ONE-WAY ANOVA FOR FUNCTIONAL DATA

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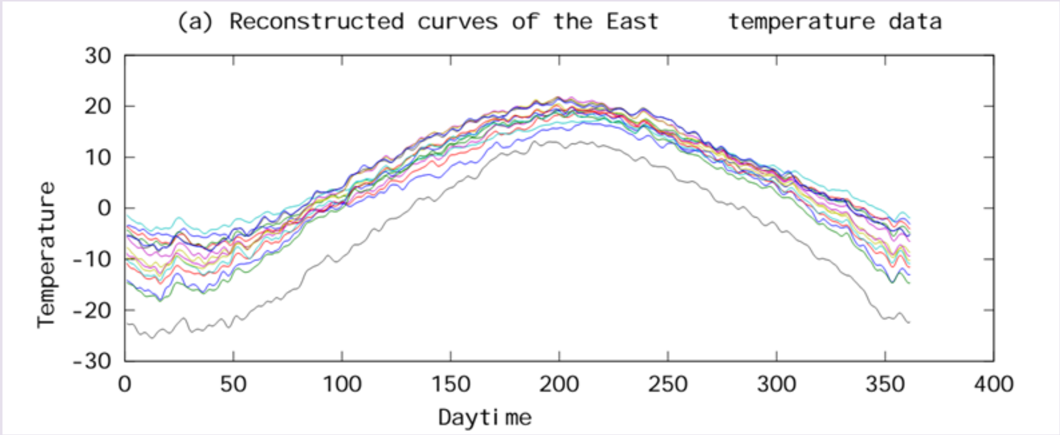
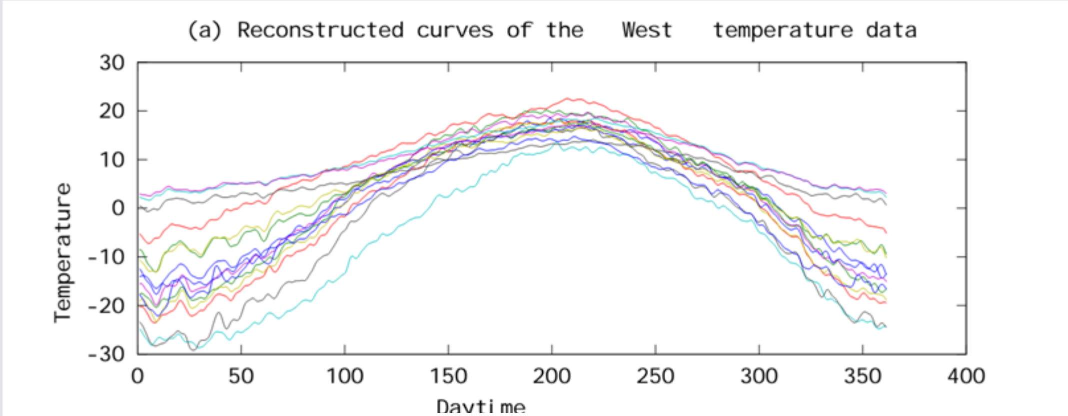
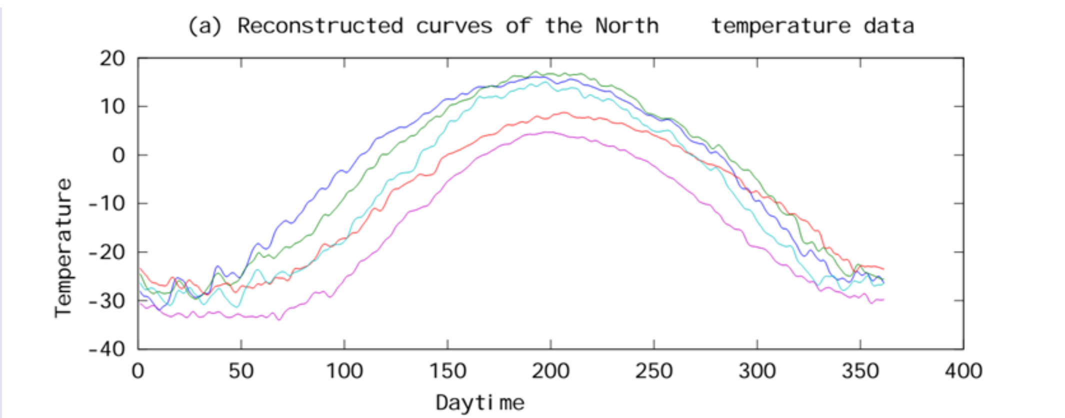

(a) Reconstructed curves of the East temperature data
Temperature
Daytime
30
20
10
0
-10
-20
-30
0
50
100
150
200
250
300
350
400


(a) Reconstructed curves of the West temperature data
Temperature
Daytime
30
20
10
0
-10
-20
-30
0
50
100
150
200
250
300
350
400


(a) Reconstructed curves of the North temperature data
Temperature
Daytime
20
10
0
-10
-20
-30
-40
0
50
100
150
200
250
300
350
400

