TWO-SAMPLE PROBLEM FOR FUNCTIONAL DATA

2015 04 14

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EXAMPLE

Example 5.1 *Figure 5.1 displays the reconstructed individual curves of the progesterone data: (a) nonconceptive and (b) conceptive, obtained by the local linear reconstruction method described in Section 3.2.3 of Chapter 3 with bandwidth* $h^* = 1.40$ *selected by the GCV rule (3.10). The progesterone data were introduced in Section 1.2.1 of Chapter 1. The nonconceptive progesterone curves were from the women who were not pregnant after the ovulation day (Day 0) when they discharged their ova, while the conceptive progesterone curves were from those women who were pregnant. The horizontal axis shows the days before and after the ovulation day. Of interest is to know if there is a significant difference between the mean functions of the nonconceptive and conceptive progesterone curves before or after the ovulation day or over the whole experimental period. This knowledge may be used to detect if a woman is pregnant after the ovulation day.*

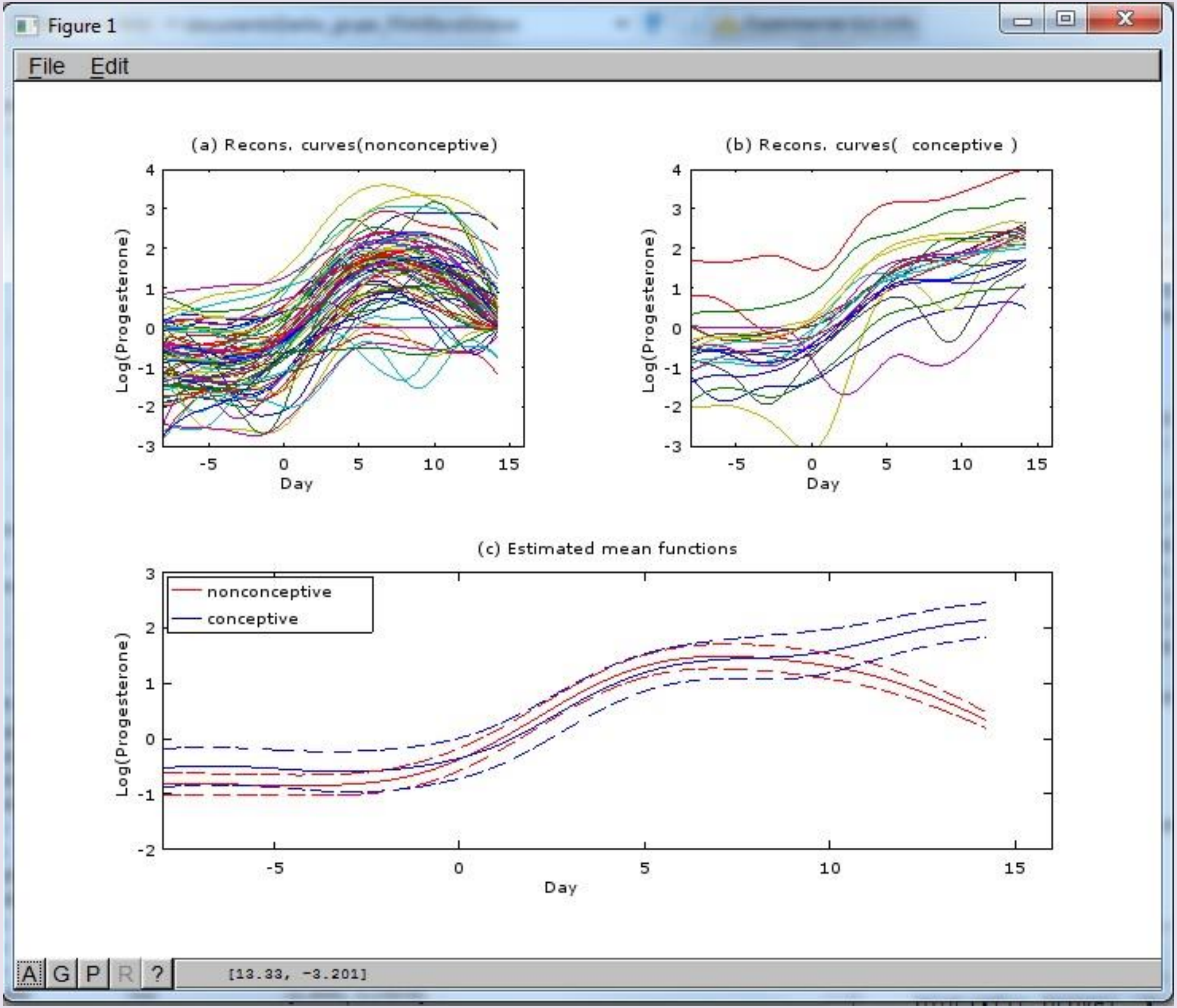


Figure 1
File Edit
(a) Recons. curves(nonconceptive)
(b) Recons. curves(conceptive)
Log(Progesterone)
Day
(c) Estimated mean functions
nonconceptive
conceptive
[13.33, -3.201]

GENERAL PROBLEM

A general two-sample problem for functional data with a common covariance function can be formulated as follows. Suppose we have two functional samples

$$y_{11}(t), \cdots, y_{1n_1}(t) \overset{i.i.d.}{\sim} \mathrm{SP}(\eta_1, \gamma), \quad y_{21}(t), \cdots, y_{2n_2}(t) \overset{i.i.d.}{\sim} \mathrm{SP}(\eta_2, \gamma), \tag{5.1}$$

where $\eta_1(t)$ and $\eta_2(t)$ are the unknown mean functions of the two samples, and $\gamma(s,t)$ is their common covariance function, which is usually unknown. We wish to test the following hypotheses:

$$\begin{array}{ll} & H_0 \quad : \eta_1(t) \equiv \eta_2(t), t \in \mathcal{T}, \\ \text{versus} & H_1 \quad : \eta_1(t) \neq \eta_2(t), \quad \text{for some } t \in \mathcal{T}, \end{array} \tag{5.2}$$

where $\mathcal{T}$ is the time period of interest, often a finite interval $[a, b]$ say with $-\infty < a < b < \infty$.

