Beveik nestacionarių procesų dalinių sumų funkcinės ribinės teoremos

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Structure

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AR(1) process

First order autoregressive $AR(1)$ process is generated according to the scheme

$$y_k = \phi y_{k-1} + \varepsilon_k, \quad k \geq 1 \tag{1}$$

where (ε_k) are innovations and ϕ is an unknown parameter.

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where (ε_k) are innovations and ϕ is an unknown parameter.

1. if $|\phi| < 1$, then (1) is stationary process ;
2. if $|\phi| > 1$, then (1) is explosive process ;
3. if $\phi = 1$, then (1) is nonstationary process.

Estimate of the coefficient ϕ

The least-squares estimate (LSE) of ϕ based on observations $y_1, \dots, y_n$ is

$$\widehat{\phi} = \frac{\sum_{k=1}^{n} y_k y_{k-1}}{\sum_{k=1}^{n} y_{k-1}^2}.$$

Alternative ways are

- Yule Walker equations (method of moments) ;
- maximum likelihood estimate.

Some notations

1. We denote $\xrightarrow[n\to\infty]{\mathrm{F}}$ the convergence in distribution in the metric space F.
2. Classical convergence in distribution of a sequence of random variables is denoted by $\xrightarrow[n\to\infty]{\mathbb{R}}$.
3. Convergence in probability is denoted by $\xrightarrow[n\to\infty]{\mathrm{P}}$.
4. $W = (W(t), t \in [0,1])$ is a standard Brownian motion.
5. $U_\gamma(s) = \int_0^s \mathrm{e}^{\gamma(s-r)} \,\mathrm{d}W(r)$ is an Ornstein-Uhlenbeck process.
6. $\mathfrak{N}(0, \sigma^2)$ denotes normal distribution with mean 0 and variance σ^2.

Asymptotic behaviour of $\widehat{\phi}$

When $|\phi| < 1$ it is well known (see, for example, Mann and Wald (1943) and Anderson (1959)) that the standardized LSE is asymptotically normal :

$$\tau_n := \left(\sum_{k=1}^{n} y_{k-1}^2 \right)^{1/2} (\widehat{\phi} - \phi) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,1).$$

Remark, that with another standardization

$$\sqrt{n}(\widehat{\phi} - \phi) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0, 1 - \phi^2).$$

Asymptotic behaviour of $\widehat{\phi}$

For $\phi > 1$, Anderson (1959) showed that

$$\tau_n := \left(\sum_{k=1}^{n} y_{k-1}^2 \right)^{1/2} (\widehat{\phi} - \phi) \xrightarrow[n \to \infty]{\mathbb{R}} \mathfrak{N}(0, 1)$$

is true when the ε_k's are i.i.d. For general ε_k's he showed that limiting distribution of τ_n may not exists.

Asymptotic behaviour of ϕ

However when $\phi = 1$, the limit distribution of the properly standardized sequence of the least-squares estimators is non-normal, and it was shown by White (1958) (see also Rao (1978)) that

$$\left(\sum_{k=1}^{n} y_{k-1}^2\right)^{1/2}(\widehat{\phi}-1) \xrightarrow[n\to\infty]{\mathbb{R}} \tau := \frac{\frac{1}{2}(W^2(1)-1)}{\left(\int_0^1 W^2(t)\,\mathrm{d}t\right)^{1/2}}$$

or

$$n(\widehat{\phi}-1) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 W(t)\,\mathrm{d}W(t)}{\int_0^1 W^2(t)\,\mathrm{d}t},$$

where $(W(t), 0 \le t \le 1)$ is a standard Brownian motion.

ϕ is "close" to 1

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3. Also

$$\left(\sum_{k=1}^{n} y_{k-1}^2\right)^{1/2} (\widehat{\phi} - 1) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\frac{1}{2}(W^2(1) - 1)}{\left(\int_0^1 W^2(t)\,\mathrm{d}t\right)^{1/2}} \qquad (3)$$

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   could be used to approximate the distribution of τ_n when ϕ is close to one (Evans and Savin (1981)).
4. However, neither (2) nor (3) seems to be intuitive approximations.