GENERAL PROBLEM

We can define the one-way ANOVA problem for functional data as follows. Suppose we have k independent samples:

$$y_{i1}(t), \cdots, y_{in_i}(t), \; i = 1, \cdots, k. \tag{5.23}$$

These k samples satisfy

$$\begin{array}{c} y_{ij}(t) = \eta_i(t) + v_{ij}(t), \; v_{ij}(t) \overset{i.i.d.}{\sim} \mathrm{SP}(0, \gamma), \\ j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k, \end{array} \tag{5.24}$$

where $\eta_1(t), \eta_2(t), \cdots, \eta_k(t)$ are the unknown group mean functions of the k samples, $v_{ij}(t), j = 1, \cdots, n_i; i = 1, 2, \cdots, k$ are the subject-effect functions, and $\gamma(s, t)$ is the common covariance function.

GENERAL PROBLEM

We wish to test the following one-way ANOVA testing problem:

$$H_0 : \eta_1(t) \equiv \eta_2(t) \equiv \cdots \equiv \eta_k(t), \ t \in \mathcal{T}, \qquad (5.25)$$

where again $\mathcal{T}$ is some time period of interest, often specified as $[a, b]$ with $-\infty < a < b < \infty$.

MAIN-EFFECT TEST

Main-Effect Test Set $\eta_i(t) = \eta(t) + \alpha_i(t), i = 1, 2, \cdots, k$, where $\eta(t)$ is known as the overall mean function of the k samples and $\alpha_i(t)$ is the ith main-effect function for $i = 1, 2, \cdots, k$. Then the model (5.24) can be further written as the following standard one-way ANOVA model for functional data:

$$y_{ij}(t) = \eta(t) + \alpha_i(t) + v_{ij}(t), \; j = 1, 2, \cdots, n_i; \; i = 1, 2, \cdots, k. \tag{5.26}$$

In this formulation, the null hypothesis (5.25) can be equivalently expressed as

$$\alpha_1(t) \equiv \alpha_2(t) \equiv \cdots \equiv \alpha_k(t) \equiv 0, t \in \mathcal{T}, \tag{5.27}$$

that is, to test if the main-effect functions are the same and are equal to 0.

POST HOC TEST

Post Hoc Test When the test (5.27) is accepted, the one-way ANOVA model (5.26) is not significant. When it is rejected, further investigation is often required. For example, one may want to know if any two main-effect functions $\alpha_i(t)$ and $\alpha_j(t)$ are the same, where i and j are any two integers such that $1 \le i < j \le k$. This test can be written as

$$\begin{aligned} & H_0 : \alpha_i(t) \equiv \alpha_j(t), t \in \mathcal{T} \\ \text{versus} \quad & H_1 : \alpha_i(t) \neq \alpha_j(t), \text{ for some } t \in \mathcal{T}. \end{aligned} \tag{5.28}$$

The above test is known as a post hoc test. Obviously, it can be equivalently written as

$$\begin{aligned} & H_0 : \eta_i(t) \equiv \eta_j(t), t \in \mathcal{T} \\ \text{versus} \quad & H_1 : \eta_i(t) \neq \eta_j(t), \text{ for some } t \in \mathcal{T}. \end{aligned} \tag{5.29}$$

CONTRAST TEST

Contrast Test The post hoc tests are special cases of contrast tests. Let $a_1, \cdots, a_k$ be k constants such that they add up to 0, that is, $\sum_{i=1}^k a_i = \mathbf{a}^T \mathbf{1}_k = 0$, where $\mathbf{a} = [a_1, \cdots, a_k]^T$ and $\mathbf{1}_k$ is a column vector of k ones. A contrast is defined as $\sum_{i=1}^k a_i \alpha_i(t) = \mathbf{a}^T \boldsymbol{\alpha}(t)$, a linear combination of the main-effect functions $\alpha_i(t), i = 1, 2, \cdots, k$, where $\boldsymbol{\alpha}(t) = [\alpha_1(t), \cdots, \alpha_k(t)]^T$ consists of all the main-effect functions. A simple contrast is the difference of two main-effect functions, for example, $\alpha_1(t) - \alpha_2(t)$. Another simple example of contrast is $\alpha_1(t) - 3\alpha_2(t) + 2\alpha_5(t)$ when $k \geq 5$. For a given $\mathbf{a} \in \mathcal{R}^k$ such that $\mathbf{a}^T \mathbf{1}_k = 0$, a contrast test is defined as

$$H_0 : \mathbf{a}^T \boldsymbol{\alpha}(t) \equiv 0, \ t \in \mathcal{T} \quad \text{versus} \quad H_1 : \mathbf{a}^T \boldsymbol{\alpha}(t) \neq 0, \ \text{for some } t \in \mathcal{T}. \quad (5.30)$$

As $\mathbf{a}^T \mathbf{1}_k = 0$, the above test can be equivalently written as

$$H_0 : \mathbf{a}^T \boldsymbol{\eta}(t) \equiv 0, t \in \mathcal{T} \quad \text{versus} \quad H_1 : \mathbf{a}^T \boldsymbol{\eta}(t) \neq 0, \ \text{for some } t \in \mathcal{T}, \quad (5.31)$$

where $\boldsymbol{\eta}(t) = [\eta_1(t), \eta_2(t), \cdots, \eta_k(t)]^T$.