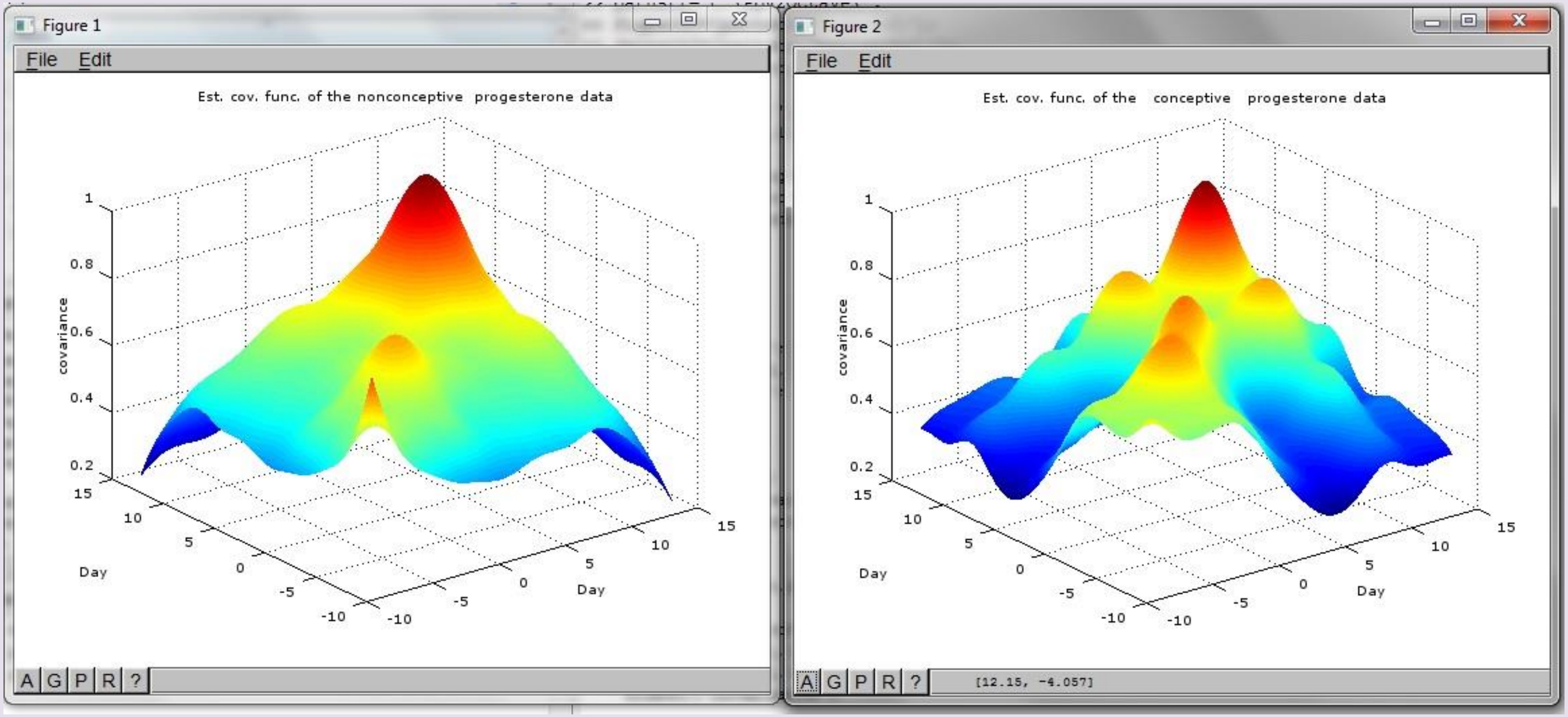
ESTIMATORS

Based on the two functional samples (5.1), the unbiased estimators of the mean functions $\eta_1(t)$, $\eta_2(t)$ and the common covariance functions $\gamma(s,t)$ are given by

$$\begin{aligned} \hat{\eta}_i(t) &= \bar{y}_i(t) = n_i^{-1} \sum_{j=1}^{n_i} y_{ij}(t), \ i = 1, 2, \\ \hat{\gamma}(s,t) &= (n-1)^{-1} \sum_{i=1}^{2} \sum_{j=1}^{n_i} \left[y_{ij}(s) - \bar{y}_i(s)\right] \left[y_{ij}(t) - \bar{y}_i(t)\right], \end{aligned} \tag{5.3}$$

which are known as the sample mean and pooled sample covariance functions of the two samples, respectively, where and throughout this section, $n = n_1 + n_2$ denotes the total sample size of the two samples.

Equal covariance



Equal covariance

Table 5.1 *Traces of the sample covariance functions* $\hat{\gamma}_1(s,t)$ *and* $\hat{\gamma}_2(s,t)$ *and their cross-square functions* $\hat{\gamma}_1^{\otimes 2}(s,t)$ *and* $\hat{\gamma}_2^{\otimes 2}(s,t)$ *calculated with resolution* $M = 1,000$ *over various periods.*

$[a,b]$	$[-8,0]$		$[-8,8]$		$[-8,15]$	
i	1	2	1	2	1	2
$\mathrm{tr}(\hat{\gamma}_i)$	251	264	512	536	728	760
$\mathrm{tr}(\hat{\gamma}_i^{\otimes 2})$	51,076	62,308	191,475	191,688	368,392	361,410

Table 5.2 *Traces of the pooled sample covariance function* $\hat{\gamma}(s,t)$ *and its cross-square function* $\hat{\gamma}^{\otimes 2}(s,t)$ *calculated with resolution* $M = 1,000$ *over various periods.*

[a,b]	$[-8,0]$	$[-8,8]$	$[-8,15]$
$\mathrm{tr}(\hat{\gamma})$	255	518	736
$\mathrm{tr}(\hat{\gamma}^{\otimes 2})$	53,577	190,990	365,544

Two-Sample Problem Assumptions (TS)

1. The two samples (5.1) are with $\eta_1(t), \eta_2(t) \in \mathcal{L}^2(\mathcal{T})$ and $\text{tr}(\gamma) < \infty$.
2. The two samples (5.1) are Gaussian.
3. As $n \to \infty$, the sample sizes satisfy $n_1/n \to \tau$ such that $\tau \in (0, 1)$.
4. The subject-effect functions $v_{ij}(t) = y_{ij}(t) - \eta_i(t), j = 1, 2, \cdots, n_i; i = 1, 2$ are i.i.d..
5. The subject-effect function $v_{11}(t)$ satisfies

$$\text{E}\|v_{11}\|^4 = \text{E}\left[\int_{\mathcal{T}} v_{11}^2(t)dt\right]^2 < \infty.$$

6. The maximum variance $\rho = \max_{t \in \mathcal{T}} \gamma(t, t) < \infty$.
7. The expectation $\text{E}[v_{11}^2(s)v_{11}^2(t)]$ is uniformly bounded. That is, for any $(s, t) \in \mathcal{T}^2$, we have $\text{E}[v_{11}^2(s)v_{11}^2(t)] < C < \infty$, where C is some constant independent of $(s, t) \in \mathcal{T}^2$.