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Nearly nonstationary process

Suppose we have first-order autoregressive process $(y_{n,k})$ given by

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \quad k \geq 1, \quad n \geq 1, \tag{4}$$

where

1. $\phi_n \to 1$, as $n \to \infty$,
2. (ε_k) is a sequence of i.i.d.random variables with $\mathbb{E}\varepsilon_k = 0$ and $\mathbb{E}\varepsilon_k^2 = 1$,
3. $y_{n,1} \ldots, y_{n,n}$ are observations and n is a sample size,
4. $y_{n,0}$ random variable.

Parametrisation of ϕ_n

Case 1

$\phi_n = \mathrm{e}^{\gamma/n}$ with constant $\gamma < 0$.
This parametrisation was suggested by Phillips (1987 m.).

Case 2

$\phi_n = 1 - \frac{\gamma_n}{n}$, $\gamma_n \to \infty$ as $n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$
This parametrisation was suggested by Phillips and Giraitis (2006 m.)

Another parametrizations :

1. $\phi_n = 1 - \frac{\gamma}{n}$, $\gamma > 0$ (Proposed by Chan & Wei (1987), Cox & Llatas (1991), etc.)
2. $\phi_n = 1 - \frac{\gamma_n}{n}$, $\gamma_n \to \gamma$ (Proposed by Andrews & Guggenberger (2007))
3. $\phi_n - 1 - \frac{\gamma_n}{n}$, $\gamma_n \to 0$ (Proposed by Andrews & Guggenberger (2007))

Some results (Phillips (1987))

If (y_k) is a nearly nonstationary process generated by (4) and $\phi_n = \mathrm{e}^{\gamma/n}$, $y_{n,0} = o_{\mathrm{P}}(n^{1/2})$ and (ε_k) are i.i.d. random variables with $\mathbb{E}\varepsilon_0 = 0$ and $\mathbb{E}\varepsilon_0^2 = \sigma^2$, then :

$$n^{-1/2} y_{[nt]} \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} \sigma U_\gamma(t),$$

$$n^{-3/2} \sum_{j=1}^{n} y_j \xrightarrow[n\to\infty]{\mathbb{R}} \sigma \int_0^1 U_\gamma(r)\,\mathrm{d}r,$$

$$n^{-2} \sum_{j=1}^{n} y_j^2 \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 \int_0^1 U_\gamma^2(r)\,\mathrm{d}r,$$

$$n^{-1} \sum_{j=1}^{n} y_{j-1}\varepsilon_j \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 \int_0^1 U_\gamma(r)\,\mathrm{d}W(r).$$

Some results (Phillips and Giraitis (2006))

Suppose (y_k) is a nearly nonstationary process generated by (4) and (ε_k) are i.i.d. random variables with $\mathbb{E}\varepsilon_0 = 0$ and $\mathbb{E}\varepsilon_0^2 = \sigma^2$. Under assumptions $n(1-\phi_n) \to \infty$, as $n \to \infty$ and $\mathbb{E}y_{n,0}^2 = o(n^{1/2})$:

$$\frac{\left(1-\phi_n^2\right)^{1/2}}{n^{1/2}} \sum_{j=1}^{n} \varepsilon_j y_{j-1} \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,\sigma^4),$$

$$\frac{1-\phi_n^2}{n} \sum_{j=1}^{n} y_{j-1}^2 \xrightarrow[n\to\infty]{\mathrm{P}} \sigma^2,$$

$$\frac{(1-\phi_n)}{n^{1/2}} \sum_{j=1}^{n} y_j \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,\sigma^2).$$