ESTIMATION OF GROUP MEAN AND COVARIANCE FUNCTIONS

UNBIASED ESTIMATORS

$$\begin{aligned} \hat{\eta}_i(t) &= \bar{y}_{i\cdot}(t) = n_i^{-1} \sum_{j=1}^{n_i} y_{ij}(t), \quad i = 1, 2, \cdots, k, \\ \hat{\gamma}(s,t) &= (n-k)^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} [y_{ij}(s) - \bar{y}_{i\cdot}(s)][y_{ij}(t) - \bar{y}_{i\cdot}(t)], \end{aligned} \tag{5.32}$$

$$\mathrm{E}\hat{\eta}_i(t) = \eta_i(t), \quad \mathrm{cov}\left[\hat{\eta}_i(s), \hat{\eta}_i(t)\right] = \gamma(s,t)/n_i, i = 1, 2, \cdots, k.$$

Set $\hat{\boldsymbol{\eta}}(t) = [\hat{\eta}_1(t), \hat{\eta}_2(t), \cdots, \hat{\eta}_k(t)]^T$. It is an unbiased estimator of $\boldsymbol{\eta}(t)$. Then we have $\mathrm{E}\hat{\boldsymbol{\eta}}(t) = \boldsymbol{\eta}(t)$ and $\mathrm{Cov}\left[\hat{\boldsymbol{\eta}}(s), \hat{\boldsymbol{\eta}}(t)\right] = \gamma(s,t)\mathbf{D}$, where $\mathbf{D} =$

$\mathrm{diag}(1/n_1, 1/n_2, \cdots, 1/n_k)$ is a diagonal matrix with diagonal entries $1/n_i, i = 1, 2, \cdots, k$. That is, $\hat{\boldsymbol{\eta}}(t) \sim \mathrm{SP}_k(\boldsymbol{\eta}, \gamma\mathbf{D})$, where $\mathrm{SP}_k(\boldsymbol{\eta}, \boldsymbol{\Gamma})$ denotes a k-dimensional stochastic process having the vector of mean functions $\boldsymbol{\eta}(t)$ and the matrix of covariance functions $\boldsymbol{\Gamma}(s,t)$.

ASSUMPTIONS

One-Way ANOVA Assumptions (KS)

1. The k samples (5.23) are with $\eta_1(t), \eta_2(t), \cdots, \eta_k(t) \in \mathcal{L}^2(\mathcal{T})$ and $\mathrm{tr}(\gamma) < \infty$.
2. The k samples (5.23) are Gaussian.
3. As $n \to \infty$, the k sample sizes satisfy $n_i/n \to \tau_i,\ i = 1, 2, \cdots, k$ such that $\tau_1, \tau_2, \cdots, \tau_k \in (0, 1)$.
4. The subject-effect functions $v_{ij}(t) = y_{ij}(t) - \eta_i(t), j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k$ are i.i.d..
5. The subject-effect function $v_{11}(t)$ satisfies $\mathrm{E}\|v_{11}\|^4 < \infty$.
6. The maximum variance $\rho = \max_{t \in \mathcal{T}} \gamma(t, t) < \infty$.
7. The expectation $\mathrm{E}[v_{11}^2(s)v_{11}^2(t)]$ is uniformly bounded.

THEOREM 5.5

1. The k samples (5.23) are with $\eta_1(t), \eta_2(t), \cdots, \eta_k(t) \in \mathcal{L}^2(\mathcal{T})$ and $\text{tr}(\gamma) < \infty$.
2. The k samples (5.23) are Gaussian.

Theorem 5.5 *Under Assumptions KS1 and KS2, we have*

$$\begin{aligned} \mathbf{D}^{-1/2}\left[\hat{\boldsymbol{\eta}}(t) - \boldsymbol{\eta}(t)\right] &\sim GP_k(\mathbf{0}, \gamma \mathbf{I}_k), \quad \textit{and} \\ (n-k)\hat{\gamma}(s,t) &\sim WP(n-k, \gamma). \end{aligned} \tag{5.33}$$

