Asymptotics of Binomial $\binom{n}{k}$

What is a good asymptotic formula for $\binom{n}{k}$ as $n \to \infty$. Well, when k is fixed it is easy:

$$\binom{n}{k} \sim \frac{n^k}{k!} \text{ for } k \text{ fixed} \tag{1}$$

But now say k also goes to infinity. There are different answers depending on the asymptotic relation between n and k. Lets restrict ourselves (until the end) to the cases when $k = o(n)$. We start by writing

$$\binom{n}{k} = A\frac{n^k}{k!}$$

where

$$A = \frac{(n)_k}{n^k} = \prod_{i=1}^{k-1}(1 - \frac{i}{n})$$

If we have an asymptotic formula for A then we will be done since we can use Stirling on the denominator $k!$. Clearly $A \leq 1$. Note that

$$\ln A = \sum_{i=1}^{k-1} \ln(1 - \frac{i}{n})$$

The main estimate is to note that as $x \to 0$, $\ln(1-x) \sim x$.
`Case 1:` $k = o(\sqrt{n})$. Then

$$\sum_{i=1}^{k-1} \ln(1 - \frac{i}{n}) \sim \sum_{i=1}^{k-1} -\frac{i}{n} \sim -\frac{k^2}{2n} = o(1)$$

Thus $A \sim 1$. Thus the asymptotic formula (1) still holds. With $k \to \infty$ employing Stirling gives:

$$\binom{n}{k} \sim \left(\frac{ne}{k}\right)^k (2\pi k)^{-1/2} \tag{2}$$

`Case 2:` $k \sim c\sqrt{n}$ with c a fixed positive real. then the argument above gives $\ln A \sim \frac{c^2}{2}$ and so

$$\binom{n}{k} \sim e^{-c^2/2}\frac{n^k}{k!} \tag{3}$$

For bigger k let us use the expansion $\ln(1-x) = -x + O(x^2)$. Applying this to the formula for $\ln A$ we have $\ln(1-\frac{i}{n}) = -\frac{i}{n} + O(\frac{i^2}{n^2})$. Summing over

$1 \le i < k$ gives $\frac{k^2}{2n}$ from the first term and the $O(k^3n^{-2})$ for the error term. We see that the error term is $o(1)$ as long as $k = o(n^{2/3})$. This gives
`Case 3:` $k = o(n^{2/3})$. Then

$$\binom{n}{k} \sim e^{-k^2/2n}\frac{n^k}{k!} \tag{4}$$

As k gets bigger we use the more precise $\ln(1-x) = -x - \frac{x^2}{2} + O(x^3)$. Applying this to the formula for $\ln A$ we have $\ln(1-\frac{i}{n}) = -\frac{i}{n} - \frac{i^2}{2n^2} + O(i^3n^{-3})$. Summing over $1 \le i < k$ gives $\frac{k^2}{2n}$ from the first term and $\frac{k^3}{6n^2}$ for the second term and $O(k^4n^{-3})$ for the error. As long as $k = o(n^{3/4})$ the error term is $o(1)$. This gives
`Case 4:` $k = o(n^{3/4})$. Then

$$\binom{n}{k} \sim e^{-k^2/2n}e^{-k^3/6n^2}\frac{n^k}{k!} \tag{5}$$

Indeed, we get an infinite sequence of formulae as we take the expansion of $\ln(1-x)$ out to more and more terms. For each integer $j \ge 2$ we have a formula for when $k = o(n^{1-(1/j)})$. In practice use of even Case 4 is extremely rare.

An often very handy upper bound is

$$\binom{n}{k} \le \frac{n^k}{k!} \le \left(\frac{ne}{k}\right)^k \tag{6}$$

This is true for *all* n, k, not just asymptotically. When k is linear in n one should use the estimates below. But when $k = o(n)$ and $k \to \infty$ this is pretty good in the sense that the k-th root of $\binom{n}{k}$ is asymptotic to $\frac{ne}{k}$. That is not as strong as a truly asymptotic estimate but for many applications it does fine.

Now lets suppose k is linear in n, that $k \sim pn$ where $0 < p < 1$. Now we write $\binom{n}{k} = n!/(k!(n-k)!)$ and apply Stirling to the three terms. (By the way, this is not a good idea to try when $k = o(n)$ – the formula for the $(n-k)!$ gets very messy and you *definitely* should use the formulas of Cases 1-2-3 when they apply.) Lets throw the square root terms into the form $2^{o(n)}$. One finds that

$$\binom{n}{k} = 2^{n(H(p)+o(1))} \text{ when } k \sim pn \tag{7}$$

where H is the entropy function

$$H(p) = -p \lg p - (1-p) \lg(1-p) \tag{8}$$

The entropy function is defined for $0 < p < 1$, is symmetric about $p = \frac{1}{2}$, has $\lim_{p \to 0+} H(p) = \lim_{p \to 1-} H(p) = 0$ (indeed, its handy to define $H(0) = H(1) = 0$) and has a maximum at $p = \frac{1}{2}$ where $H(1/2) = 1$. We do note that this is not the full asymptotics since the $o(1)$ is in the exponent multiplied by n. More accurate formulae are available but this one is quite handy.

Finally, let us look at the region when k is very close to $\frac{n}{2}$. (When n is odd look at $\lfloor n/2 \rfloor$ and all the asymptotics will be the same.) Direct application of Stirling's Formula gives

$$\binom{n}{\frac{n}{2}} \sim \sqrt{\frac{2}{\pi}} n^{-1/2} 2^n$$

(Note this means the probability of having precisely half heads with n flips of a fair coin is $\sim \sqrt{\frac{2}{\pi}} n^{-1/2}$.) Now let us write

$$k = \frac{n+i}{2}$$

One then has

$$\binom{n}{k} \sim \binom{n}{\frac{n}{2}} e^{-(i^2/2n)} \tag{9}$$

as long as $i = o(n^{2/3})$. In particular, if you parametrize (good idea!)

$$k = \frac{n + c\sqrt{n}}{2}$$

then we have the asymtotics

$$\binom{n}{k} \sim \binom{n}{\frac{n}{2}} e^{-c^2/2} \tag{10}$$

and this will be valid for any constant c and even $c = c(n) \to \infty$ as long as $c(n) = o(n^{1/6})$.

One particular case is often handy. If we set (another good idea!) $c = 100\sqrt{\ln n}$ then we have

$$\binom{n}{k} = \Theta(2^n n^{-1/2} n^{-5000})$$

The Binomial coefficients increase to the middle and then decrease and there are only $O(n)$ terms so we get

$$\sum_{s\geq k}\binom{n}{s}=\Theta(n2^n n^{-1/2}n^{-5000})$$

In many applications this makes an excellent cut, as this sum is so small. To take an example, suppose you want the asymptotics of $\sum_{s=0}^{n}\binom{n}{k}^3$. We see that for $k=\frac{n+100\sqrt{\ln n}}{2}$ as above

$$\binom{n}{k}^3=O(\binom{n}{\frac{n}{2}}^3 n^{-15000})$$

and so

$$\sum_{s\geq k}\binom{n}{s}^3=O(\binom{n}{\frac{n}{2}}^3 n^{-14999})$$

so that all these terms together are negligible compared to the biggest (the middle) term. By symmetry the same holds for the sum over $s\leq k^-$ where $k^-=n-k$. So the asymptotics of the sum over all s is the same as the asymptotics of the sum over the limited range $k^-\leq s\leq k$. Of course, we still have to find that sum but we are now helped by the approximations of Case 3.