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If (y_k) is a nearly nonstationary process generated by (4) and $\phi_n = 1 - \frac{\gamma_n}{n}$, where $\gamma_n \to \gamma \in [0, \infty)$, (ε_k) are i.i.d. random variables with $\mathbb{E}\varepsilon_0 = 0$ and $\mathbb{E}\varepsilon_0^2 = \sigma^2$, then :

$$n^{-1/2} y_{[nt]} \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} \sigma U_\gamma(t),$$

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$$n^{-2} \sum_{j=1}^{n} y_j^2 \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 \int_0^1 U_\gamma^2(r)\,\mathrm{d}r,$$

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If (y_k) is a nearly nonstationary process generated by (4) and $\phi_n = 1 - \frac{\gamma_n}{n}$, where $\gamma_n \to \gamma \in [0, \infty)$, (ε_k) are i.i.d. random variables with $\mathbb{E}\varepsilon_0 = 0$ and $\mathbb{E}\varepsilon_0^2 = \sigma^2$ and $y_{n,0} = \sum_{j=0}^{\infty} \phi_n^j \varepsilon_{-j}$ then :

$$n^{-1/2} y_{[nt]} \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} \sigma U_\gamma^*(t),$$

$$n^{-3/2} \sum_{j=1}^{n} y_j \xrightarrow[n\to\infty]{\mathbb{R}} \sigma \int_0^1 U_\gamma^*(r)\,\mathrm{d}r,$$

$$n^{-2} \sum_{j=1}^{n} y_j^2 \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 \int_0^1 (U_\gamma^*)^2(r)\,\mathrm{d}r,$$

$$n^{-1} \sum_{j=1}^{n} y_{j-1}\varepsilon_j \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 \int_0^1 U_\gamma^*(r)\,\mathrm{d}W(r),$$

where $U_\gamma^*(t) = U_\gamma(t) + (2\gamma)^{-1/2}\mathrm{e}^{-\gamma t} Z$ and Z is standard normal random variable.

Some results (Andrews & Guggenberger (2007))

If (y_k) is a nearly nonstationary process generated by (4) and $\phi_n = 1 - \frac{\gamma_n}{n}$, where $\gamma_n \to 0$, (ε_k) are i.i.d. random variables with $\mathbb{E}\varepsilon_0 = 0$ and $\mathbb{E}\varepsilon_0^2 = \sigma^2$ and $y_{n,0} = \sum_{j=0}^{\infty} \phi_n^j \varepsilon_{-j}$ then :

$$(2\gamma_n)^{1/2} n^{-3/2} \sum_{j=1}^{n} y_{n,j-1} \xrightarrow[n\to\infty]{\mathbb{R}} \sigma Z,$$

$$2\gamma_n n^{-2} \sum_{j=1}^{n} y_{n,j-1}^2 \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 Z^2,$$

$$(2\gamma_n)^{1/2} n^{-1} \sum_{j=1}^{n} y_{j-1}\varepsilon_j \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 Z Z^*,$$

where Z and Z^* are independent standard normal random variables.

The asymptotic behaviour of ϕ_n

Phillips (1987) :

$$n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 U_\gamma(r)\,\mathrm{d}W(r)}{\int_0^1 U_\gamma^2(r)\,\mathrm{d}r}$$

Phillips and Giraitis (2006) :

$$\frac{n^{1/2}}{(1-\phi_n^2)^{1/2}}(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,1)$$

The asymptotic behaviour of ϕ_n

Chan and Wei (1987) :

$$\left(\sum_{k=1}^{n} y_{k-1}^2\right)^{1/2} (\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \mathcal{L}(\gamma)$$

where

$$\mathcal{L}(\gamma) = \frac{\int_0^1 (1 + (\mathrm{e}^{2/\gamma} - 1)t)^{-1} W(t)\,\mathrm{d}W(t)}{\left(\int_0^1 (1 + (\mathrm{e}^{2/\gamma} - 1)t)^{-2} W^2(t)\,\mathrm{d}t\right)^{1/2}}.$$

