TOPIC 8
More on Rank Arguments
(Related sections: 15.2, 16)

Summary: A simple rank lower bounds argument for monotone formulas and monotone span programs. Proof that no self-avoiding family of subsets of an n-element set can have more than $n^{O(\log n)}$ members.

We already know (see Theorem 16.8 in the book) that if the family $\mathcal{A}$ of minterms of a monotone boolean function f is self-avoiding, then f cannot be computed by a monotone span program of size $|\mathcal{A}| - 1$. In this lecture we will discuss a recent observation, made by Pudlák, that this (and even more general) result can be obtained by a direct and very simple rank argument.

So far, it was not known whether self-avoiding families can have exponentially many members (see Problem 1 on p. 216 of the book). Combining Pudlák's argument with some known results in communication complexity, Gál has recently shown that no self-avoiding family of subsets of an n-element set can have more than $n^{O(\log n)}$ members.

8.1. Locally intersecting families are hard to separate

Let $f(x_1, \ldots, x_n)$ be a monotone boolean function. Recall that a *minterm* (resp. *maxterm*) of f is a minimal subset $A \subseteq \{1, \ldots, n\}$ such that the constant 1 (resp. constant 0) to all variables x_i with $i \in A$ forces the function to output 1 (resp. 0), independent of the values of the remaining variables. Hence, if $\mathcal{A}$ is a subset of minterms and $\mathcal{B}$ a subset of maxterms of f, then the pair $\mathcal{A}, \mathcal{B}$ is *cross-intersecting*, that is $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Say that the pair $\mathcal{A}, \mathcal{B}$ is *locally intersecting* if we can divide each set $B \in \mathcal{B}$ into two parts B_0 and B_1 so that every $A \in \mathcal{A}$ has a nonempty intersection with exactly one of these parts.

Given a locally intersecting pair $\mathcal{A}, \mathcal{B}$, define its *disjointness matrix* to be an $|\mathcal{A}|$ by $|\mathcal{B}|$ matrix R, with its rows indexed by sets $A \in \mathcal{A}$ and its columns indexed by sets $B \in \mathcal{B}$, such that the entry $R_{A,B}$ is defined by

$$R_{A,B} = \begin{cases} 0 & \text{if } A \cap B_0 \neq \emptyset, \\ 1 & \text{if } A \cap B_1 \neq \emptyset. \end{cases}$$

Recall that a monotone span program (over some field $\mathbb{F}$) is given by a matrix M over $\mathbb{F}$ with its rows labeled by variables $x_1, \ldots, x_n$. Such a matrix *accepts* a subset $A \subseteq \{1, \ldots, n\}$ if and only if the rows of M, labeled by variables x_i with $i \in A$, span the all-1 vector (or any other fixed in advance vector). A span program *separates* $\mathcal{A}, \mathcal{B}$ if it accepts all members of $\mathcal{A}$ and rejects the complements $\overline{B}$ of all members $B \in \mathcal{B}$.

Theorem 8.1 (Pudlák 2002). *If the pair $\mathcal{A}, \mathcal{B}$ is locally intersecting, then any monotone span program over $\mathbb{F}$ separating this pair has size at least* $\mathrm{rank}_{\mathbb{F}}(R)$.

Proof. Let M be a monotone span program separating $\mathcal{A}, \mathcal{B}$. Let d be the number of rows in M. The idea is to show that the intersection matrix R of $\mathcal{A}, \mathcal{B}$ is a matrix of scalar products of vectors of dimension at most d; this yields $\mathrm{rank}_{\mathbb{F}}(R) \leq d$ (see Exercise 8.2).

For every $A \in \mathcal{A}$, let $v_A \in \mathbb{F}^d$ be a vector witnessing the fact that A must be accepted, which means that $v_A \cdot M = v_0$, where v_0 is the all-1 vector and v_A is a vector which is nonzero only in coordinates corresponding to elements of A. Let $B \in \mathcal{B}$. Since $\overline{B}$ cannot be accepted, no linear combination of the rows of $M_{\overline{B}}$ can give v_0. Hence (see Exercise 8.3) for each $B \in \mathcal{B}$ there is a vector u_B such that $\langle v_0, u_B \rangle = 1$ and $M_{\overline{B}} \cdot u_B = 0$ (the all-0 vector). Let $w_B \in \mathbb{F}^d$ be the vector obtained from the vector $M \cdot u_B$ by replacing to 0 all its elements, corresponding to the rows labeled by elements of B_0; note that $w_B(i) \neq 0$ only if $i \in B_1$: the elements $w_B(i)$ with $i \in \overline{B}$ are zero because $M_{\overline{B}} \cdot u_B = 0$, and the elements $w_B(i)$ with $i \in B_0$ are zero by the definition of w_B. We claim that

