).$$

Further, we call

$$x\{e\} := \min\{x(e), x[e]\}$$

the *bottleneck weight* of e. The answers to the aforementioned questions 1–5 are given by the corresponding Theorems 1 to 5 in the next section. All necessary matroid concepts are recalled in Section 3.

2. Results

Let $M = (E, \mathscr{I})$ be a *loopless* matroid, that is, no element $e \in E$ belongs to all bases and no singleton set $\{e\}$ is dependent. All our results concern the minimum weight basis problem on M. Given a weighting $x : E \to \mathbb{R}$, a basis B is x-*optimal* (or simply *optimal* if the weighting is clear from the context) if its x-weight $x(B) = \sum_{e\in B} x(e)$ is minimal among all bases. The weight of such a basis, that is, the number $\tau_M(x)$, is the *optimal value* under the weighting x.

Contraction and deletion. Given a ground element $e \in E$, the independent sets of the matroid M/e, obtained by *contracting* the element e, are all sets $I-e$ with $I \in \mathscr{I}$ and $e \in I$, while those of the matroid $M\setminus e$, obtained by *deleting* the element e, are all sets $I \in \mathscr{I}$ with $e \notin I$. Since the matroid M is loopless, each of these two matroids contains at least one nonempty independent set. Note that the set of ground elements of both matroids $M\setminus e$ and M/e is $E-e$. For every basis B of M, either $B-e$ is a basis of M/e (if $e \in B$), or B is a basis of $M\setminus e$ (if $e \notin B$). This gives us a known recursion

$$\tau_M(x) = \min\left\{\tau_{M/e}(x) + x(e),\ \tau_{M\setminus e}(x)\right\}.$$

But what about the opposite direction: if we already know the optimal value $\tau_M(x)$ in the matroid M, what are the optimal values $\tau_{M/e}(x)$ and $\tau_{M\setminus e}(x)$ in the two submatroids M/e and $M\setminus e$? Our main result (Theorem 1) gives the answer.

Theorem 1 (Tropical Kirchhoff's formula)**.** *Let $M = (E, \mathscr{I})$ be a loopless matroid, and $e \in E$ be a ground element. For every weighting $x : E \to \mathbb{R}$, the following equalities hold:*

(a) $\tau_{M/e}(x) = \tau_M(x) - x\{e\}$;
(b) $\tau_{M\setminus e}(x) = \tau_M(x) - x\{e\} + x[e]$.

In particular, (a) and (b) yield the equality $\tau_{M\setminus e}(x) - \tau_{M/e}(x) = x[e]$. Thus, the min-max weight of an element e in the matroid M is determined by the minimum weights of bases in the submatroids $M\setminus e$ and M/e.

Remark 1. In the special case of *graphic* matroids (see Example 1 in Section 3), Theorem 1(a) gives us the tropical $(\min, +, -)$ version of the classical arithmetic $(+, \times, /)$ effective conductance formula for electrical networks proved by Kirchhoff [7] already in 1847. The *spanning tree polynomial* of an undirected connected graph G is $\kappa_G(x) = \sum_T \prod_{e\in T} x_e$, where the sum is over all spanning trees T of G. Kirchhoff's formula (see also [12, Theorem 8] for a detailed exposition) states that, when the edges e of G are interpreted as electrical resistors and their weights x_e as electrical conductances (reciprocals of electrical resistances), then the effective conductance between the endpoints of any edge e is exactly the ratio $\kappa_G(x)/\kappa_{G/e}(x)$, where G/e is the graph obtained from G by contracting the edge e.

In the tropical semifield $(\mathbb{R}, \min, +, -)$ "addition" means taking the minimum, "multiplication" means adding the numbers, and "division" turns into subtraction. In particular, the spanning tree polynomial $\kappa_G(x)$ of a graph G turns into the tropical polynomial $\tau_G(x) = \min_T \sum_{e\in T} x_e$. The function computed by $\tau_G(x)$ is the well-known minimum weight spanning tree problem. The *ratio* $\kappa_G(x)/\kappa_{G/e}(x)$ of polynomials in Kirchhoff's formula turns into the *difference* $\tau_G(x) - \tau_{G/e}(x)$ of their tropical versions. So, a natural question arises: what is the *tropical* analogue of the effective conductance between the endpoints of an edge e? Theorem 1(a) gives the answer (even in general matroids): this is exactly the bottleneck weight $x\{e\} = \min\{x(e), x[e]\}$ of e.

Postoptimality. If we change the weight of a single ground element $e \in E$, what is the optimal value $\tau_M(x')$ under the new weighting x'? We show that the difference (of the new and the old optimal values) is determined by the bottleneck weights of the element e under the old and the new weighting.

Theorem 2 (Postoptimality). *Let $e \in E$ and let $x, x' : E \to \mathbb{R}$ be weightings that differ only in the weights given to the element e. Then*

$$\tau_M(x') - \tau_M(x) = x'\{e\} - x\{e\}.$$

Since the min-max weight $x[e]$ does not depend on the weight of the element e itself, we have $x'[e] = x[e]$ and, hence, $x'\{e\} = \min\{x'(e), x[e]\}$. Thus, when a weighting $x : E \to \mathbb{R}$ and an element $e \in E$ are fixed, and a new weighting x' gives weight $x'(e) = \theta \in \mathbb{R}$ to the element e, then the behavior of the function

$$f(\theta) := \tau_M(x') - \tau_M(x) = x'\{e\} - x\{e\} = \min\{\theta, x[e]\} - \min\{x(e), x[e]\}$$

only depends on whether $\theta \leqslant x[e]$ or not. If $\theta \leqslant x[e]$, then $f(\theta) = \theta - c$ for the constant $c = \min\{x(e), x[e]\}$. If $\theta \geqslant x[e]$, then $f(\theta)$ is constant 0 or constant $x[e] - x(e)$, whichever of these two numbers is larger.

If all weights are nonnegative, and if the new weighting x' gives weight 0 to the element e, then $x'\{e\} = \min\{x'(e), x'[e]\} = \min\{0, x'[e]\} = 0$, and Theorem 2 directly yields the following consequence.