Chan (1988) showed that

$$\mathcal{L}(\gamma) \equiv_{\mathcal{D}} \frac{\int_0^1 U_\gamma(r)\,\mathrm{d}W(r)}{\left(\int_0^1 U_\gamma^2(r)\,\mathrm{d}r\right)^{1/2}}.$$

The asymptotic behaviour of ϕ_n

Andrews & Guggenberger (2007), $\phi_n = 1 - \gamma_n/n$ and $\gamma_n \to 0$, $y_{n,0} = \sum_{j=0}^{\infty} \phi_n^j \varepsilon_{-j}$:

$$(2\gamma_n)^{-1/2} n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} C$$

where C is a Cauchy random variable.

Andrews & Guggenberger (2007), $\phi_n = 1 - \gamma_n/n$ and $\gamma_n \to \gamma \in (0, \infty]$, $y_{n,0} = \sum_{j=0}^{\infty} \phi_n^j \varepsilon_{-j}$:

1. $\gamma \in (0, \infty)$

$$n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 U_\gamma^*(t)\,\mathrm{d}W(r)}{\int_0^1 (U_\gamma^*)^2(t)\,\mathrm{d}t}$$

2. $\gamma = \infty$

$$(1 - \phi_n^2)^{-1/2} n^{1/2}(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} Z.$$

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Process built on the y_k's

We focus on polygonal line processes built on the y_k's :

$$S_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} y_{k-1} + (nt - [nt]) y_{[nt]}, \quad t \in [0,1], \quad n \geq 1. \qquad (5)$$

Remark

The definition of the S_n^{pl} is quite unusual with a general term y_{k-1} were one would expect y_k. However, asymptotic results remains true with y_{k-1} replaced by y_k as well.

Function spaces

The polygonal line process S_n^{pl} can be viewed as a random element either in $\mathrm{C}[0,1]$ or in $\mathrm{H}_\alpha^o[0,1]$. Continuous function space $\mathrm{C}[0,1]$ is endowed with the uniform norm

$$\|f\|_\infty = \sup_{0\le t\le 1} |f(t)| \quad f \in \mathrm{C}[0,1].$$

For $\alpha \in (0,1)$ the Hölder space

$$\mathrm{H}_\alpha^o[0,1] := \left\{ f \in C[0,1] : \lim_{\delta\to 0} \omega_\alpha(f,\delta) = 0 \right\},$$

endowed with the norm $\|f\|_\alpha := |f(0)| + \omega_\alpha(f,1)$, where

$$\omega_\alpha(f,\delta) := \sup_{\substack{s,t\in[0,1]\\ 0<t-s<\delta}} \frac{|f(t)-f(s)|}{|t-s|^\alpha}$$

is a separable Banach space.

Process built on the ε_k's

The polygonal line process built on i.i.d. random variables (ε_j) is

$$W_n^{\mathrm{pl}}(t) = \sum_{j=1}^{[nt]} \varepsilon_j + (nt - [nt])\varepsilon_{[nt]+1}, \quad t \in [0,1].$$

By classical Donsker-Prohorov invariance principle

$$n^{-1/2} W_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W.$$

Invariance principle in Hölder space

By the classical Levy's result on the modulus of continuity of W, $W \in \mathrm{H}_\alpha^o[0,1]$ with probability one for every $0 \le \alpha < 1/2$.
Račkauskas and Suquet (2004) proved that for $0 < \alpha < 1/2$ the convergence

$$n^{-1/2}\sigma^{-1}W_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^o[0,1]} W \tag{6}$$

holds if and only if

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| \ge t) = 0. \tag{7}$$

Initial condition

Let us associate to each autoregressive process $(y_{n,k})$ satisfying

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \quad k \geq 1, \quad n \geq 1, \tag{8}$$

the process $(y'_{n,k})$ defined by

$$y'_{n,k} = y_{n,k} - \phi_n^k y_{k,0}.$$

Then $(y'_{n,k})$ satisfies (8) with initialization $y'_{n,0} = 0$.