$$R_{A,B} = \langle v_A, w_B \rangle.$$

Indeed, if $A \cap B_0 \neq \emptyset$, then $A \cap B_1 = \emptyset$, and hence, the vectors v_A and w_B have no element on which they both are nonzero; hence, in this case $\langle v_A, w_B \rangle = 0$. If $A \cap B_1 \neq \emptyset$, then $A \cap B_0 = \emptyset$, and hence, in this case, $\langle v_A, w_B \rangle = \langle v_A, M \cdot u_B \rangle$, implying that

$$\langle v_A, w_B \rangle = \langle v_A, M \cdot u_B \rangle = \langle v_A \cdot M, u_B \rangle = \langle v_0, u_B \rangle = 1.$$

This shows that R is a matrix of scalar products of vectors of dimension d, implying that $\mathrm{rank}_{\mathbb{F}}(R) \leq d$. □

Ex. 8.2. Let $M : X \times Y \to \mathbb{F}$ be an $n \times m$ matrix with rows labeled by elements of X and columns labeled by elements of Y. Suppose that there exists a sequence of functions $a(x) = (a_1(x), \ldots, a_d(x))$ from X to $\mathbb{F}$ and a sequence of functions $b(y) = (b_1(y), \ldots, b_d(y))$ from Y to $\mathbb{F}$ such that

$$M(x, y) = \langle a(x), b(y) \rangle = \sum_{i=1}^{d} a_i(x) b_i(y).$$

Show that then $\mathrm{rank}_{\mathbb{F}}(M) \leq d$. *Hint*: Let M_i be the matrix defined by $M_i(x, y) := a_i(x) b_i(y)$. Note that the x-th row of M_i is $a_i(x)$ times one and the same vector r whose y-th entry is $b_i(y)$. Use the sub-additivity of rank (see (14.1) in the book)

Ex. 8.3. (Extension of Ex. 16.1 from the book to arbitrary fields.) Let $A \subseteq \mathbb{F}^n$ and 1^n be the all-1 vector in $\mathbb{F}^n$. Show that $1^n \in \mathrm{span}(A) \iff A \cdot u = 0$ for some $u \in \mathbb{F}^n$ such that $\langle 1^n, u \rangle = 1$. *Hint*: Let $E := \{u \in \mathbb{F}^n : \langle 1^n, u \rangle \neq 1\}$ be the set of all "even" vectors. Then $1^n \in \mathrm{span}(A)$ iff $\mathrm{span}(A)^{\perp} \subseteq E$.

Since, as shown in the book (see Sect. 16.2), monotone span programs are not weaker than switching networks (and hence, are not weaker than monotone formulas) Theorem 8.1 directly yields the following

Theorem 8.4. *If the pair $\mathcal{A}, \mathcal{B}$ is locally intersecting, then any monotone formula separating this pair has size at least* $\mathrm{rank}_{\mathbb{F}}(R)$.

8.2. Self-avoiding families

Let $\mathcal{A}$ be a family of minterms of some monotone boolean function $f(X)$. For a subset $Y \subseteq X$, let $S(Y)$ be the union of all members of $\mathcal{A}$ intersecting Y. Recall that a family $\mathcal{A}$ is *self-avoiding* if one can associate a subset $T_A \subseteq A$ with each $A \in \mathcal{A}$, called the *core* of A, such that

1. T_A determines A in the family. That is, no other set in the family $\mathcal{A}$ contains T_A as a subset.
2. For every $A \in \mathcal{A}$ and every subset $Y \subseteq T_A$, the set $S(Y) \setminus Y$ does not contain any member of $\mathcal{A}$.

In Sect. 16.4 of the book we have proved the following general lower bounds criterion for monotone span programs.

Theorem 8.5. *Let f be a monotone boolean function and $\mathcal{A}$ be the family of its minterms. If $\mathcal{A}$ is self-avoiding then for every field $\mathbb{F}$, every monotone span program over $\mathbb{F}$ computing f has size at least $|\mathcal{A}|$.*

Here we will show that this theorem is a special case of Theorem 8.1.

Proof. Define $\mathcal{B}$ to be the family of all sets $B = B_0 \cup B_1$ where $B_0 \subseteq T_A$ for some $A \in \mathcal{A}$ and $B_1 = X \setminus S(B_0)$. The theorem follows from Theorem 8.1 and the following two claims.

Claim 8.6. *The pair $\mathcal{A}, \mathcal{B}$ is locally intersecting.*

Proof of Claim 8.6. Take an arbitrary $A' \in \mathcal{A}$ and an arbitrary $B \in \mathcal{B}$; hence, $B = B_0 \cup B_1$ where $B_0 \subseteq T_A$ for some $A \in \mathcal{A}$ and $B_1 = X \setminus S(B_0)$.