Corollary 1. *If the weights are nonnegative, and if the weight of a single ground element is dropped down to zero, then the minimum weight of a basis decreases by exactly the bottleneck weight of this element.*

Corollary 1 allows us to compute the optimal value $\tau_M(x)$ under an arbitrary nonnegative weighting $x : E \to \mathbb{R}_+$ by computing the bottleneck weights of the elements of *any* (fixed in advance) basis.

Corollary 2. *Let $B = \{e_1, \dots, e_r\}$ be a basis. Given a nonnegative weighting $x : E \to \mathbb{R}_+$, consider the sequence of weightings $x_0, x_1, \dots, x_r$, where $x_0 = x$, and each next weighting x_i is obtained from x by setting the weights of the elements $e_1, \dots, e_i$ to zero. Then*

$$\tau_M(x) = x_0\{e_1\} + x_1\{e_2\} + \cdots + x_{r-1}\{e_r\}.$$

Proof. Corollary 1 gives us the recursion $\tau_M(x_i) = \tau_M(x_{i+1}) + x_i\{e_{i+1}\}$ which rolls out into $\tau_M(x) = \tau_M(x_r) + x_{r-1}\{e_r\} + \cdots + x_1\{e_2\} + x_0\{e_1\}$. Since the weighting x_r gives weight 0 to all elements $e_1, \dots, e_r$ of the basis B, we have $x_r(B) = 0$. Since the weights are nonnegative, the basis B is of minimum weight under the weighting x_r. Hence, $\tau_M(x_r) = x_r(B) = 0$. □

Sensitivity. Given a weighting $x : E \to \mathbb{R}$ the *sensitivity* question is: what changes of the weights preserve optimality of optimal bases. That is, if $x' : E \to \mathbb{R}$ is a new weighting, under what conditions do x-optimal bases remain x'-optimal? In the case when x' only changes the weight of a *single* element $e \in E$, this question was answered by Libura [8] in terms of fundamental circuits and cuts relative to an x-optimal basis (see Remark 7 in Section 7); in the case of graphic matroids (where bases are spanning trees, see Example 1 in Section 3), the same answer was given earlier by Tarjan [11]. We answer this question in terms of the min-max weight $x[e]$ of e. Namely, we associate with every element $e \in E$ its *tolerance* under the (old) weighting:

$$t_x(e) := |x[e] - x(e)|\,.$$

Theorem 3 (Sensitivity, local change). *Let $e \in E$ and let $x, x' : E \to \mathbb{R}$ be weightings that only differ in the weights given to the element e, and B be an x-optimal basis.*

(a) *If $e \in B$, then B is x'-optimal if and only if $x'(e) \leqslant x[e]$.*
(b) *If $e \notin B$, then B is x'-optimal if and only if $x'(e) \geqslant x[e]$.*
(c) *If $|x'(e) - x(e)| \leqslant t_x(e)$, then B is x'-optimal.*

By Theorem 3(c), the weight of a single element e can be changed by $t_x(e)$ while preserving optimality. In contrast, if we allow the weights of two or more elements to be changed, then only changes by at most $\frac{1}{2} t_x(e)$ preserve optimality.

Theorem 4 (Sensitivity, global change). *Let $x : E \to \mathbb{R}$ be a weighting, and B be an x-optimal basis.*

(a) *If a weighting $x' : E \to \mathbb{R}$ satisfies $|x'(e) - x(e)| \leqslant \frac{1}{2} t_x(e)$ for all $e \in E$, then the basis B is x'-optimal.*
(b) *For every $\epsilon > 0$ there is a weighting $x' : E \to \mathbb{R}$ such that $|x'(e) - x(e)| \leqslant \frac{1}{2} t_x(e) + \epsilon$ holds for all elements $e \in E$ but the basis B is not x'-optimal.*

Claim (b) shows that the upper bound in claim (a) is tight.

Persistency. Given a weighting $x : E \to \mathbb{R}$, the set E of ground elements is split into three (not necessarily nonempty) subsets:

$E_{\mathrm{all}}(x)$ = elements belonging to all x-optimal bases;
$E_{\mathrm{none}}(x)$ = elements not belonging to any x-optimal basis;
$E_{\mathrm{some}}(x)$ = elements that belong to some but not to all x-optimal bases.

The *persistency problem* is to determine this partition. Elements $e \in E_{\mathrm{all}}(x) \cup E_{\mathrm{none}}(x)$ are called *persistent*. Knowing which ground elements belong to which of these three subsets may be helpful when constructing an optimal basis. Namely, we can contract all elements of $E_{\mathrm{all}}(x)$ (that is, include them into the solution), remove all elements of $E_{\mathrm{none}}(x)$, and try to extend our partial solution $E_{\mathrm{all}}(x)$ to an optimal basis by only treating the elements of $E_{\mathrm{some}}(x)$.

Cechlárová and Lacko [3] characterized the sets $E_{\mathrm{all}}(x)$ and $E_{\mathrm{none}}(x)$ in terms of the rank function of the underlying matroid: $e \in E_{\mathrm{all}}(x)$ iff removing e from the set of all elements *not heavier* than e decreases the rank of this set, and $e \in E_{\mathrm{none}}(x)$ iff adding e to the set of all elements *lighter* than e leaves the rank of this set unchanged. We characterize these sets in terms of min-max weights of ground elements.

Theorem 5 (Persistency). *Let $x : E \to \mathbb{R}$ be a weighting, and $e \in E$ be a ground element.*

(1) *$e \in E_{\mathrm{all}}(x)$ if and only if $x[e] > x(e)$;*
(2) *$e \in E_{\mathrm{none}}(x)$ if and only if $x[e] < x(e)$;*
(3) *$e \in E_{\mathrm{some}}(x)$ if and only if $x[e] = x(e)$.*
(4) *If all weights are distinct, then $B = \{e \in E : x[e] > x(e)\}$ is the unique optimal basis.*

Organization. In Section 3, we briefly recall main matroid concepts and results used in this paper. Section 4 is devoted to the proof of our main technical tool (Lemma 1). Given Lemma 1, the proofs of Theorems 1 to 5 are fairly simple, and are given in the subsequent Sections 5 to 9.