Initial condition

Proposition 1

Let $S_n^{\mathrm{pl}'}$ be the polygonal line process obtained by substituting in (5) the $y_{n,j}$'s by the $y'_{n,j}$'s. Assume that $c_n S_n^{\mathrm{pl}'}$ converges in distribution in $\mathrm{H}_\alpha^o[0,1]$, where the c_n's are some positive normalizing constants. Then $c_n S_n^{\mathrm{pl}}$ converges in distribution in $\mathrm{H}_\alpha^o[0,1]$ to the same limit provided that

$$\frac{c_n n^\alpha}{1-\phi_n} y_{n,0} \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{9}$$

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Convergence in $\mathrm{C}[0,1]$ and $\mathrm{H}^o_\alpha[0,1]$ spaces

Theorem 1

In the case 1 where (y_k) is generated by (4) with $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$, suppose that the sequence of polygonal lines $(n^{-1/2} W_n^{\mathrm{pl}})$ converges weakly to the standard Brownian motion W either in $\mathrm{C}[0,1]$ or in $\mathrm{H}^o_\alpha[0,1]$ for some $0 < \alpha < 1/2$. Suppose moreover that $y_{n,0} = o_P(n^{1/2})$ or $y_{n,0} = o_P(n^{1/2-\alpha})$ according to the function space considered. Then $n^{-3/2} S_n^{\mathrm{pl}}$ converges weakly, as $n \to \infty$, in the space under consideration to the integrated Ornstein-Uhlenbeck process J defined by :

$$J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s, \quad 0 \le t \le 1, \tag{10}$$

where $U_\gamma(s) = \int_0^s \mathrm{e}^{\gamma(s-r)}\,\mathrm{d}W(r)$.

Convergence with i.i.d. innoovations

Taking into account the classical Donsker-Prohorov invariance principle and the functional central limit theorem proved by R&S (2004) we have the following corollary.

Corollary 2

Assume that (y_k) is generated by (4) with $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$ and that the ε_k's are i.i.d. and centered. Then the weak convergence of $n^{-3/2} S_n^{\mathrm{pl}}$ to J holds

- *in $\mathrm{C}[0,1]$ provided that $\mathbb{E}\varepsilon_1^2 < \infty$ and $y_{n,0} = o_P(n^{1/2})$;*
- *in $\mathrm{H}_\alpha^o[0,1]$ for $0 < \alpha < 1/2$ under condition*

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| \geq t) = 0.$$

and $y_{n,0} = o_P(n^{1/2-\alpha})$.

Essential lemma

Lemma 3

Suppose that the process (y_k) is generated by (4) and $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of nonnegative numbers such that $\gamma_n \to \infty$ and $\gamma_n/n \to 0$ as $n \to \infty$. Suppose, moreover, that $y_0 = 0$. Let $p \geq 2$. Assume that the innovations (ε_k) satisfy

$$\lim_{t\to\infty} t^p \mathbb{P}(|\varepsilon_1| > t) = 0 \quad \text{if} \quad p > 2,$$
$$\mathbb{E}\varepsilon_1^2 < \infty \quad \text{if} \quad p = 2.$$

For $p \geq 2$, put $\alpha = 1/2 - 1/p$. Then

$$n^{-1/2}\gamma_n^{\alpha} \max_{1\leq k\leq n} |y_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

Convergence in $\mathrm{C}[0,1]$

Theorem 4

Suppose that the process (y_k) is generated by (4) and $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of nonnegative numbers such that $\gamma_n \to \infty$ and $\gamma_n/n \to 0$ as $n \to \infty$. Assume also, that innovations (ε_k) are i.i.d. with $\mathbb{E}\varepsilon_1 = 0$ and $\mathbb{E}\varepsilon_1^2 = 1$ and $y_0 = o_{\mathrm{P}}(n^{1/2})$. Then