THEOREM 5.1

1. The two samples (5.1) are with $\eta_1(t), \eta_2(t) \in \mathcal{L}^2(\mathcal{T})$ and $\mathrm{tr}(\gamma) < \infty$.
2. The two samples (5.1) are Gaussian.

Theorem 5.1 *Under Assumptions TS1 and TS2, we have*

$$\Delta(t) \sim GP(\eta_\Delta, \gamma), \quad \text{and} \quad (n-2)\hat{\gamma}(s,t) \sim WP(n-2, \gamma). \tag{5.6}$$

Proof of Theorem 5.1

Proof of Theorem 5.1 Under the Gaussian assumption, the first assertion is obvious. To show the second assertion, notice that $(n-2)\hat{\gamma}(s,t) = (n_1-1)\hat{\gamma}_1(s,t) + (n_2-1)\hat{\gamma}_2(s,t)$, where $\hat{\gamma}_1(s,t)$ and $\hat{\gamma}_2(s,t)$ are the sample covariance functions of the two functional samples (5.1), respectively. By Theorem 4.14 in Chapter 4, we have $(n_1-1)\hat{\gamma}_1(s,t) \sim \text{WP}(n_1-1,\gamma)$, $(n_2-1)\hat{\gamma}_2(s,t) \sim \text{WP}(n_2-1,\gamma)$ and they are independent. By Theorem 4.4, we have $(n-2)\hat{\gamma}(s,t) \sim \text{WP}(n-2,\gamma)$. The theorem is proved.

Theorem 4.4 *Let $W_i(s,t) \sim WP(n_i,\gamma)$, $i = 1, 2, \cdots, k$. Then we have*

$$W_1(s,t) + W_2(s,t) + \cdots W_k(s,t) \sim WP(n_1 + n_2 + \cdots + n_k, \gamma).$$

PIVOTAL TEST FUNCTION

To test the two-sample problem (5.2) based on the two samples (5.1), a natural pivotal test function is

$$\Delta(t) = \sqrt{n_1 n_2/n}\Big[\bar{y}_1(t) - \bar{y}_2(t)\Big], \tag{5.4}$$

sample problem (5.2). Notice that $\Delta(t)$ has its mean and covariance functions as

$$\eta_\Delta(t) = \mathrm{E}\Delta(t) = \sqrt{n_1 n_2/n}\Big[\eta_1(t) - \eta_2(t)\Big], \text{ and} \quad \mathrm{Cov}\left[\Delta(s), \Delta(t)\right] = \gamma(s,t). \tag{5.5}$$

Under the null hypothesis of (5.2), we have $\mathrm{E}\Delta(t) \equiv 0, t \in \mathcal{T}$.

In fact, under H_0 and the Gaussian assumption TS2, by Theorem 5.1, it is easy to see that

$$\sqrt{n_1 n_2/n}\Big[\bar{y}_1(t) - \bar{y}_2(t)\Big] \sim \mathrm{GP}(0, \gamma). \tag{5.7}$$

THEOREM 5.2

1. The two samples (5.1) are with $\eta_1(t), \eta_2(t) \in \mathcal{L}^2(\mathcal{T})$ and $\mathrm{tr}(\gamma) < \infty$.

3. As $n \to \infty$, the sample sizes satisfy $n_1/n \to \tau$ such that $\tau \in (0, 1)$.
4. The subject-effect functions $v_{ij}(t) = y_{ij}(t) - \eta_i(t), j = 1, 2, \cdots, n_i; i = 1, 2$ are i.i.d..

Theorem 5.2 *Under Assumptions TS1, TS3, and TS4, as $n \to \infty$, we have*

$$\Delta(t) - \eta_\Delta(t) \xrightarrow{d} GP(0, \gamma),$$

where $\eta_\Delta(t)$ is as defined in (5.5).

Proof of Theorem 5.2

Proof of Theorem 5.2 Notice that

$$\Delta(t) - \eta_\Delta(t) = \sqrt{n_2/n}\sqrt{n_1}\left[\hat{\eta}_1(t) - \eta_1(t)\right] + \sqrt{n_1/n}\sqrt{n_2}\left[\hat{\eta}_2(t) - \eta_2(t)\right].$$

Under Assumptions TS1, TS3, and TS4, as $n \to \infty$, we have $n_1/n \to \tau, n_2/n \to 1-\tau$ and by Theorem 4.15 in Chapter 4, we have

$$\sqrt{n_1}\left[\hat{\eta}_1(t) - \eta_1(t)\right] \xrightarrow{d} \mathrm{GP}(0,\gamma),\ \sqrt{n_2}\left[\hat{\eta}_2(t) - \eta_2(t)\right] \xrightarrow{d} \mathrm{GP}(0,\gamma).$$

The theorem is then proved.

Theorem 4.15 *Under Assumption OS1, as $n \to \infty$, we have*

$$\sqrt{n}\left\{\hat{\eta}(t) - \eta(t)\right\} \xrightarrow{d} GP(0,\gamma),$$

where and throughout, "$\xrightarrow{d}$" denotes "convergence in distribution."

THEOREM 5.3

1. The two samples (5.1) are with $\eta_1(t), \eta_2(t) \in \mathcal{L}^2(\mathcal{T})$ and $\mathrm{tr}(\gamma) < \infty$.
3. As $n \to \infty$, the sample sizes satisfy $n_1/n \to \tau$ such that $\tau \in (0, 1)$.
4. The subject-effect functions $v_{ij}(t) = y_{ij}(t) - \eta_i(t), j = 1, 2, \cdots, n_i; i = 1, 2$ are i.i.d..
5. The subject-effect function $v_{11}(t)$ satisfies

$$\mathrm{E}\|v_{11}\|^4 = \mathrm{E}\left[\int_{\mathcal{T}} v_{11}^2(t)dt\right]^2 < \infty.$$

6. The maximum variance $\rho = \max_{t \in \mathcal{T}} \gamma(t, t) < \infty$.

Theorem 5.3 *Under Assumptions TS1 and TS3 through TS6, as $n \to \infty$, we have*

$$\sqrt{n}\left\{\hat{\gamma}(s, t) - \gamma(s, t)\right\} \xrightarrow{d} GP(0, \varpi),$$

where $\varpi\{(s_1, t_1), (s_2, t_2)\} = E\{v_{11}(s_1)v_{11}(t_1)v_{11}(s_2)v_{11}(t_2)\} - \gamma(s_1, t_1)\gamma(s_2, t_2)$.