3. Preliminaries

We use standard matroid terminology as, for example, in Oxley's book [9]. A *matroid* on a finite set E of *ground elements* is a pair $M = (E, \mathscr{I})$, where $\mathscr{I} \subseteq 2^E$ is a nonempty downward closed collection of subsets of E, called *independent sets*, with the *augmentation property*: whenever I and J are independent sets of cardinalities $|I| < |J|$, there is an element $e \in J \setminus I$ such that the set $I + e$ is independent; as customary, we abbreviate $I \cup \{e\}$ to $I + e$ and write $J - e$ for $J \setminus \{e\}$.

Bases and circuits. An independent set is a *basis* if it is contained in no other independent set. The augmentation property implies that all bases have the same cardinality. This property also yields the *basis exchange axiom*: if A and B are bases, then for every element $e \in A$ there is an element $f \in B$ such that $A - e + f$ is a basis. The following two important refinements of the basis exchange axiom are known as the *symmetric basis exchange* and the *bijective basis exchange*.

Proposition 1 (Brualdi [1], Brylawski [2])**.** *Let A and B be bases.*

(a) *For every $e \in A$ there is an $f \in B$ such that both $A - e + f$ and $B - f + e$ are bases.*
(b) *There is a bijection $\phi : A \to B$ such that the set $A - e + \phi(e)$ is a basis for every $e \in A$.*

A subset of E is *dependent* if it is not independent. A *circuit* is a dependent set whose proper subsets are all independent. For a ground element e, an *e-circuit* is a circuit containing e. An element e is a *loop* if the set $\{e\}$ is dependent, and is a *coloop* if e belongs to all bases. To avoid pathological situations, we assume that our matroid is *loopless*: no ground element is a loop or a coloop. We only need this assumption to ensure two properties: every circuit contains at least two elements, and for every ground element e at least one e-circuit exists.

Fundamental paths and cuts. Let B be a basis, and $e \in E$ a ground element. If $e \notin B$, then the set $B + e$ must contain at least one e-circuit, because B is independent but $B + e$ is dependent. An important fact, shown by Brualdi [1, Lemma 1] (see also Oxley [9, Proposition 1.1.4]) and known as the *unique circuit property*, is that the set $B + e$ contains a *unique* circuit C. Since B is independent, this circuit C is an e-circuit (that is, C contains e). This unique circuit $C = C(e, B)$ is known as the *fundamental circuit of e relative to B*. Motivated by graphic matroids (cf. Example 1), we call the independent set

$$\mathrm{Path}(e, B) := C(e, B) - e \subseteq B$$

the *fundamental path of e relative to B*. If $e \in B$ is a basis element, then the set

$$\mathrm{Cut}(e, B) := \{f \in E \setminus B : e \in C(f, B)\}$$

is known as the *fundamental cut of e relative to B*. Note that $e \notin \mathrm{Path}(e, B)$ and $f \notin \mathrm{Cut}(f, B)$. Also note the duality: if $e \notin B$ and $f \in B$, then

$$e \in \mathrm{Cut}(f, B) \text{ if and only if } f \in \mathrm{Path}(e, B).$$

The unique circuit property yields the following equivalent definition of $\mathrm{Cut}(e, B)$ and $\mathrm{Path}(e, B)$.

Proposition 2. *Let B be a basis, and $e \in E$ a ground element.*

(a) *If $e \in B$, then* $\mathrm{Cut}(e, B) = \{f \in E \setminus B : B - e + f$ *is a basis*$\}$.
(b) *If $e \notin B$, then* $\mathrm{Path}(e, B) = \{f \in B : B - f + e$ *is a basis*$\}$.

That is, if $e \in B$, then $\mathrm{Cut}(e,B)$ consists of all elements $f \in E \setminus B$ from outside the basis B that *can replace* e in B. If $e \notin B$, then $\mathrm{Path}(e,B)$ consists of all basis elements $f \in B$ that *can be replaced* by e in B.

Proof. Claim (a) follows from claim (b) and the aforementioned duality. To show claim (b), let $e \notin B$ and $P = \mathrm{Path}(e,B)$; hence, $P \subseteq B$. Take an arbitrary element $f \in B$. If $f \notin P$, then $B - f + e$ cannot be a basis because it contains the circuit $P + e$. If $f \in P$, then $A = B - f + e$ is a basis because f is removed from the unique circuit $P + e$ contained in $B + e$ (the set A is independent and has the same cardinality as B). □

Remark 2. If $e \notin B$, then $\mathrm{Path}(e,B)$ is nonempty, because e is not a loop (the set $\{e\}$ is independent). If $e \in B$, then Proposition 2 implies that the set $\mathrm{Cut}(e,B)$ is also nonempty. Indeed, since e is not a coloop (does not belong to all bases), $e \notin A$ holds for some basis $A \neq B$. By the basis exchange axiom, there is some element $f \in A \setminus B$ such that $B - e + f$ is a basis. By Proposition 2(b), the element f belongs to the set $\mathrm{Cut}(e,B)$; hence, $\mathrm{Cut}(e,B) \neq \emptyset$.

Proposition 3. *Let B be a basis, $e \in B$ and C an e-circuit. Then $(C - e) \cap \mathrm{Cut}(e,B) \neq \emptyset$.*

Proof. Let C be an e-circuit. Since the set $I = C - e$ is independent, it lies in some basis A, and $e \notin A$ holds since $I + e = C$ is already dependent. By Proposition 1(a), there is an $f \in A$ such that both sets $B - e + f$ and $A - f + e$ are bases. By Proposition 2, this is equivalent to $f \in \mathrm{Cut}(e,B)$ and $f \in \mathrm{Path}(e,A)$. Thus, $\mathrm{Cut}(e,B) \cap \mathrm{Path}(e,A) \neq \emptyset$. Since A is a basis, and both circuits C and $\mathrm{Path}(e,A) + e$ lie in $A + e$, the uniqueness of fundamental circuits yields $\mathrm{Path}(e,A) = C - e$. Hence, $\mathrm{Cut}(e,B) \cap (C - e) \neq \emptyset$, as claimed. □

Min-max and bottleneck weights of elements. A *weighting* is an assignment $x : E \to \mathbb{R}$ of real weights to the ground elements. The *weight* of a set $F \subseteq E$ is the sum $x(F) := \sum_{f \in F} x(f)$ of the weights of its elements. The *minimum weight basis problem* on a matroid $M = (E, \mathscr{I})$ is, given a weighting $x : E \to \mathbb{R}$, to determine the minimum weight of a basis:

$$\tau_M(x) := \min_{B \text{ basis}} \sum_{f \in B} x(f).$$

We call the number $\tau_M(x)$ the *optimal value* (under the weighting x), and call a basis B *x-optimal* (or just *optimal*, if the weighting is clear from the context) if $x(B) = \tau_M(x)$ holds, that is, if the basis B is of minimal x-weight.