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W.$$

Convergence in H^o_β

Theorem 5

Suppose that the process (y_k) *is generated by* (4) *and* $\phi_n = 1 - \gamma_n/n$, *where* (γ_n) *is a sequence of nonnegative numbers such that* $\gamma_n \to \infty$ *and* $\gamma_n/n \to 0$ *as* $n \to \infty$. *Assume also, that innovations* (ε_k) *are i.i.d. and satisfy condition*

$$\lim_{t\to\infty} t^p \mathbb{P}(|\varepsilon_1| > t) = 0 \quad \text{if} \quad p > 2$$

for some $p > 2$. *Put* $\alpha = 1/2 - 1/p$. *Then for* $0 < \beta < \alpha$,

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}^o_\beta[0,1]} W$$

provided that $y_0 = o_{\mathrm{P}}(n^{1/2-\beta})$ *and*

$$\liminf_{n\to\infty} \gamma_n n^{\beta/\alpha} > 0.$$

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Theorem 1

In the case 1 where (y_k) is generated by (4) with $\phi_n = \mathrm{e}^{\gamma/n}$, $\gamma < 0$, suppose that the sequence of polygonal lines $(n^{-1/2} W_n^{\mathrm{pl}})$ converges weakly to the standard Brownian motion W either in $\mathrm{C}[0,1]$ or in $\mathrm{H}^o_\alpha[0,1]$ for some $0 < \alpha < 1/2$. Suppose moreover that $y_{n,0} = o_P(n^{1/2})$ or $y_{n,0} = o_P(n^{1/2-\alpha})$ according to the function space considered. Then $n^{-3/2} S_n^{\mathrm{pl}}$ converges weakly, as $n \to \infty$, in the space under consideration to the integrated Ornstein-Uhlenbeck process J defined by :

$$J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s, \quad 0 \le t \le 1, \tag{10}$$

where $U_\gamma(s) = \int_0^s \mathrm{e}^{\gamma(s-r)}\,\mathrm{d}W(r)$.

Ideas of the proof 1

1. Since the Banach spaces $(C[0,1], \| \|_\infty)$ and $(\mathrm{H}_0^o, \| \|_0)$ are isomorphic, the unified proof proposed here for the spaces $\mathrm{H}_\alpha^o[0,1]$, $0 \le \alpha < 1/2$, includes the special case of the space $C[0,1]$.
2. Using Proposition 1 and our assumption $y_{n,0} = o_P(n^{1/2-\alpha})$, it is enough to give the proof in the case where $y_{n,0} = 0$.
3. Hölder norms of polygonal line is reached at two vertices.

Ideas of the proof 2

Lemma 6

Let $\rho : [0,1] \to \mathbb{R}$ *be a weight function satisfying the following properties.*

i) ρ *is concave.*

ii) $\rho(0) = 0$ *and* ρ *is positive on* $(0,1]$.

iii) ρ *is non decreasing on* $[0,1]$.

Let $t_0 = 0 < t_1 < \cdots < t_n = 1$ *be a partition of* $[0,1]$ *and* f *be a real valued polygonal line function on* $[0,1]$ *with vertices at the* t_i*'s, i.e.* f *is continuous on* $[0,1]$ *and its restriction to each interval* $[t_i, t_{i+1}]$ *is an affine function. Define*

$$R(s,t) := \frac{|f(t) - f(s)|}{\rho(t-s)}, \quad 0 \le s < t \le 1.$$

Then

$$\sup_{0 \le s < t \le 1} R(s,t) = \max_{0 \le i < j \le n} R(t_i, t_j). \tag{11}$$