Proof of Theorem 5.3

Proof of Theorem 5.3 Notice that

$$\sqrt{n}\,[\hat{\gamma}(s,t) - \gamma(s,t)] = a_n\sqrt{n_1}\,[\hat{\gamma}_1(s,t) - \gamma(s,t)] + b_n\sqrt{n_2}\,[\hat{\gamma}_2(s,t) - \gamma(s,t)]\,,$$

where $\hat{\gamma}_1(s,t)$ and $\hat{\gamma}_2(s,t)$ are the sample covariance functions of the two functional samples (5.1), respectively, and $a_n = [\sqrt{n}(n_1 - 1)]/[\sqrt{n_1}(n-2)]$ and $b_n = [\sqrt{n}(n_2 - 1)]/[\sqrt{n_2}(n-2)]$. As $n \to \infty$, we have $a_n \to \sqrt{\tau}$ and $b_n \to \sqrt{1-\tau}$. In addition, under the given conditions, by Theorem 4.16 in Chapter 4, we have

$$\sqrt{n_1}[\hat{\gamma}_1(s,t) - \gamma(s,t)] \xrightarrow{d} \mathrm{GP}(0,\varpi), \quad \sqrt{n_2}[\hat{\gamma}_2(s,t) - \gamma(s,t)] \xrightarrow{d} \mathrm{GP}(0,\varpi),$$

where $\varpi\{(s_1,t_1),(s_2,t_2)\} = \mathrm{E}v_{11}(s_1)v_{11}(t_1)v_{11}(s_2)v_{11}(t_2) - \gamma(s_1,t_1)\gamma(s_2,t_2)$. The theorem is then proved.

POINTWISE TESTS

IDEA OF THE TEST

The key idea of a pointwise test is to test the null hypothesis at each time point $t \in \mathcal{T}$. For any fixed $t \in \mathcal{T}$, the sub-problem is

$$H_{0t} : \eta_1(t) = \eta_2(t), \quad \text{versus} \quad H_{1t} : \eta_1(t) \neq \eta_2(t). \tag{5.8}$$

Based on the sample mean functions and the pooled sample covariance function given in (5.3), the pivotal test statistic for (5.8) is

$$z(t) = \frac{[\bar{y}_1(t) - \bar{y}_2(t)]}{\sqrt{(1/n_1 + 1/n_2)\hat{\gamma}(t,t)}} = \frac{\Delta(t)}{\sqrt{\hat{\gamma}(t,t)}}, \tag{5.9}$$

where $\Delta(t)$ is the pivotal test function defined in (5.4).

UNDER H_0

In many situations, the two samples (5.1) may be approximately Gaussian. That is, Assumption TS2 is approximately satisfied. By Theorem 5.1 and under H_{0t}, we have

$$z(t) \sim t_{n-2}, \quad t \in \mathcal{T}. \tag{5.10}$$

When the Gaussian assumption is not satisfied, for large samples, one may use the pointwise z-test instead. As $n_1, n_2 \to \infty$, by Theorem 5.2, we have

$$z(t) \xrightarrow{d} N(0,1), \text{ for any fixed } t \in \mathcal{T}. \tag{5.12}$$

BOOTSTRAP

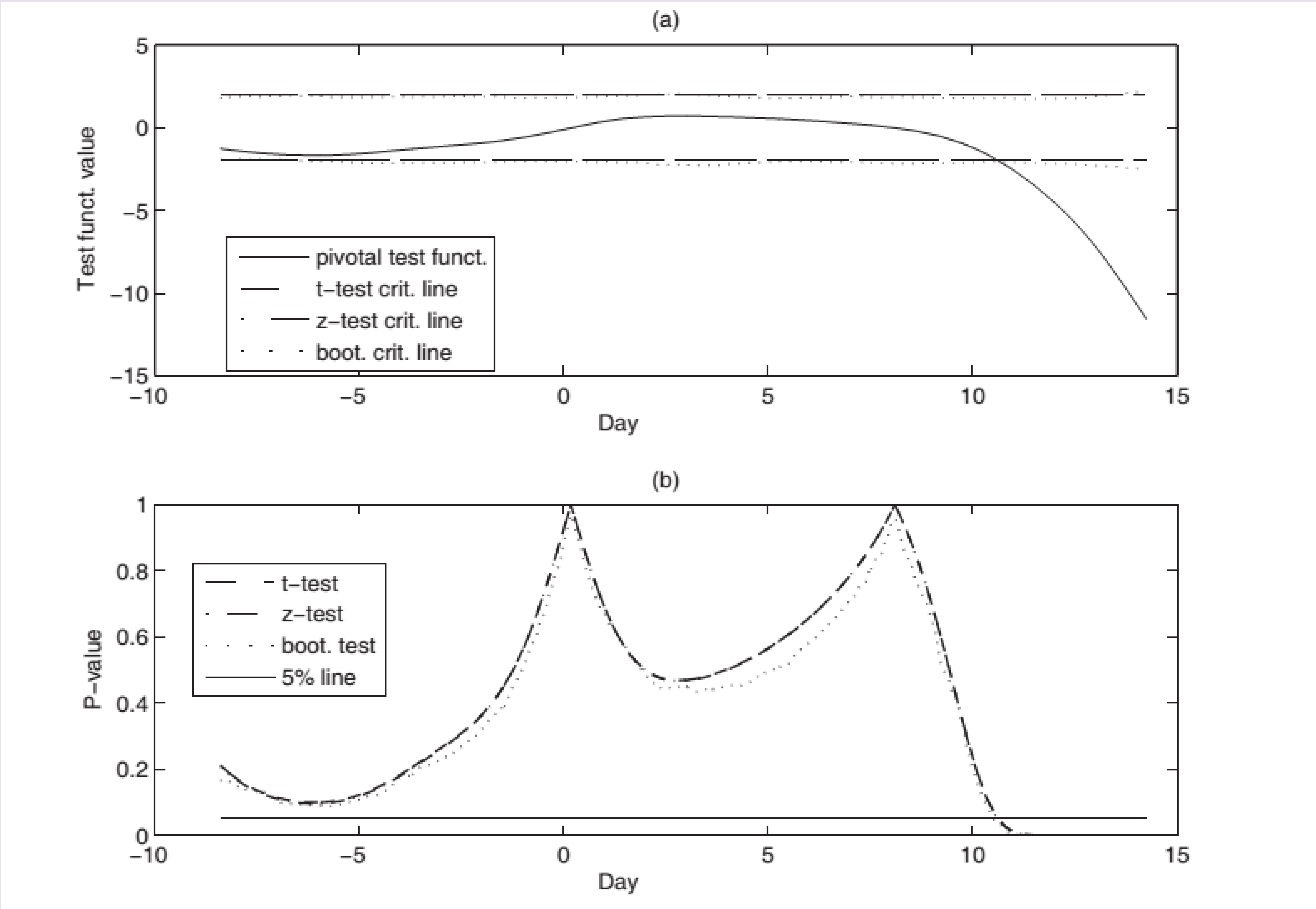
When the two samples (5.1) are not Gaussian while both n_1 and n_2 are small, the above pointwise t- and z-tests are not preferred. In this case, one may resort to a pointwise bootstrap test. Let $v_{ij}^*(t), j = 1, 2, \cdots, n_i; i = 1, 2,$ be bootstrapped from the estimated subject-effect functions $\hat{v}_{ij}(t) = y_{ij}(t) - \hat{\eta}_i(t), j = 1, 2, \cdots, n_i; i = 1, 2.$ Set