Remark 3. Note that in the context of the minimum weight basis problem our assumption that no ground element e is a loop or a coloop is quite natural. If e is a loop, then it belongs to *none* of the bases, and the element e contributes nothing to the optimal value $\tau_M(x)$. If e is a coloop, then it belongs to *all* bases, and the contribution $x(e)$ of the element e to the optimal value $\tau_M(x)$ is predetermined.

The *min-max weight* $x[e]$ of an element $e \in E$ under a weighting $x : E \to \mathbb{R}$ is the minimum, over all e-circuits C, of the maximum weight of an element in the (independent) set $C - e$:

$$x[e] := \min_{C \text{ } e\text{-circuit}} \max_{f \in C - e} x(f).$$

Since e is not a loop (the set $\{e\}$ is independent), the set $C - e$ is nonempty for every e-circuit C. Moreover, since e is not a coloop, at least one e-circuit C exists. So, the min-max weight is well-defined. Note that the min-max weight $x[e]$ of e does not depend on the weight $x(e)$ of the element e itself: it only depends on the weights of the remaining elements. So, all three relations $x[e] < x(e)$, $x[e] = x(e)$ and $x[e] > x(e)$ are possible. We call

$$x\{e\} := \min\{x(e), x[e]\}$$

the *bottleneck weight* of e.

Remark 4. Thanks to a classical "bottleneck extrema" result of Edmonds and Fulkerson [5], the min-max weight $x[e]$ of a ground element $e \in E$ has an equivalent definition as the "max-min" weight in terms of cocircuits:

$$x[e] = \max_{D\ e\text{-cocircuit}} \min_{f \in D-e} x(f). \tag{1}$$

A set $D \subseteq E$ is a *cocircuit* if it intersects every basis, and no proper subset has this property. In other words, cocircuits in a matroid M are circuits of the dual matroid M^*; bases in M^* are complements of the bases of M. An *e-cocircuit* is a cocircuit containing the element e. To obtain Eq. (1) from the main theorem of Edmonds and Fulkerson [5], one has only to verify that the family $\mathscr{D}_e = \{D - e : D \text{ is an } e\text{-cocircuit}\}$ is a blocking family for the family $\mathscr{C}_e = \{C - e : C \text{ is an } e\text{-circuit}\}$, i.e., that members of $\mathscr{D}_e$ are minimal (under set inclusion) subsets of E intersecting all members of $\mathscr{C}_e$. This follows from a well-known fact that $|C \cap D| \neq 1$ holds for every circuit C and every cocircuit D (see, for example [9, Proposition 2.1.11]). In this paper, we will not use this equivalent "max-min" definition Eq. (1) of $x[e]$: we only mention it for interested readers.

Example 1. The *graphic matroid* (or *cycle matroid*) $M(G)$ determined by an undirected connected graph $G = (V, E)$ has edges of G as its ground elements. Independent sets are forests, bases are spanning trees of G, and circuits are simple cycles in G. A loop is an edge with identical endpoints, and a coloop is an edge e whose deletion destroys the connectivity of G (such edges are also called *bridges*). The min-max weight $x[e]$ of an edge $e \in E$ is the minimum, over all simple paths in G of length at least two between the endpoints of e, of the maximum weight of an edge in this path. The bottleneck weight $x\{e\} = \min\{x(e), x[e]\}$ of an edge e is also known as the *bottleneck distance* between the endpoints of e. If T is a spanning tree of G and $e \notin T$ is an edge of G, then the set $\mathrm{Path}(e, T)$ consists of all edges of the unique path in T between the endpoints of e. If $f \in T$, then $\mathrm{Cut}(f, T)$ consists of all edges of G lying between the two trees of $T - f$, except the edge f itself.

For the rest of the paper, let $M = (E, \mathscr{I})$ be an arbitrary loopless matroid.

4. Main lemma

The following lemma (illustrated in Figure 1) is our main technical tool.

Lemma 1 (Main lemma). *Let $x : E \to \mathbb{R}$ be a weighting, $e \in E$ a ground element, and B an x-optimal basis.*

(a) *If $e \in B$, then*

- $x(e) \leqslant x[e] =$ *the minimum weight of an element in* $\mathrm{Cut}(e, B)$*;*
- *the minimum weight of a basis avoiding the element e is $x(B) - x(e) + x[e]$.*

(b) *If $e \notin B$, then*

- $x(e) \geqslant x[e] =$ *the maximum weight of an element in* $\mathrm{Path}(e, B)$*;*
- *the minimum weight of a basis containing the element e is $x(B) - x[e] + x(e)$.*

Given a weighting $x : E \to \mathbb{R}$ and an element $e \in E$, by an e-circuit *witnessing* the min-max weight $x[e]$ of an element e we will mean an e-circuit C on which the min-max weight of the element e is achieved, that is, for which $x[e]$ is the maximum weight $x(f)$ of an element $f \in C - e$.

Proof of Lemma 1(a). Let $x : E \to \mathbb{R}$ be a weighting, $e \in E$ a ground element, and B an optimal basis. Assume that $e \in B$, and let c_0 be a lightest element in $\mathrm{Cut}(e, B)$. Our goal is to show that

(i) $x(e) \leqslant x[e] = x(c_0)$, and

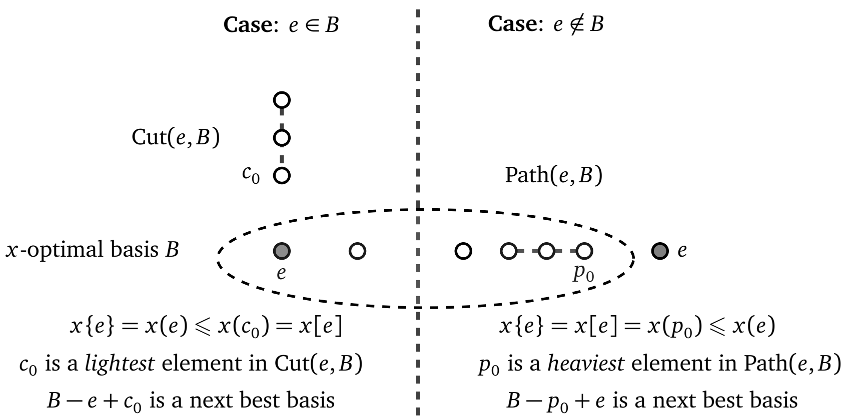

Figure 1: A schematic summary of Lemma 1. A *lightest* (resp., *heaviest*) element of a set $S \subseteq E$ is an element of S of the minimum (resp., maximum) weight (there may be several such elements). If $e \in B$, then a *next best* basis is a basis which has the minimal weight under all bases *avoiding* the element e. If $e \notin B$, then a *next best* basis is a basis which has the minimal weight under all bases *containing* the element e.