If $A' \cap B_0 \neq \emptyset$, then $A' \subseteq S(B_0)$, and hence, $A' \cap B_1 = \emptyset$. Thus, A' intersects at most one of the sets B_0 and B_1.

If $A' \cap B_0 = \emptyset$, then $A' \cap B_1 \neq \emptyset$, because otherwise we would have that $A' \subseteq S(B_0)$ and, since $A' \cap B_0 = \emptyset$, also that $A' \subseteq S(B_0) \setminus B_0$. Thus, A' intersects exactly one of the sets B_0 and B_1, implying that the pair $\mathcal{A}, \mathcal{B}$ is locally intersecting. □

Claim 8.7. *The disjointness matrix R of the pair $\mathcal{A}, \mathcal{B}$ has full row-rank over $\mathbb{F}$.*

Proof of Claim 8.7. Recall that the rows of R are labeled by the members A of $\mathcal{A}$, the columns are labeled by the members B of $\mathcal{B}$; moreover, each B consists of two parts B_0 and B_1 where $B_0 \subseteq T_{A'}$ for some $A' \in \mathcal{A}$ and $B_1 = X \setminus S(B_0)$. The entry $R_{A,B}$ is 0 if $A \cap B_0 \neq \emptyset$, and is 1 otherwise.

Fix an $A \in \mathcal{A}$ and let M be the submatrix of R consisting of all columns indexed by subsets Y of T_A. Since, by our assumption, *every* subset of T_A appears as a column of M, the rows of M are rows[1] of the full disjointness matrix[2] D_m with $m = |T_A|$, some of them repeated. Observe that the row indexed by A occurs in M only once. This is because this row has 0's in all columns indexed by one element subsets $\{x\}$ with $x \in T_A$. Hence, if for some $A' \in \mathcal{A}$, the row of M indexed by A' would be the same then we would have that $T_A \subseteq A'$ and the first property of self-avoiding families would imply that $A' = A$.

We know that full disjointness matrix D_m has full row-rank (see Exercise 16.3 in the book). Since the row indexed by A occurs in M only once, this implies that

[1]That is, each row of M is a row of D_m, although some rows of D_m may not appear in M.

[2]Recall that D_m is a 0-1 matrix whose rows and columns are indexed by subsets of $\{1, \ldots, m\}$, and the entry corresponding to A and B is 1 precisely when $A \cap B = \emptyset$.

this row cannot be a linear combination of others. Thus, the corresponding row of the entire matrix R cannot be a linear combination of others, as well. $\square$

Claims 8.6 and 8.7 also imply the following

Lemma 8.8. *If $\mathcal{A}$ is self-avoiding then there is a family $\mathcal{B}$ such that the pair $\mathcal{A}, \mathcal{B}$ is locally intersecting and the disjointness matrix R of this pair has rank $|\mathcal{A}|$ (over any field).*

8.3. The limits of the approach

The highest lower bound on the size of monotone span programs, obtained by using Theorem 8.5 have had the form $n^{\Omega(\log n)}$ (see Sect. 16.5 in the book). A natural question was whether one can prove larger lower bounds when using the more general Theorem 8.1.

Unfortunately, as observed by Anna Gál, Theorem 8.1 (and hence, Theorem 8.5) *cannot* yield lower bounds larger than $n^{O(\log n)}$. This observation also implies that every self-avoiding family can have at most so many members, thus answering the first of four open problems formulated at the end of Sect. 16.5 of the book.

Proposition 8.9 (Gál 2002). *Let $\mathcal{A}, \mathcal{B}$ be a locally intersecting pair of families, and R be its disjointness matrix over a field $\mathbb{F}$. Then*

$$\operatorname{rank}_{\mathbb{F}}(R) \leq n^{O(\log n)}.$$

Proof. Let $\delta(R)$ be the minimum number of mutually disjoint monochromatic submatrices of R whose union covers all entries of R. Hence, $\operatorname{rank}_{\mathbb{F}}(R) \leq \delta(R)$. On the other hand, Aho, Ullman and Yannakakis (1983) has proved that, for every matrix R, $\log \delta(R) \leq O((\log \alpha(R))^2$ where $\alpha(R)$ is the minimum number of monochromatic (but not necessarily disjoint!) submatrices whose union covers all entries of R. (This is an interesting result in itself, and we will prove it in the next section.)