PROOF OF THE THEOREM 5.5

Proof of Theorem 5.5 Notice that $\mathbf{D}^{-1/2}[\hat{\boldsymbol{\eta}}(t)-\boldsymbol{\eta}(t)] = [z_1(t), z_2(t), \cdots, z_k(t)]^T$, where $z_i(t) = \sqrt{n_i}[\hat{\eta}_i(t) - \eta_i(t)], i = 1, 2, \cdots, k$ are independent and by Theorem 4.14, we have $z_i(t) \sim \text{GP}(0, \gamma), i = 1, 2, \cdots, k$. It follows that $\mathbf{D}^{-1/2}[\hat{\boldsymbol{\eta}}(t)-\boldsymbol{\eta}(t)] \sim \text{GP}_k(\mathbf{0}, \gamma\mathbf{I}_k)$ as desired. To show the second assertion, notice that $(n-k)\hat{\gamma}(s,t) = \sum_{i=1}^{k}(n_i-1)\hat{\gamma}_i(s,t)$, where $\hat{\gamma}_i(s,t), i = 1, 2, \cdots, k$ are the sample covariance functions of the k functional samples (5.23). By Theorem 4.14, we have $(n_i-1)\hat{\gamma}_i(s,t) \sim \text{WP}(n_i-1, \gamma), i = 1, 2, \cdots, k$ and they are independent. Then by Theorem 4.4, we have $(n-k)\hat{\gamma}(s,t) \sim \text{WP}(n-k, \gamma)$. The theorem is proved.

Theorem 4.4 *Let* $W_i(s,t) \sim WP(n_i, \gamma),\ i = 1, 2, \cdots, k$. *Then we have*

$$W_1(s,t) + W_2(s,t) + \cdots W_k(s,t) \sim WP(n_1 + n_2 + \cdots + n_k, \gamma).$$

THEOREM 5.6

1. The k samples (5.23) are with $\eta_1(t), \eta_2(t), \cdots, \eta_k(t) \in \mathcal{L}^2(\mathcal{T})$ and $\mathrm{tr}(\gamma) < \infty$.

3. As $n \to \infty$, the k sample sizes satisfy $n_i/n \to \tau_i$, $i = 1, 2, \cdots, k$ such that $\tau_1, \tau_2, \cdots, \tau_k \in (0, 1)$.
4. The subject-effect functions $v_{ij}(t) = y_{ij}(t) - \eta_i(t), j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k$ are i.i.d..

Theorem 5.6 *Under Assumptions KS1, KS3, and KS4, as $n \to \infty$, we have*

$$\mathbf{D}^{-1/2}\left[\hat{\boldsymbol{\eta}}(t) - \boldsymbol{\eta}(t)\right] \xrightarrow{d} GP_k(\mathbf{0}, \gamma\mathbf{I}_k).$$

PROOF OF THE THEOREM 5.6

Proof of Theorem 5.6 Notice that $\mathbf{D}^{-1/2}[\hat{\boldsymbol{\eta}}(t)-\boldsymbol{\eta}(t)] = [z_1(t), z_2(t), \cdots, z_k(t)]^T$, where $z_i(t) = \sqrt{n_i}[\hat{\eta}_i(t) - \eta_i(t)], i = 1, 2, \cdots, k$ are independent. Under the given conditions and by Theorem 4.15, as $n \to \infty$, we have $z_i(t) \xrightarrow{d} \mathrm{GP}(0, \gamma), i = 1, 2, \cdots, k$. It follows that $\mathbf{D}^{-1/2}[\hat{\boldsymbol{\eta}}(t) - \boldsymbol{\eta}(t)] \xrightarrow{d} \mathrm{GP}_k(\mathbf{0}, \gamma\mathbf{I}_k)$ as desired. The theorem is then proved.

THEOREM 5.7

1. The k samples (5.23) are with $\eta_1(t), \eta_2(t), \cdots, \eta_k(t) \in \mathcal{L}^2(\mathcal{T})$ and $\text{tr}(\gamma) < \infty$.

3. As $n \to \infty$, the k sample sizes satisfy $n_i/n \to \tau_i$, $i = 1, 2, \cdots, k$ such that $\tau_1, \tau_2, \cdots, \tau_k \in (0, 1)$.
4. The subject-effect functions $v_{ij}(t) = y_{ij}(t) - \eta_i(t), j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k$ are i.i.d..
5. The subject-effect function $v_{11}(t)$ satisfies $\text{E}\|v_{11}\|^4 < \infty$.
6. The maximum variance $\rho = \max_{t \in \mathcal{T}} \gamma(t, t) < \infty$.

Theorem 5.7 *Under Assumptions KS1, KS3, KS4, KS5, and KS6, as* $n \to \infty$, *we have*

$$\sqrt{n}\{\hat{\gamma}(s,t) - \gamma(s,t)\} \xrightarrow{d} GP(0, \varpi), \tag{5.34}$$

where $\varpi\{(s_1, t_1), (s_2, t_2)\} = E\{v_{11}(s_1)v_{11}(t_1)v_{11}(s_2)v_{11}(t_2)\} - \gamma(s_1, t_1)\gamma(s_2, t_2)$.