(ii) the set $B - e + c_0$ is a lightest basis among all bases avoiding the element e.

(i) To show the inequality $x(e) \leqslant x(c_0)$, suppose for a contradiction that $x(c_0) < x(e)$ holds. Since $c_0 \in \mathrm{Cut}(e,B)$, Proposition 2 implies that the set $B - e + c_0$ is a basis. But then its weight is smaller than that of B, contradicting the optimality of B.

To show the inequality $x[e] \geqslant x(c_0)$, let C be an e-circuit witnessing the min-max weight $x[e]$ of the element e. Hence, $x[e] = x(f_0)$, where f_0 is a *heaviest* element of $C - e$. By Proposition 3, there is an element g in the intersection $(C - e) \cap \mathrm{Cut}(e,B)$. Then $x(g) \leqslant x(f_0) = x[e]$ because $g \in C - e$ and f_0 is a heaviest element of $C - e$, and $x(g) \geqslant x(c_0)$ because $g \in \mathrm{Cut}(e,B)$ and c_0 is a lightest element of Cut(e,B). Hence, $x[e] \geqslant x(c_0)$.

To show the opposite inequality $x[e] \leqslant x(c_0)$, consider the fundamental circuit $C = \mathrm{Path}(c_0,B)+c_0$ of the element c_0 relative to the basis B. Since $c_0 \in \mathrm{Cut}(e,B)$, we have $e \in \mathrm{Path}(c_0,B)$. Thus, both e and c_0 belong to the same circuit C. Let p_0 be a *heaviest* element in $C - e = \mathrm{Path}(c_0,B) + c_0 - e$. Since in the definition of the min-max weight $x[e]$ we take the *minimum* over all circuits containing e, we have $x[e] \leqslant x(p_0)$. So, it remains to show that $x(p_0) \leqslant x(c_0)$ holds. Suppose for a contradiction that we have a strict inequality $x(p_0) > x(c_0)$. Then (clearly) $p_0 \neq c_0$ and, hence, $p_0 \in \mathrm{Path}(c_0,B)$. By Proposition 2, the set $A = B - p_0 + c_0$ is a basis. But the weight of this basis is $x(A) = x(B) - x(p_0) + x(c_0) < x(B)$, contradicting the optimality of the basis B. Thus, $x[e] \leqslant x(c_0)$, as desired.

(ii) Let $\mathscr{B}_0$ be the family of all bases *avoiding* the element e; hence, $B \notin \mathscr{B}_0$. Since c_0 belongs to Cut(e,B), Proposition 2 implies that the set $B - e + c_0$ is a basis, and this basis belongs to $\mathscr{B}_0$. Our goal is to show that this specific basis has the smallest weight among all bases in $\mathscr{B}_0$.

So, let B' be a lightest basis in $\mathscr{B}_0$; hence, $e \in B \setminus B'$. By Proposition 1(b), there is a bijection $\phi : B \to B'$ such that the set $B - f + \phi(f)$ is a basis for every element $f \in B$. Since the basis B is optimal, this yields $x(f) \leqslant x(\phi(f))$ for every $f \in B$. Consider the basis $A = B - e + c$ where $c := \phi(e) \in B'$. Its weight is

$$x(A) = x(c) + \sum_{f \in B-e} x(f) \leqslant x(c) + \sum_{f \in B-e} x(\phi(f)) = x(c) + x(B' - c) = x(B').$$

Since A is a basis, Proposition 2 implies that $c \in \mathrm{Cut}(e,B)$. Since both elements c and c_0 belong to Cut(e,B), and since c_0 is a lightest element of Cut(e,B), we have $x(c_0) \leqslant x(c)$. So $x(B - e + c_0) \leqslant x(B - e + c) = x(A) \leqslant x(B')$, meaning that $B - e + c_0$ is a lightest basis in $\mathscr{B}_0$, as claimed. □

Remark 5. The inequality $x[e] \geqslant x(c_0)$ in part (i) also follows from the equivalent definition Eq. (1) of the min-max weight $x[e]$ (see Remark 4). For this, it is enough to verify that the set $\mathrm{Cut}(e,B)+e$ is a cocircuit. The set $\mathrm{Cut}(e,B)$ intersects every basis A with $e \notin A$: by the basis exchange axiom, $B-e+a$ is a basis for some $a \in A$; hence, $a \in \mathrm{Cut}(e,B)$. Moreover, no proper subset of $\mathrm{Cut}(e,B)$ has this property: for every $f \in \mathrm{Cut}(e,B)$, the set $\mathrm{Cut}(e,B)-f$ does not intersect the basis $B-e+f$.

Proof of Lemma 1(b). Let $x: E \to \mathbb{R}$ be a weighting, $e \in E$ a ground element, and B an optimal basis. Assume that $e \notin B$, and let p_0 be a heaviest element in $\mathrm{Path}(e,B)$. Our goal is to show that

(i) $x(e) \geqslant x[e] = x(p_0)$, and

(ii) the set $B-p_0+e$ is a lightest basis among all bases containing the element e.

(i) To show the inequality $x(e) \geqslant x(p_0)$, suppose for a contradiction that $x(e) < x(p_0)$. Since $p_0 \in \mathrm{Path}(e,B)$, Proposition 2 implies that the set $B-p_0+e$ is a basis. But then its weight is smaller than that of B, contradicting the optimality of B.