$$y_{ij}^*(t) = \hat{\eta}_i(t) + v_{ij}^*(t), j = 1, 2, \cdots, n_i; i = 1, 2. \tag{5.13}$$

Then we can compute the sample mean functions and the pooled sample covariance function $\bar{y}_i^*(t), i = 1, 2,$ and $\hat{\gamma}^*(s, t)$ as in (5.3) but now based on the two bootstrapped samples (5.13). For the pointwise bootstrap test, we compute

$$z^*(t) = \frac{([\bar{y}_1^*(t) - \bar{y}_2^*(t)] - [\bar{y}_1(t) - \bar{y}_2(t)])}{\sqrt{(\frac{1}{n_1} + \frac{1}{n_2})\hat{\gamma}^*(t, t)}}.$$

EXAMPLE



(a)
Test funct. value
pivotal test funct.
t−test crit. line
z−test crit. line
boot. crit. line
Day
(b)
P−value
t−test
z−test
boot. test
5% line
Day

L2 NORM-BASED TEST

IDEA OF THE TEST

L^2-Norm-Based Test For the two-sample problem (5.2), the L^2-norm-based test uses the squared L^2-norm of the pivotal test function $\Delta(t)$ (5.4) as the test statistic:

$$T_n = \int_{\mathcal{T}} \Delta^2(t)dt = \frac{n_1 n_2}{n} \int_{\mathcal{T}} [\bar{y}_1(t) - \bar{y}_2(t)]^2 dt. \qquad (5.14)$$

UNDER H_0

If sample is Gaussian

$$T_n \stackrel{d}{=} \sum_{r=1}^{m} \lambda_r A_r, \; A_r \stackrel{i.i.d.}{\sim} \chi_1^2,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_m$ are all the positive eigenvalues of the common covariance function $\gamma(s,t)$. It follows that the null distribution of T_n can be approximated using the methods described in Section 4.3 of Chapter 4. In fact, by the Welch-Satterthwaite χ^2-approximation method described there, we have

$$T_n \sim \beta \chi_d^2 \text{ approximately}, \quad \text{where } \beta = \frac{\operatorname{tr}(\gamma^{\otimes 2})}{\operatorname{tr}(\gamma)}, \; d = \kappa = \frac{\operatorname{tr}^2(\gamma)}{\operatorname{tr}(\gamma^{\otimes 2})}. \quad (5.15)$$

APPROXIMATION

$$T_n \sim \hat{\beta}\chi^2_{\hat{d}} \text{ approximately.}$$

NAIVE →

$$\hat{\beta} = \frac{\mathrm{tr}(\hat{\gamma}^{\otimes 2})}{\mathrm{tr}(\hat{\gamma})}, \quad \hat{d} = \hat{\kappa} = \frac{\mathrm{tr}^2(\hat{\gamma})}{\mathrm{tr}(\hat{\gamma}^{\otimes 2})},$$

BIAS - REDUCED →

$$\hat{\beta} = \frac{\frac{(n-2)^2}{(n-3)n}\left[\mathrm{tr}(\hat{\gamma}^{\otimes 2}) - \frac{\mathrm{tr}^2(\hat{\gamma})}{n-2}\right]}{\mathrm{tr}(\hat{\gamma})},$$

$$\hat{d} = \hat{\kappa} = \frac{(n-1)\left[\mathrm{tr}^2(\hat{\gamma}) - \frac{2\mathrm{tr}(\hat{\gamma}^{\otimes 2})}{n-1}\right]}{(n-2)\left[\mathrm{tr}(\hat{\gamma}^{\otimes 2}) - \frac{\mathrm{tr}^2(\hat{\gamma})}{n-2}\right]}.$$

CONSISTENCY

Theorem 5.4 *Under Assumptions TS1 and TS3 through TS7, as* $n \to \infty$*, we have* $tr(\hat{\gamma}) \xrightarrow{p} tr(\gamma)$ *and* $tr(\hat{\gamma}^{\otimes 2}) \xrightarrow{p} tr(\gamma^{\otimes 2})$*. Furthermore, as* $n \to \infty$*, we have*

$$\hat{\beta} \xrightarrow{p} \beta, \quad \hat{\kappa} \xrightarrow{p} \kappa,$$

where $\hat{\beta}$ *and* $\hat{\kappa}$ *are the naive or bias-reduced estimators of* β *and* κ*, respectively.*

Proof

Proof of Theorem 5.4 Under the given conditions, by Theorem 5.3, as $n \to \infty$, we have $\mathrm{E}[\hat{\gamma}(s,t) - \gamma(s,t)]^2 = \frac{\omega[(s,t),(s,t)]}{n}[1 + o(1)]$. By Assumptions TS6 and TS7, we have

$$|\omega[(s,t),(s,t)]| \le \mathrm{E}[v_{11}^2(s)v_{11}^2(t)] + \gamma^2(s,t) \le C + \rho, \quad \text{for all } (s,t) \in \mathcal{T}^2.$$