PROOF OF THE THEOREM 5.7

Proof of Theorem 5.7 Notice that

$$\sqrt{n}\,[\hat{\gamma}(s,t)-\gamma(s,t)] = \sum_{i=1}^{k} a_i\sqrt{n_i}\,[\hat{\gamma}_i(s,t)-\gamma(s,t)]\,,$$

where $\hat{\gamma}_i(s,t), i = 1, 2, \cdots, k$ are the sample covariance functions of the k

functional samples (5.23) respectively, and $a_i = [\sqrt{n}(n_i-1)]/[\sqrt{n_i}(n-k)], i = 1, 2, \cdots, k$. As $n \to \infty$, we have $a_i \to \sqrt{\tau_i}$ and $\sum_{i=1}^{k} a_i^2 \to 1$. In addition, under the given conditions, by Theorem 4.16, we have

$$\sqrt{n_i}[\hat{\gamma}_i(s,t)-\gamma(s,t)] \xrightarrow{d} \mathrm{GP}(0,\varpi),\ i = 1, 2, \cdots, k,$$

where $\varpi((s_1,t_1),(s_2,t_2)) = \mathrm{E}v_{11}(s_1)v_{11}(t_1)v_{11}(s_2)v_{11}(t_2) - \gamma(s_1,t_1)\gamma(s_2,t_2)$. The theorem is then proved.

NOTATIONS

For the main-effect, post hoc, or contrast tests, we do not need to identify the main-effect functions $\alpha_i(t), i = 1, 2, \cdots, k$ defined in (5.26). In fact, they are not identifiable unless some constraint is imposed. If we do want to estimate these main-effect functions, the most commonly used constraint is

$$\sum_{i=1}^{k} n_i \alpha_i(t) = 0, \tag{5.35}$$

involving the k sample sizes. Under this constraint, it is easy to show that the unbiased estimators of the main-effect functions are

$$\hat{\alpha}_i(t) = \bar{y}_{i.}(t) - \bar{y}_{..}(t), \; i = 1, 2, \cdots, k, \tag{5.36}$$

where

$$\bar{y}_{..}(t) = n^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}(t) = n^{-1} \sum_{i=1}^{k} n_i \bar{y}_{i.}(t) \tag{5.37}$$

NOTATIONS

Let

$$\begin{aligned} \mathrm{SSH}_n(t) &= \textstyle\sum_{i=1}^{k} n_i[\bar{y}_{i\cdot}(t) - \bar{y}_{\cdot\cdot}(t)]^2, \text{ and} \\ \mathrm{SSE}_n(t) &= \textstyle\sum_{i=1}^{k} \sum_{j=1}^{n_i} [y_{ij}(t) - \bar{y}_{i\cdot}(t)]^2, \end{aligned} \qquad (5.38)$$

denote the pointwise between-subject and within-subject variations, respectively, where $\bar{y}_{i\cdot}(t), i = 1, 2, \cdots, k$ are the group sample mean functions as

$$\hat{\eta}_i(t) = \bar{y}_{i\cdot}(t) = n_i^{-1} \textstyle\sum_{j=1}^{n_i} y_{ij}(t), \quad i = 1, 2, \cdots, k,$$

and $\bar{y}_{\cdot\cdot}(t)$ is the sample grand mean function as

$$\bar{y}_{\cdot\cdot}(t) = n^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} y_{ij}(t) = n^{-1} \sum_{i=1}^{k} n_i \bar{y}_{i\cdot}(t)$$

NOTATIONS

$$\sum_{i=1}^{k} n_i \alpha_i(t) = 0,$$

Under the constraint (5.35), it is easy to see that

$$\mathrm{SSH}_n(t) = \sum_{i=1}^{k} n_i \hat{\alpha}_i^2(t),$$

From (5.32), we can see that

$$\mathrm{SSE}_n(t) = (n-k)\hat{\gamma}(t,t). \tag{5.40}$$

THEOREM 5.8

1. The k samples (5.23) are with $\eta_1(t), \eta_2(t), \cdots, \eta_k(t) \in \mathcal{L}^2(\mathcal{T})$ and $\text{tr}(\gamma) < \infty$.
2. The k samples (5.23) are Gaussian.

Theorem 5.8 *Suppose Assumptions KS1 and KS2 hold. Then under the null hypothesis (5.25), we have*

$$\begin{array}{rcl} \int_{\mathcal{T}} SSH_n(t)dt & \stackrel{d}{=} & \sum_{r=1}^m \lambda_r A_r, \ A_r \stackrel{i.i.d.}{\sim} \chi^2_{k-1}, \\ \int_{\mathcal{T}} SSE_n(t)dt & \stackrel{d}{=} & \sum_{r=1}^m \lambda_r E_r, \ E_r \stackrel{i.i.d.}{\sim} \chi^2_{n-k}, \end{array}$$

where $A_r, E_r, r = 1, 2, \cdots, m$ are independent of each other, and $\lambda_1, \cdots, \lambda_m$ are all the positive eigenvalues of $\gamma(s,t)$.