The inequality $x[e] \leqslant x(p_0)$ holds because $\mathrm{Path}(e,B)+e$ is an e-circuit, and $x[e]$ takes the *minimum* (of the maximum weights) over *all* e-circuits. To show the opposite inequality $x[e] \geqslant x(p_0)$, suppose for a contradiction that we have a strict inequality $x[e] < x(p_0)$, and let C be an e-circuit witnessing $x[e]$. Hence, $x(f) < x(p_0)$ holds for all $f \in C-e$. Since $p_0 \in \mathrm{Path}(e,B)$, Proposition 2 implies that $A = B-p_0+e$ is also a basis. Since $e \in A$ and C is an e-circuit, Proposition 3 implies that some element $f_0 \in C-e$ belongs to $\mathrm{Cut}(e,A)$. So, by Proposition 2, the set $A' = A-e+f_0 = B-p_0+f_0$ is a basis. But since $x(f_0) < x(p_0)$, we have $x(A') < x(B)$, contradicting the optimality of B.

(ii) Let $\mathscr{B}_1$ be the family of all bases *containing* the element e; hence, $B \notin \mathscr{B}_1$. Since p_0 belongs to $\mathrm{Path}(e,B)$, Proposition 2 implies that the set $B-p_0+e$ is a basis, and this basis belongs to $\mathscr{B}_1$. Our goal is to show that this specific basis has the smallest weight among all bases in $\mathscr{B}_1$.

So, let B' be a lightest basis in $\mathscr{B}_1$; hence, $e \in B' \setminus B$. By Proposition 1(b), there is a bijection $\phi: B' \to B$ such that the set $B-\phi(f)+f$ is a basis for every element $f \in B'$. Since the basis B is optimal, this yields $x(\phi(f)) \leqslant x(f)$ for every $f \in B'$. Consider the basis $A = B-p+e$ where $p := \phi(e) \in B$. Its weight is

$$x(A) = x(e) + \sum_{f \in B'-e} x(\phi(f)) \leqslant x(e) + \sum_{f \in B'-e} x(f) = x(e) + x(B'-e) = x(B').$$

Since A is a basis, Proposition 2 implies that $p \in \mathrm{Path}(e,B)$. Since both elements p and p_0 belong to $\mathrm{Path}(e,B)$, and since p_0 is a *heaviest* element of $\mathrm{Path}(e,B)$, we have $x(p_0) \geqslant x(p)$. So $x(B-p_0+e) \leqslant x(B-p+e) = x(A) \leqslant x(B')$, meaning hat $B-p_0+e$ is a lightest basis in $\mathscr{B}_1$, as claimed. □

Remark 6 (From optimal bases to bottleneck weights). Having an optimal basis B, we can determine the bottleneck weight $x\{e\} = \min\{x(e), x[e]\}$ of any ground element $e \in E$ from the weights of elements of B: if $e \in B$, then $x\{e\} = x(e)$ (by Lemma 1(a)), and if $e \notin B$, then $x\{e\} = x[e]$ is the weight of a heaviest element in $\mathrm{Path}(e,B)$ (by Lemma 1(b)). The following proposition shows how to find such a heaviest element in $\mathrm{Path}(e,B)$.

Proposition 4. *Let $x: E \to \mathbb{R}$ be a weighting and $B = \{f_1,\ldots,f_r\}$ be an optimal basis with $x(f_1) \leqslant \ldots \leqslant x(f_r)$. If $e \notin B$, then $x[e] = x(f_i)$, where i is the smallest index for which the set $\{f_1,\ldots,f_i,e\}$ is dependent.*

Proof. For $j = 1,\ldots,r$, let $B_j = \{f_1,\ldots,f_j\}$ be the set of the j lightest elements of B, and let $B_0 = \emptyset$. The set $B_0+e = \{e\}$ is independent because e is not a loop, and the set $B_r+e = B+e$ is dependent, because B is a basis and $e \notin B$. So, there is a unique index $i \in \{1,\ldots,r\}$ such that the set $B_{i-1}+e$ is independent but B_i+e is dependent. Our goal is to show that $x[e] = x(f_i)$ holds for this i.

Since B_i is independent but B_i+e is dependent, the set B_i+e contains an e-circuit C. Since $C \subseteq B_i+e \subseteq B+e$, the uniqueness of fundamental circuits yields $C = C(e,B)$; hence, $\mathrm{Path}(e,B) =$

$C(e,B)-e \subseteq B_i$. Since the set $B_{i-1}+e$ is independent, the last element f_i of B_i must be contained in $\text{Path}(e,B)$. Since $\text{Path}(e,B) \subseteq B_i$ and since f_i is a heaviest element of B_i, f_i is also a heaviest element of $\text{Path}(e,B)$. Thus, Lemma 1(b) gives $x[e]=x(f_i)$. □

5. Proof of Theorem 1

Let, as before, $M=(E,\mathscr{I})$ be a loopless matroid, and $e \in E$ be a ground element. Recall that the independent sets of the matroid M/e, obtained by *contracting* the element e, are all sets $I-e$ with $I \in \mathscr{I}$ and $e \in I$, while those of the matroid $M \setminus e$, obtained by *deleting* the element e, are all sets $I \in \mathscr{I}$ with $e \notin I$. Our goal is to show that, for every weighting $x : E \to \mathbb{R}$, the following equalities hold:

(a) $\tau_{M/e}(x)=\tau_M(x)-x\{e\}$;

(b) $\tau_{M\setminus e}(x)=\tau_M(x)-x\{e\}+x[e]$.

Take an arbitrary optimal basis B of M; hence, $\tau_M(x)=x(B)$. In the proof of both equalities (a) and (b), we distinguish two cases depending on whether our element e belongs to B or not.

(a) If $e \in B$, then Lemma 1(a) yields $x\{e\}=x(e)$ and, since then $B-e$ is an optimal basis of M/e, we obtain $\tau_{M/e}(x)=x(B)-x(e)=\tau_M(x)-x\{e\}$. If $e \notin B$, then consider a basis A of minimum weight among all bases of M containing the element e. By Lemma 1(b), we have $x\{e\}=x[e]$ and $x(A)=x(B)-x[e]+x(e)$. Since then $A-e$ is an optimal basis of M/e, we obtain $\tau_{M/e}(x)=x(A-e)=x(B)-x[e]=\tau_M(x)-x\{e\}$.