Each entry $R_{A,B}$ of R is either 0 or 1 depending on whether A intersects B in its first block B_0 or in the second one B_1. Hence, all entries of R can be covered by $2n$ submatrices $M_i^{\varepsilon} = \{(A, B) : i \in A \cap B_{\varepsilon}\}$, $i = 1, \ldots, n$ and $\varepsilon = 0, 1$. It is clear that these submatrices cover all entries of R. Moreover, since the pair $\mathcal{A}, \mathcal{B}$ is locally intersecting, we have that all the entries of M_i^{ε} are equal to ε. Hence, we have found a covering of R by $2n$ monochromatic submatrices M_i^{ε}, implying that $\alpha(R) \leq 2n$. By the above mentioned result of Aho et al, $\log \operatorname{rank}_{\mathbb{F}}(R) \leq \log \delta(R) \leq O((\log \alpha(R))^2 \leq O((\log n)^2$. $\square$

The proposition above, together with Lemma 8.8, yield the following

Corollary 8.10. *Let $\mathcal{A}$ be a family of subsets of some n-element set. If $\mathcal{A}$ is self-avoiding then $|\mathcal{A}| \leq n^{O(\log n)}$.*

8.4. Disjoint vs. non-disjoint covers of matrices

Let M be an n by n 0-1 matrix. For $\varepsilon \in \{0, 1\}$, let $\alpha_{\varepsilon}(M)$ denote the minimum number of all-ε submatrices of M whose union covers all entries ε of M. Let $\delta(M)$ be the minimum number of *mutually disjoint* monochromatic submatrices of M whose union covers all entries of M.

Theorem 8.11 (Aho–Ullman–Yannakakis 1983). *Let M be a 0-1 matrix with $\alpha_0(M) = r$ and $\alpha_1(M) = s$. Then $\delta(M) \leq r^{O(\log s)}$.*

Proof. We can look at the matrix M as a function $M : X \times Y \to \{0,1\}$ and consider the communication game in which Alice and Bob are trying to compute this function (see lecture "Communication Games"). Let $C(M)$ be the deterministic communication complexity of this game. It is easy to show (do this!) that $\delta(M) \leq 2^{C(M)}$.

Let $M_1, \ldots, M_r$ be the $r = \alpha_0(M)$ all-0 submatrices of M covering all 0's of M. For every matrix M_i, consider the matrix A_i formed by the rows of M intersecting M_i, and B_i be the matrix formed by the columns of M intersecting M_i. Since M_i consists only of 0's, we have that

$$\alpha_1(A_i) + \alpha_1(B_i) \leq \alpha_1(M) \qquad (*)$$

The protocol consists of $\log s = \log \alpha_1(M)$ rounds. In each round, the players do the following. First, Alice checks whether there is an index i such that her row intersects M_i and $\alpha_1(A_i) \leq \frac{1}{2}\alpha_1(M)$. If yes, then (using $1 + \log r$ bits) she sends "1" and the index i to Bob. If not, then she sends "0". Now Bob checks whether there is an index i such that his column intersects M_i and $\alpha_1(B_i) \leq \frac{1}{2}\alpha_1(M)$. If yes, then (using $1 + \log r$ bits) he sends "1" and the index i to Bob. If not, then she sends "0".

If either Alice or Bob find a suitable index i in this round then, by communicating at most $2 + \log r$ bits, they have restricted the problem to a matrix M' ($= A_i$ or B_i) for which $\alpha_1(M') \leq \frac{1}{2}\alpha_1(M)$. Hence, in this case, the theorem follows by induction.

If both players have sent "0" in this round, then they can finish the protocol: the answer is "1". Indeed, if there would be a 0 in the intersection of Alice's row and Bob's column, then this 0 would belong to some submatrix M_i. However, for these submatrices we have on the one hand $\alpha_1(A_i) > \frac{1}{2}\alpha_1(M)$ (since they did not suit Alice), on the other hand $\alpha_1(B_i) > \frac{1}{2}\alpha_1(M)$ since they did not suit Bob. But this contradicts (*).

Thus, we have shown that $C(M) \leq (2 + \log r) \cdot \log s = O((\log r) \cdot (\log s)$. Since $\delta(M) \leq 2^{C(M)}$, the desired upper bound $\delta(M) \leq r^{O(\log s)}$ follows. □

Remark. Note that the measure $\alpha_1(M)$ in this theorem can be replaced by any measure $\mu(M)$ satisfying (*).

> **Ex. 8.12.** Let $\mathcal{A}, \mathcal{B}$ be a locally intersecting pair of families. Suppose that each member of $\mathcal{A}$ has at most a elements, and each member of $\mathcal{B}$ has at most b elements. Show that then the disjointness matrix R of this pair contains an $N \times M$ submatrix of rank 1 with $N \geq |\mathcal{A}|/2b$ and $M \geq |\mathcal{B}|/2a$.
> *Hint*: Use Theorem 18.6 from the book.