It follows that, as $n \to \infty$, we have $\hat{\gamma}(s,t) = \gamma(s,t) + O_{UP}(n^{-1/2})$, $(s,t) \in \mathcal{T}^2$, where O_{UP} means "uniformly bounded in probability." Thus, as $n \to \infty$, we have $\hat{\gamma}(s,t) \xrightarrow{p} \gamma(s,t)$ uniformly over $\mathcal{T}^2$. Therefore, as $n \to \infty$, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{tr}(\hat{\gamma}) &= \int_{\mathcal{T}} \lim_{n\to\infty} \hat{\gamma}(t,t)dt = \int_{\mathcal{T}} \gamma(t,t)dt = \mathrm{tr}(\gamma), \\
\lim_{n\to\infty} \mathrm{tr}(\hat{\gamma}^{\otimes 2}) &= \int_{\mathcal{T}}\int_{\mathcal{T}} \lim_{n\to\infty} \hat{\gamma}^2(s,t)dsdt \\
&= \int_{\mathcal{T}}\int_{\mathcal{T}} \gamma^2(s,t)dsdt = \mathrm{tr}(\gamma^{\otimes 2}),
\end{aligned}$$

in probability. It follows from (5.16) and (5.19) that as $n \to \infty$, $\hat{\beta} \xrightarrow{p} \beta$ and $\hat{\kappa} \xrightarrow{p} \kappa$. The theorem is proved.

EXAMPLE

Table 5.3 *The L^2-norm-based test for the two-sample problem (5.2) for the progesterone data with resolution $M = 1,000$.*

Method	$[a, b]$	T_n	$\hat{\beta}$	$\hat{d} = \hat{\kappa}$	P-value
Naive	$[-8, 0]$	407.4	210.4	1.21	0.208
	$[-8, 8]$	513.2	368.6	1.41	0.347
	$[-8, 15]$	3,751	496.7	1.48	0.012
Bias-reduced	$[-8, 0]$	407.4	205.2	1.22	0.203
	$[-8, 8]$	513.2	358.9	1.42	0.343
	$[-8, 15]$	3,751	483.2	1.50	0.011

Note: The P-values by the naive method are generally comparable with those by the bias-reduced method although the latter are generally smaller than the former.

F TYPE TEST

IDEA OF THE TEST

$$F_n = \frac{\|\Delta\|^2}{\mathrm{tr}(\hat{\gamma})} = \frac{\frac{n_1 n_2}{n} \int_{\mathcal{T}} [\bar{y}_1(t) - \bar{y}_2(t)]^2 dt}{\mathrm{tr}(\hat{\gamma})}. \qquad (5.21)$$

$$F_n \sim F_{\kappa,(n-2)\kappa} \text{ approximately,}$$

EXAMPLE

Table 5.4 *The F-type test for the two-sample problem (5.2) for the progesterone data with resolution $M = 1,000$.*

Method	$[a, b]$	F_n	$\hat{\kappa}$	$(n-2)\hat{\kappa}$	P-value
Naive	$[-8, 0]$	1.60	1.21	108	0.211
	$[-8, 8]$	0.99	1.41	125	0.349
	$[-8, 15]$	5.10	1.48	132	0.014
Bias-reduced	$[-8, 0]$	1.60	1.22	108	0.211
	$[-8, 8]$	0.99	1.42	126	0.350
	$[-8, 15]$	5.10	1.50	134	0.014

Note: The P-values by the naive method are generally comparable with those by the bias-reduced method.

BOOTSTRAP TEST

IDEA OF THE TEST

For the L^2-norm-based bootstrap test, we compute the bootstrap test statistic $T_n^* = \|\Delta^*\|^2$ with the pivotal test function

$$\Delta^*(t) = \sqrt{\frac{n_1 n_2}{n}} \left[(\bar{y}_1^*(t) - \bar{y}_2^*(t)) - (\bar{y}_1(t) - \bar{y}_2(t)) \right].$$

$$F_n^* = \frac{\|\Delta^*\|}{\operatorname{tr}(\hat{\gamma}^*)}.$$

EXAMPLE

Table 5.5 *The bootstrap tests for the two-sample problem (5.2) for the progesterone data with resolution* $M = 1,000$.

	L^2-norm-based bootstrap test		F-type bootstrap test	
$[a, b]$	T_n	P-value	F_n	P-value
$[-8, 0]$	407.4	0.203	1.60	0.214
$[-8, 8]$	513.2	0.340	0.99	0.344
$[-8, 15]$	3,751	0.014	5.10	0.016

Note: The number of bootstrap replicates is $N = 10,000$. *The P-values by the bootstrap method are generally comparable with those by the* L^2*-norm-based test and by the* F*-type test, which are presented in Tables 5.3 and 5.4, respectively.*