Proof of Theorem 5.8 Under the one-way ANOVA model (5.26) and the null hypothesis (5.25), we can further express

$$\mathrm{SSH}_n(t) = \sum_{i=1}^{k} n_i[\bar{v}_{i.}(t) - \bar{v}_{..}(t)]^2 = \mathbf{z}_n(t)^T(\mathbf{I}_k - \mathbf{b}_n\mathbf{b}_n^T/n)\mathbf{z}_n(t), \qquad (5.145)$$

where $\bar{v}_{i.}(t) = n_i^{-1}\sum_{j=1}^{n_i} v_{ij}(t)$ and $\bar{v}_{..}(t) = n^{-1}\sum_{i=1}^{k} n_i\bar{v}_{i.}(t)$, $\mathbf{b}_n = [n_1^{1/2}, n_2^{1/2}, \cdots, n_k^{1/2}]^T$, and $\mathbf{z}_n(t) = [n_1^{1/2}\bar{v}_{1.}(t), \cdots, n_k^{1/2}\bar{v}_{k.}(t)]^T$. On the one hand, under Assumption KS2, the k samples (5.23) are Gaussian, we have $\mathbf{z}_n(t) \sim \mathrm{GP}_k(\mathbf{0}, \gamma\mathbf{I}_k)$. On the other hand, it is easy to verify that $\mathbf{I}_k - \mathbf{b}_n\mathbf{b}_n^T/n$ is an idempotent matrix of rank $k-1$. The first assertion of the theorem follows immediately from Theorem 4.10 of Chapter 4.

To show the second assertion of the theorem, notice that by (5.40), we have $\int_{\mathcal{T}} \mathrm{SSE}_n(t)dt = (n-k)\mathrm{tr}(\hat{\gamma})$ and by Theorem 5.5, we have $(n-k)\hat{\gamma}(s,t) \sim \mathrm{WP}(n-k, \gamma)$. By Theorem 4.5(b), we have $\int_{\mathcal{T}} \mathrm{SSE}_n(t)dt \stackrel{d}{=} \sum_{r=1}^{m} \lambda_r E_r$, $E_r \stackrel{i.i.d.}{\sim} \chi^2_{n-k}$. Notice that $\hat{\eta}_i(t), i = 1, 2, \cdots, k$ and $\hat{\gamma}(s,t)$ are independent. So are $A_r, r = 1, 2, \cdots, m$ and $E_r, r = 1, 2, \cdots, m$. The theorem is proved.

THEOREM 5.9

1. The k samples (5.23) are with $\eta_1(t), \eta_2(t), \cdots, \eta_k(t) \in \mathcal{L}^2(\mathcal{T})$ and $\mathrm{tr}(\gamma) < \infty$.

3. As $n \to \infty$, the k sample sizes satisfy $n_i/n \to \tau_i,\ i = 1, 2, \cdots, k$ such that $\tau_1, \tau_2, \cdots, \tau_k \in (0, 1)$.
4. The subject-effect functions $v_{ij}(t) = y_{ij}(t) - \eta_i(t), j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k$ are i.i.d..

Theorem 5.9 *Suppose Assumptions KS1, KS3, and KS4 hold. Then under the null hypothesis (5.25), as $n \to \infty$, we have*

$$\int_{\mathcal{T}} SSH_n(t)dt \xrightarrow{d} \sum_{i=1}^{m} \lambda_r A_r,\ A_r \overset{i.i.d.}{\sim} \chi^2_{k-1},$$

where $\lambda_1, \cdots, \lambda_m$ are all the positive eigenvalues of $\gamma(s, t)$.

Proof of Theorem 5.9 By (5.145), we have $\text{SSH}_n(t) = \mathbf{z}_n(t)^T(\mathbf{I}_k - \mathbf{b}_n\mathbf{b}_n^T/n)\mathbf{z}_n(t)$, where $\mathbf{z}_n(t)$ and $\mathbf{b}_n$ are as defined in the proof of Theorem 5.8. It is easy to see that as $n \to \infty$, we have

$$\mathbf{I}_k - \mathbf{b}_n\mathbf{b}_n^T/n \to \mathbf{I}_k - \mathbf{b}\mathbf{b}^T,$$

where $\mathbf{b} = \lim_{n\to\infty} \mathbf{b}_n/\sqrt{n} = [\tau_1^{1/2}, \tau_2^{1/2}, \cdots, \tau_k^{1/2}]^T$. It is obvious that $\mathbf{I}_k - \mathbf{b}\mathbf{b}^T$ is an idempotent matrix of rank $k-1$ and has the singular value decomposition

$$\mathbf{I}_k - \mathbf{b}\mathbf{b}^T = \mathbf{U}\text{diag}(\mathbf{I}_{k-1}, 0)\mathbf{U}^T,$$

where $\mathbf{U}$ is an orthonormal matrix. In addition, under the given conditions, by Theorem 5.6, and under the null hypothesis (5.25), as $n \to \infty$, we have $\mathbf{z}_n(t) \xrightarrow{d} \text{GP}_k(\mathbf{0}, \gamma\mathbf{I}_k)$. It follows that we have

$$\text{SSH}_n(t) \xrightarrow{d} \mathbf{z}(t)^T(\mathbf{I}_k - \mathbf{b}\mathbf{b}^T)\mathbf{z}(t) \stackrel{d}{=} \mathbf{w}(t)^T\mathbf{w}(t), \tag{5.146}$$

where $\mathbf{w}(t) \sim \text{GP}_{k-1}(\mathbf{0}, \gamma\mathbf{I}_{k-1})$, consisting of the first $(k-1)$ component of $\mathbf{U}^T\mathbf{z}(t) \sim \text{GP}_k(\mathbf{0}, \gamma\mathbf{I}_k)$. The theorem then follows immediately from Theorem 4.10 of Chapter 4.