(b) If $e \notin B$, then Lemma 1(b) yields $x\{e\}=x[e]$ and, since then B is also an optimal basis of $M\setminus e$, we obtain $\tau_{M\setminus e}(x)=x(B)=\tau_M(x)=\tau_M(x)-x\{e\}+x[e]$. If $e \in B$, then consider a basis A of minimum weight among all bases of M avoiding the element e. By Lemma 1(a), we have $x\{e\}=x(e)$ and $x(A)=x(B)-x(e)+x[e]$. Since A is an optimal basis of $M\setminus e$, we obtain $\tau_{M\setminus e}(x)=x(A)=x(B)-x(e)+x[e]=\tau_M(x)-x\{e\}+x[e]$. □

6. Proof of Theorem 2

Let $e \in E$ and let $x,x' : E \to \mathbb{R}$ be weightings that differ only in the weights given to the element e. Our goal is to show the equality $\tau_M(x')-\tau_M(x)=x'\{e\}-x\{e\}$.

Recall that the independent sets of the matroid M/e are all sets $I-e$ with $I \in \mathscr{I}$ and $e \in I$. By Theorem 1(a), the equality $\tau_{M/e}(z)=\tau_M(z)-z\{e\}$ holds for every weighting $z : E \to \mathbb{R}$. Since the weighting x' does not change the weight of elements in $E-e$, we have $\tau_{M/e}(x')=\tau_{M/e}(x)$. So, Theorem 1(a) yields

$$\tau_M(x')-x'\{e\}=\tau_{M/e}(x')=\tau_{M/e}(x)=\tau_M(x)-x\{e\},$$

from which $\tau_M(x')-\tau_M(x)=x'\{e\}-x\{e\}$ follows. □

7. Proof of Theorem 3

Fix a ground element $e \in E$, and let $x,x' : E \to \mathbb{R}$ be weightings that only differ in the weights given to e. Since the min-max weight of e only depends on the weights of the elements in $E-e$, and since the weighting x' leaves these weights unchanged, we have $x'[e]=x[e]$, that is, the min-max weight of the element e does not change. Thus, the bottleneck weight of e under the new weighting x' is $x'\{e\}=\min\{x'(e),x[e]\}$. Recall that the *tolerance* of the element e under the weighting x is $t_x(e)=|x[e]-x(e)|$.

Let B be an x-optimal basis. Our goal is to prove the following three assertions.

(a) If $e \in B$, then B is x'-optimal if and only if $x'(e) \leqslant x[e]$.

(b) If $e \notin B$, then B is x'-optimal if and only if $x'(e) \geqslant x[e]$.

(c) If $|x'(e)-x(e)| \leqslant t_x(e)$, then B is x'-optimal.

Proof. (a) Let $e \in B$. Then $x'(B)=x(B)+x'(e)-x(e)$ and, by Lemma 1(a), $x\{e\}=x(e)$. Theorem 2 yields $\tau_M(x')=\tau_M(x)+x'\{e\}-x\{e\}=x(B)+\min\{x'(e),x[e]\}-x(e)$. The basis B is x'-optimal iff $x'(B)=\tau_M(x')$, which happens precisely when $\min\{x'(e),x[e]\}=x'(e)$, that is, when $x'(e)\leqslant x[e]$.

(b) Let $e \notin B$. Then $x'(B)=x(B)$ and, by Lemma 1(b), $x\{e\}=x[e]$. Theorem 2 yields $\tau_M(x')=\tau_M(x)+x'\{e\}-x\{e\}=x(B)+\min\{x'(e),x[e]\}-x[e]$. The basis B is x'-optimal iff $x'(B)=\tau_M(x')$, which happens precisely when $\min\{x'(e),x[e]\}=x[e]$, that is, when $x'(e)\geqslant x[e]$.

(c) Assume $|x'(e)-x(e)| \leqslant t_x(e)$, i.e., $x(e)-t_x(e)\leqslant x'(e)\leqslant x(e)+t_x(e)$. If $e\in B$, then Lemma 1(a) implies $x(e)\leqslant x[e]$ and, hence, $t_x(e)=x[e]-x(e)$. Thus, $x'(e)\leqslant x(e)+t_x(e)=x[e]$, and claim (a) ensures that the basis B is x'-optimal. If $e\notin B$, then Lemma 1(b) implies $x(e)\geqslant x[e]$ and, hence, $t_x(e)=x(e)-x[e]$. Thus, $x'(e)\geqslant x(e)-t_x(e)=x[e]$, and claim (b) ensures that the basis B is x'-optimal. □

Remark 7. Given Lemma 1, claims (a) and (b) of Theorem 3 also follow from a result of Libura [8, Lemma 4] stating that B is x'-optimal iff $x'(e)\leqslant x(c_0)$ holds for a lightest element c_0 of $\mathrm{Cut}(e,B)$ (when $e\in B$) or $x'(e)\geqslant x(p_0)$ holds for a heaviest element p_0 in $\mathrm{Path}(e,B)$ (when $e\notin B$). By Lemma 1, $x[e]=x(c_0)$ (when $e\in B$) and $x[e]=x(p_0)$ (when $e\notin B$).

8. Proof of Theorem 4

We will need the following simple fact.

Proposition 5. *Let $x : E\to\mathbb{R}$ be a weighting, and B a basis. If B is not x-optimal, then $x(e)>x(f)$ holds for some elements $e\in B$ and $f\in \mathrm{Cut}(e,B)$.*

Proof. Let A be an x-optimal basis; hence, $x(B)>x(A)$. By Proposition 1(b), there is a bijection $\phi : B\to A$ such that the set $B-e+\phi(e)$ is a basis for every $e\in B$. Hence, by Proposition 2, $\phi(e)\in\mathrm{Cut}(e,B)$ holds for every $e\in B\setminus A$. Finally, since $\sum_{e\in B}x(e)=x(B)>x(A)=\sum_{e\in B}x(\phi(e))$, a strict inequality $x(e)>x(\phi(e))$ must hold for at least one element $e\in B$. □

Proof of Theorem 4(a). Let $x : E\to\mathbb{R}$ be a weighting, and let $x' : E\to\mathbb{R}$ be a weighting satisfying $|x'(e)-x(e)|\leqslant \frac{1}{2}t_x(e)$ for all $e\in E$. Our goal is to show that then every x-optimal basis is also x'-optimal.