TEMPERATURE EXAMPLE

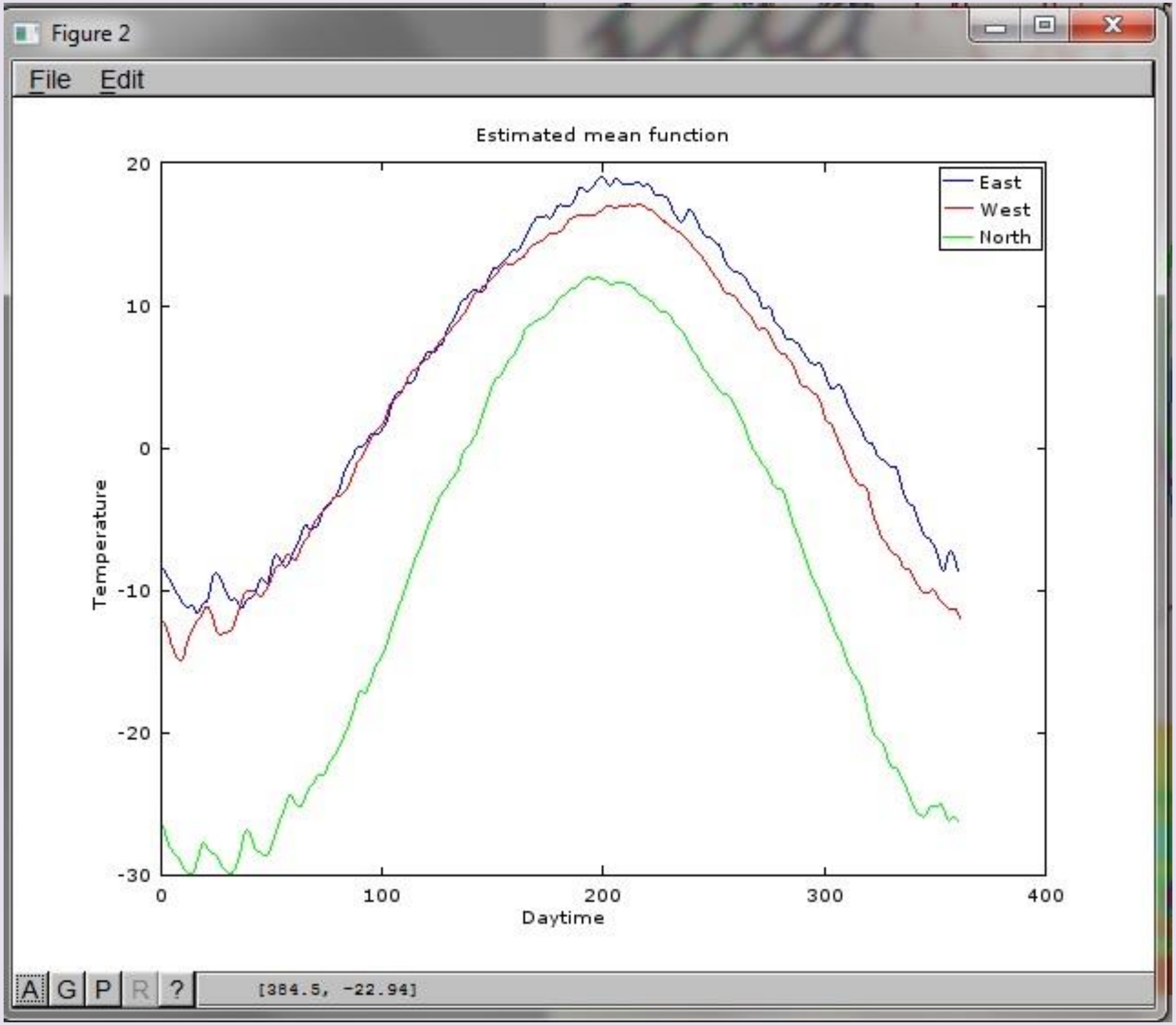


Figure 2
File Edit
Estimated mean function
East
West
North
Temperature
Daytime
20
10
0
-10
-20
-30
0
100
200
300
400
A G P R ?
[384.5, -22.94]

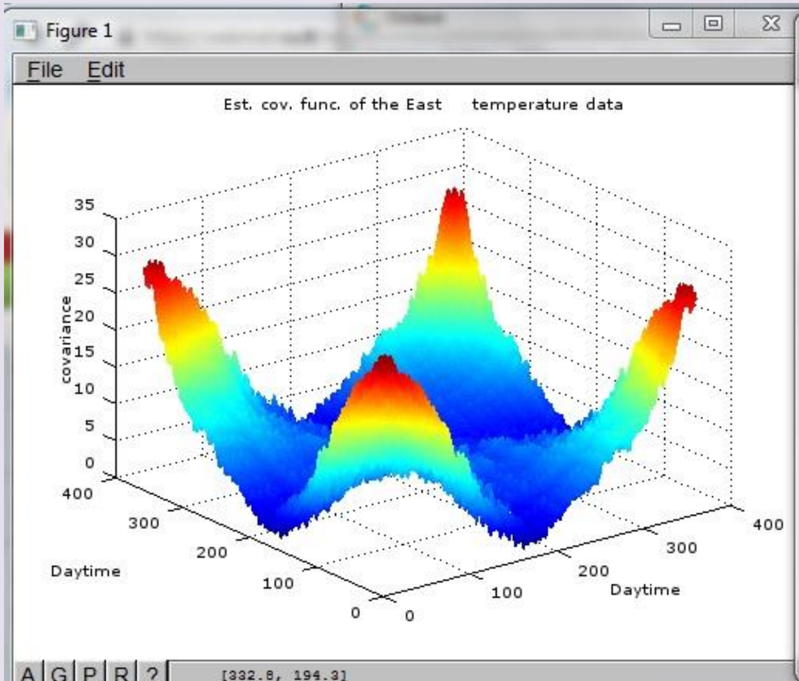


Figure 1
File Edit
Est. cov. func. of the East temperature data
covariance
35
30
25
20
15
10
5
0
400
300
200
100
0
Daytime
0
100
200
300
400
Daytime
A G P R ?
[332.8, 194.3]

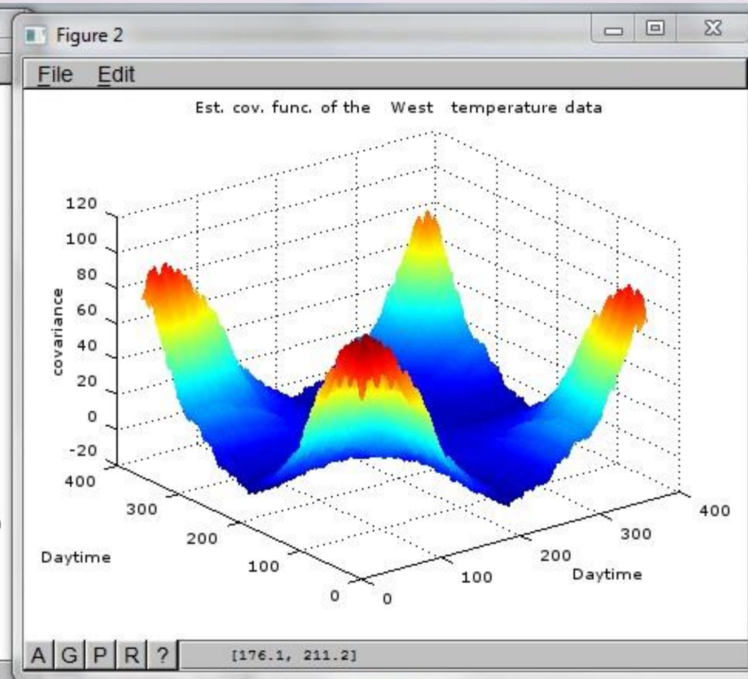


Figure 2
File Edit
Est. cov. func. of the West temperature data
covariance
120
100
80
60
40
20
0
-20
400
300
200
100
0
Daytime
0
100
200
300
400
Daytime
A G P R ?
[176.1, 211.2]