MAIN-EFFECT TEST

POINTWISE TESTS

TEST DEFINITION

Pointwise Tests We consider the pointwise F-test, the pointwise χ^2-test, and the pointwise bootstrap test. The pointwise F-test for (5.25) was adopted by Ramsay and Silverman (2005, Section 13.2.2., Chapter 13), naturally extending the classical F-test to the context of functional data analysis. The pointwise F-test is conducted for (5.25) at each $t \in \mathcal{T}$ using the following pointwise F statistic:

$$F_n(t) = \frac{\mathrm{SSH}_n(t)/(k-1)}{\mathrm{SSE}_n(t)/(n-k)}. \tag{5.41}$$

GAUSSIAN SAMPLE

From the classical linear model theory, it is easy to see that when the k samples (5.23) are Gaussian, under the null hypothesis (5.25), we have

$$F_n(t) \sim F_{k-1,n-k}, t \in \mathcal{T}. \quad (5.42)$$

The pointwise F-test is then conducted by rejecting (5.25) at each $t \in \mathcal{T}$ whenever $F_n(t) > F_{k-1,n-k}(1-\alpha)$ for any given significance level α or by computing the pointwise P-values at each $t \in \mathcal{T}$ based on the pointwise F-distribution (5.42).

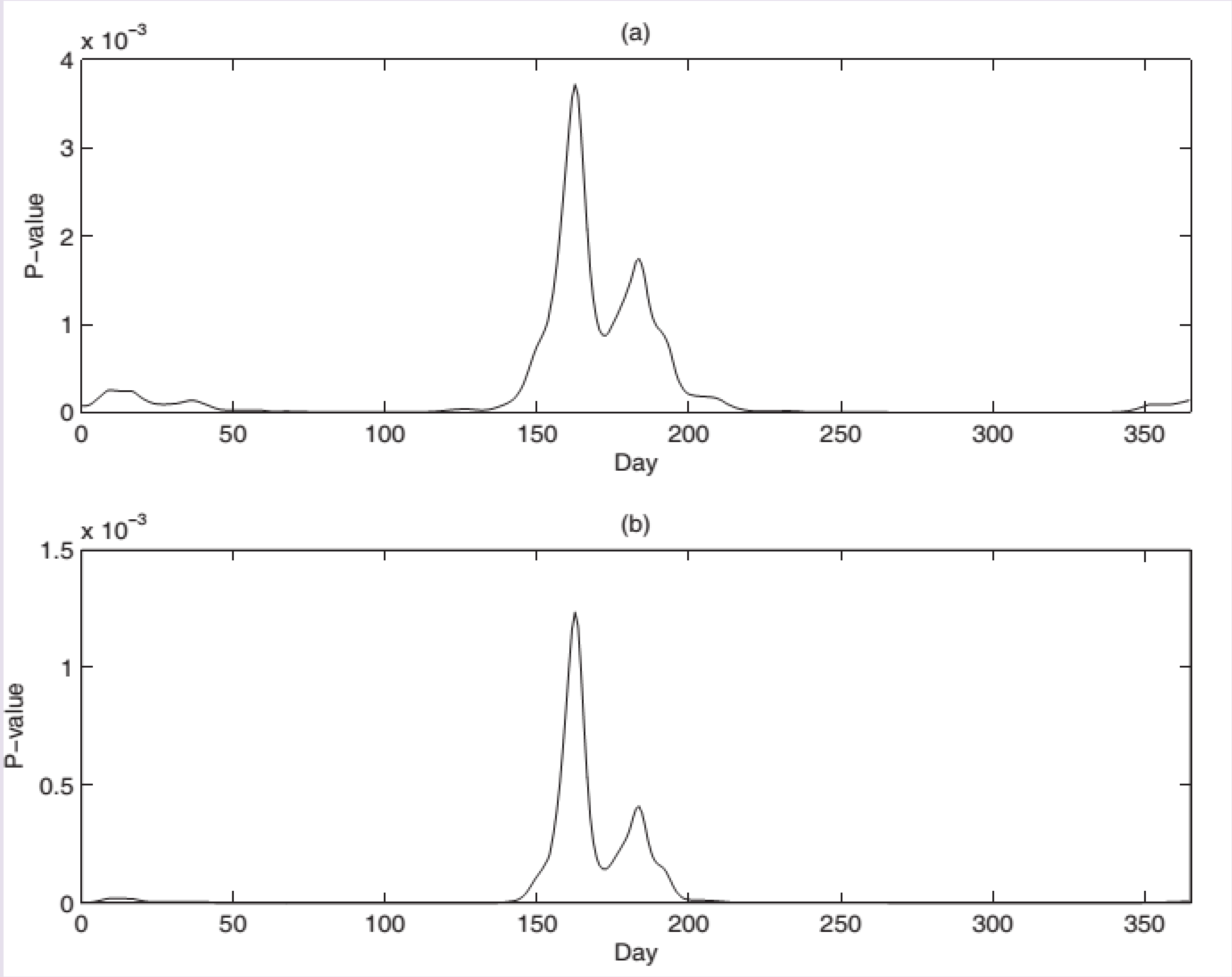
NON GAUSSIAN SAMPLE

When the k-samples are not Gaussian, for large samples, one may use the pointwise χ^2-test. It is easy to see that as $n_{\min} = \min_{i=1}^{k} n_i \to \infty$, asymptotically we have

$$F_n(t) \sim \chi^2_{k-1}/(k-1), \quad t \in \mathcal{T}. \tag{5.43}$$

This is because as $n_{\min} \to \infty$, the denominator $\text{SSE}_n(t)/(n-k) = \hat{\gamma}(t,t)$ of $F_n(t)$ tends to $\gamma(t,t)$ almost surely while the numerator $\text{SSH}_n(t)/(k-1)$ tends to $\gamma(t,t)\chi^2_{k-1}/(k-1)$. The pointwise χ^2-test is conducted by rejecting (5.25) at any given t whenever $F_n(t) > \chi^2_{k-1}(1-\alpha)/(k-1)$ or by computing the pointwise P-values of $F_n(t)$ at any given t based on the distribution (5.43).

EXAMPLE


(a)
x 10^-3
4
3
2
1
0
P–value
0
50
100
150
200
250
300
350
Day
(b)
x 10^-3
1.5
1
0.5
0
P–value
0
50
100
150
200
250