Ideas of the proof 3

The idea is to approximate the polygonal line $n^{-3/2}S_n^{\mathrm{pl}}$ by some linear interpolation of a smooth process J_n

$$J_n(t) := \int_0^t n^{-1/2} W_n^{\mathrm{pl}}(s)\,\mathrm{d}s + \gamma \int_0^t \int_0^s \mathrm{e}^{\gamma(s-r)} n^{-1/2} W_n^{\mathrm{pl}}(r)\,\mathrm{d}r\,\mathrm{d}s.$$

which is a functional of $n^{-1/2}W_n^{\mathrm{pl}}$, continuous in Hölder topology, with $\left\| n^{-3/2}S_n^{\mathrm{pl}} - J_n \right\|_\alpha = o_P(1)$.
As the functional

$$\mathrm{H}_\alpha^o[0,1] \to \mathrm{H}_\alpha^o[0,1] \quad : \quad x \longmapsto \int_0^\bullet x(s)\,\mathrm{d}s + \gamma \int_0^\bullet \int_0^s \mathrm{e}^{\gamma(s-r)} x(r)\,\mathrm{d}r\,\mathrm{d}s$$

is continuous on $\mathrm{H}_\alpha^o[0,1]$, the convergence of J_n to J follows from the convergence of $n^{-1/2}W_n^{\mathrm{pl}}$ to W.

Convergence in $\mathrm{C}[0,1]$

Theorem 4

Suppose that the process (y_k) *is generated by* (4) *and* $\phi_n = 1 - \gamma_n/n$, *where* (γ_n) *is a sequence of nonnegative numbers such that* $\gamma_n \to \infty$ *and* $\gamma_n/n \to 0$ *as* $n \to \infty$. *Assume also, that innovations* (ε_k) *are i.i.d. with* $\mathbb{E}\varepsilon_1 = 0$ *and* $\mathbb{E}\varepsilon_1^2 = 1$ *and* $y_0 = o_{\mathrm{P}}(n^{1/2})$. *Then*

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W.$$

Ideas of the proof

1. Using Proposition 1 and assumption $y_0 = o_{\mathrm{P}}(n^{1/2})$, it suffices to prove the result when $y_0 = 0$;
2. In view of the Donsker-Prohorov invariance principle, it suffices to show that

$$\left\| n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} - n^{-1/2}W_n^{\mathrm{pl}} \right\|_\infty \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

3. As supremum norm is reached at one of its vertices and

$$(1-\phi_n)\sum_{j=1}^{k} y_{j-1} = -y_k + \sum_{j=1}^{k} \varepsilon_j$$

so it is enough to show that

$$n^{-1/2} \max_{1\le k\le n} |y_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0. \qquad (12)$$

4. And (12) is true because of Lemma 1 with $p = 2$.

Convergence in H_β^o

Theorem 5

Suppose that the process (y_k) is generated by (4) and $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of nonnegative numbers such that $\gamma_n \to \infty$ and $\gamma_n/n \to 0$ as $n \to \infty$. Assume also, that innovations (ε_k) are i.i.d. and satisfy condition

$$\lim_{t\to\infty} t^p \mathbb{P}(|\varepsilon_1| > t) = 0 \quad \text{if} \quad p > 2$$

for some $p > 2$. Put $\alpha = 1/2 - 1/p$. Then for $0 < \beta < \alpha$,

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\beta^o[0,1]} W$$

provided that $y_0 = o_{\mathrm{P}}(n^{1/2-\beta})$ and

$$\liminf_{n\to\infty} \gamma_n n^{\beta/\alpha} > 0.$$

Ideas of the proof

1. In view of Hölderian invariance principle it is enough to show that

$$\left\| n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} - n^{-1/2}W_n^{\mathrm{pl}} \right\|_\beta \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

2. As Hölderian norms of the polygonal line process is reached at two vertices, so the proof reduces to

$$n^{\beta-1/2} \max_{1\le k\le n} |y_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{13}$$

3. The convergence (13) holds due to Proposition 1 and Lemma 1 with condition

$$\limsup_{n\to\infty} n^\beta/\gamma_n^\alpha < \infty.$$

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