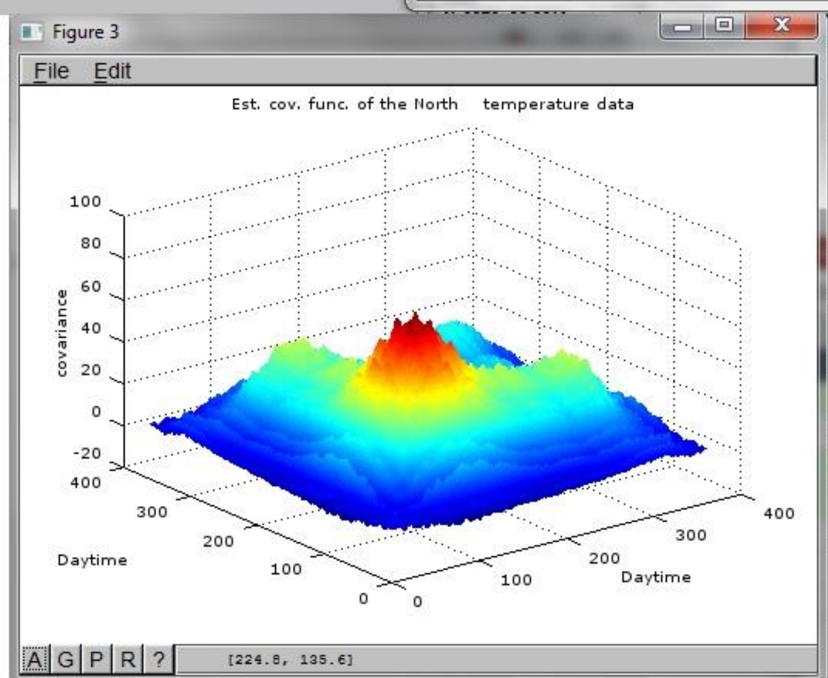


Figure 3
File Edit
Est. cov. func. of the North temperature data
covariance
100
80
60
40
20
0
-20
400
300
200
100
0
Daytime
0
100
200
300
400
Daytime
A G P R ?
[224.8, 135.6]

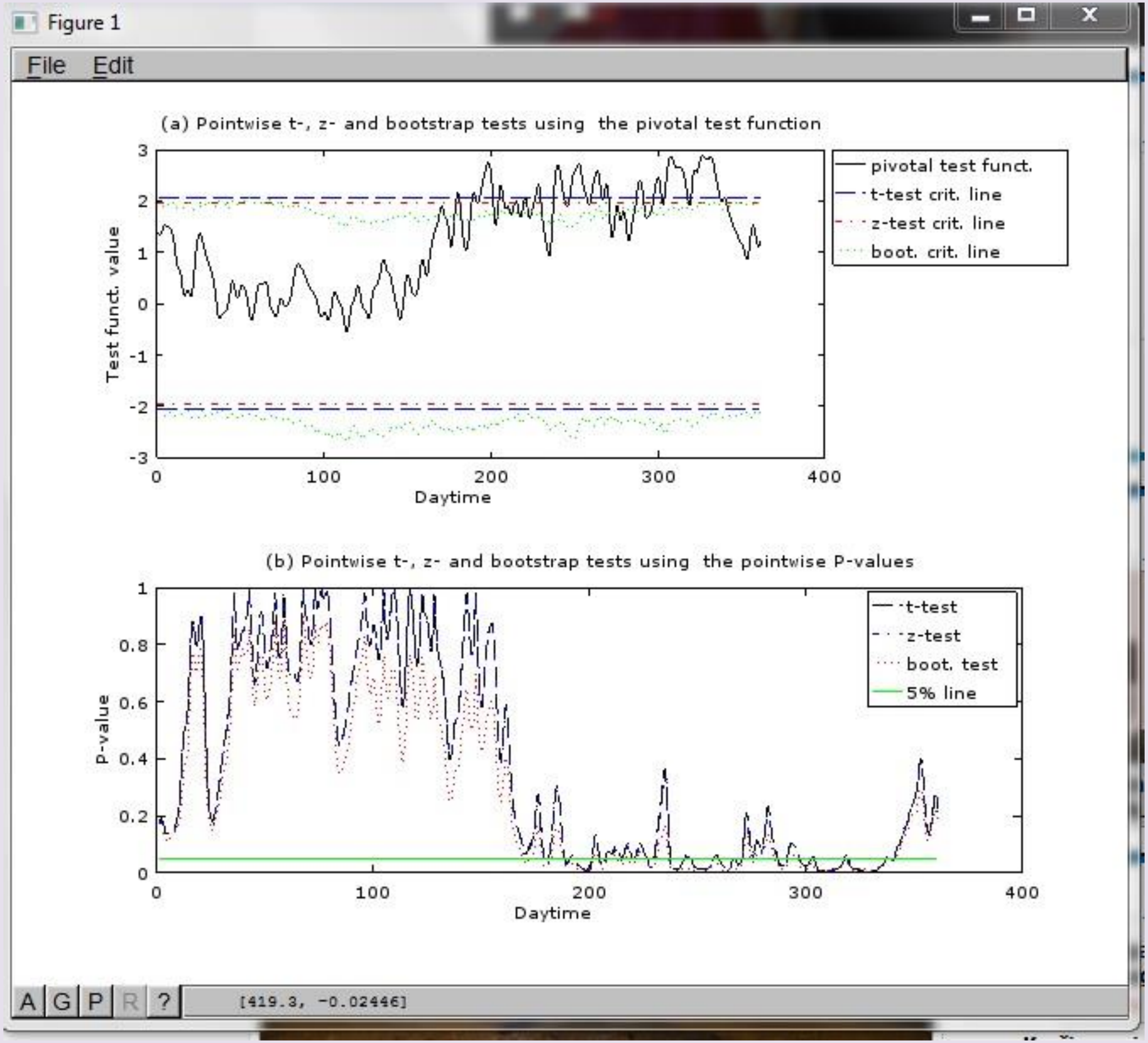


Figure 1
File Edit
(a) Pointwise t-, z- and bootstrap tests using the pivotal test function
pivotal test funct.
t-test crit. line
z-test crit. line
boot. crit. line
Test funct. value
Daytime
(b) Pointwise t-, z- and bootstrap tests using the pointwise P-values
t-test
z-test
boot. test
5% line
P-value
Daytime
[419.3, -0.02446]

EXAMPLE

	Method	T_n	$\widehat{\beta}$	$\widehat{d} = \hat{\kappa}$	P-value
Mean(East) = Mean(West)	Naive	40576	18787	1.29	0.19674
	Bias-reduced	40576	17343	1.33	0.18309
	Bootstrap	40576	---	---	0.1892

	Method	F test	$\hat{\kappa}$	$(n-1)\hat{\kappa}$	P-value
Mean(East) = Mean(West)	Naive	1.668	1.2948	36.255	0.20676
	Bias-reduced	1.668	1.3312	37.273	0.20651
	Bootstrap	1.668	---	---	0.2127

NEXT TIME

ONE-WAY ANOVA