Assume to the contrary that some x-optimal basis B is not x'-optimal. Then, by Proposition 5, $x'(f)<x'(e)$ holds for some elements $e\in B$ and $f\in\mathrm{Cut}(e,B)$; hence, we also have $e\in\mathrm{Path}(f,B)$. Since $e\in B$ and $f\in\mathrm{Cut}(e,B)$, Lemma 1(a) yields $x(e)\leqslant x[e]\leqslant x(f)$. Since $f\notin B$ and $e\in\mathrm{Path}(f,B)$, Lemma 1(b) yields $x(f)\geqslant x[f]\geqslant x(e)$. In particular, $t_x(e)=x[e]-x(e)$ and $t_x(f)=x(f)-x[f]$. Putting everything together, we get

$$\frac{x[e]+x(e)}{2}\leqslant\frac{x(f)+x[f]}{2}=x(f)-\frac{t_x(f)}{2}\leqslant x'(f)<x'(e)\leqslant x(e)+\frac{t_x(e)}{2}=\frac{x(e)+x[e]}{2},$$

a contradiction. □

Proof of Theorem 4(b). Let $\epsilon>0$ and let $x : E\to\mathbb{R}$ be a weighting. Take an arbitrary x-optimal basis B. Our goal is to show that there is a weighting $x' : E\to\mathbb{R}$ such that $|x'(e)-x(e)|\leqslant\frac{1}{2}t_x(e)+\epsilon$ holds for all elements $e\in E$ but the basis B is not x'-optimal.

Consider all pairs (e,f) such that $e\in B$ and $f\in\mathrm{Cut}(e,B)$; hence, $e\in\mathrm{Path}(f,B)$. Since the basis B is x-optimal, Proposition 2 implies that $x(f)\geqslant x(e)$ holds for every such pair. So, let (e,f) be a pair for which the difference $x(f)-x(e)$ is smallest possible. Then f is a lightest element in $\mathrm{Cut}(e,B)$ and e is a heaviest element in $\mathrm{Path}(f,B)$. By Lemma 1, $x(e)\leqslant x[e]=x(f)$ and $x(f)\geqslant x[f]=x(e)$.

Hence, $t_x(e) = x[e] - x(e) = x(f) - x(e) = x(f) - x[f] = t_x(f)$, that is, both elements e and f have the same tolerance $t := t_x(e) = t_x(f)$ under the weighting x.

Now, let $x' : E \to \mathbb{R}$ be the weighting with $x'(e) := x(e) + \frac{1}{2}t + \epsilon$, $x'(f) := x(f) - \frac{1}{2}t - \epsilon$, and $x'(g) := x(g)$ for all other elements g. So, $|x'(g) - x(g)| = \frac{1}{2}t_x(g) + \epsilon$ for $g \in \{e, f\}$, and $|x'(g) - x(g)| = 0 < \frac{1}{2}t_x(g) + \epsilon$ for all $g \notin \{e, f\}$. Then $x'(e) - x'(f) = x(e) - x(f) + t + 2\epsilon = 2\epsilon > 0$, and $x'(f) < x'(e)$ implies that the basis $B - e + f$ has smaller x'-weight than B, so B cannot be x'-optimal. □

9. Proof of Theorem 5

Every weighting $x : E \to \mathbb{R}$ yields the partition $E = E_{\mathrm{all}}(x) \cup E_{\mathrm{none}}(x) \cup E_{\mathrm{some}}(x)$ of ground elements into three (not necessarily nonempty) subsets $E_{\mathrm{all}}(x)$ (elements belonging to all x-optimal bases), $E_{\mathrm{none}}(x)$ (elements not belonging to any x-optimal basis), and $E_{\mathrm{some}}(x)$ (elements belonging to some but not to all x-optimal bases). Our goal is to prove the following claims:

(1) $e \in E_{\mathrm{all}}(x)$ if and only if $x[e] > x(e)$;

(2) $e \in E_{\mathrm{none}}(x)$ if and only if $x[e] < x(e)$;

(3) $e \in E_{\mathrm{some}}(x)$ if and only if $x[e] = x(e)$.

(4) If all weights are distinct, then $B = \{e \in E : x[e] > x(e)\}$ is the unique optimal basis.

Proof. (1) To show the direction ($\Rightarrow$), let $e \in E_{\mathrm{all}}(x)$ and take any optimal basis B; hence, $e \in B$. By Lemma 1(a), we then have $x(e) \leqslant x[e] = x(c_0)$, where c_0 is a lightest element in $\mathrm{Cut}(e, B)$. By Proposition 2, the set $A = B - e + c_0$ is a basis. If the equality $x(e) = x[e]$ held, then this basis would be optimal, too. But $e \notin A$, a contradiction with $e \in E_{\mathrm{all}}(x)$. Hence $x[e] > x(e)$ holds. The opposite direction ($\Leftarrow$) follows directly from Lemma 1(b): if the element e is avoided by some optimal basis, then $x[e] \leqslant x(e)$ holds.

(2) The proof of this claim is similar. To show the direction ($\Rightarrow$), let $e \in E_{\mathrm{none}}(x)$ and take any optimal basis B; hence, $e \notin B$. By Lemma 1(b), we then have $x(e) \geqslant x[e] = x(p_0)$, where $p_0 \in B$ is a heaviest element in $\mathrm{Path}(e, B)$. By Proposition 2, the set $A = B - p_0 + e$ is a basis. If the equality $x(e) = x[e]$ held, then this basis would be optimal, too. But $e \in A$, a contradiction with $e \in E_{\mathrm{none}}(x)$. Hence, $x[e] < x(e)$ holds. The opposite direction ($\Leftarrow$) in (2) follows directly from Lemma 1(a): if the element e is contained in some optimal basis, then $x[e] \geqslant x(e)$ holds.

(3) Follows directly from claims (1) and (2).

(4) Assume that all weights are distinct. Then the optimal basis B is *unique*: if there were two distinct optimal bases, then (by the basis exchange axiom) a heaviest element, lying in one basis but not in the other, could be replaced by a (strictly) lighter element of the other basis, contradicting the optimality of the former basis. Since the basis B is unique, we have $B = E_{\mathrm{all}}(x)$ and, by (1), $B = \{e \in E : x[e] > x(e)\}$, as claimed. □

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