300
350
Day

L2 NORM-BASED TEST

TEST DEFINITION

L^2-Norm-Based Test The L^2-norm-based test for the two-sample problem (5.2) can now be extended for the main-effect testing problem (5.25). The associated test statistic is defined as the integral of the pointwise between-subject variations:

$$T_n = \int_{\mathcal{T}} \mathrm{SSH}_n(t)dt = \sum_{i=1}^{k} n_i \int_{\mathcal{T}} [\bar{y}_{i\cdot}(t) - \bar{y}_{\cdot\cdot}(t)]^2 dt. \qquad (5.46)$$

UNDER H_0

Under the null hypothesis (5.25) and under the conditions of Theorem 5.8 or under the conditions of Theorem 5.9, we have or approximately have

$$T_n = \sum_{r=1}^{m} \lambda_r A_r,\ A_r \overset{i.i.d.}{\sim} \chi^2_{k-1},$$

where $\lambda_r, r = 1, 2, \cdots, m$ are all the positive eigenvalues of $\gamma(s,t)$. It follows that we can approximate the null distribution of T_n by the Welch-Satterthwaite χ^2-approximation method described in Section 4.3 of Chapter 4. By that method, we obtain

$$T_n \sim \beta \chi^2_{(k-1)\kappa} \text{ approximately, where } \beta = \frac{\mathrm{tr}(\gamma^{\otimes 2})}{\mathrm{tr}(\gamma)},\ \ \kappa = \frac{\mathrm{tr}^2(\gamma)}{\mathrm{tr}(\gamma^{\otimes 2})}. \tag{5.47}$$

APPROXIMATION

$$\hat{\beta} = \frac{\mathrm{tr}(\hat{\gamma}^{\otimes 2})}{\mathrm{tr}(\hat{\gamma})}, \quad \hat{\kappa} = \frac{\mathrm{tr}^2(\hat{\gamma})}{\mathrm{tr}(\hat{\gamma}^{\otimes 2})}, \tag{5.48}$$

and by the bias-reduced method, we have

$$\hat{\beta} = \frac{\widehat{\mathrm{tr}(\gamma^{\otimes 2})}}{\mathrm{tr}(\hat{\gamma})}, \quad \hat{\kappa} = \frac{\widehat{\mathrm{tr}^2(\gamma)}}{\widehat{\mathrm{tr}(\gamma^{\otimes 2})}}, \tag{5.49}$$

with

$$\begin{aligned} \widehat{\mathrm{tr}^2(\gamma)} &= \frac{(n-k)(n-k+1)}{(n-k-1)(n-k+2)} \left[\mathrm{tr}^2(\hat{\gamma}) - \frac{2\mathrm{tr}(\hat{\gamma}^{\otimes 2})}{n-k+1} \right], \\ \widehat{\mathrm{tr}(\gamma^{\otimes 2})} &= \frac{(n-k)^2}{(n-k-1)(n-k+2)} \left[\mathrm{tr}(\hat{\gamma}^{\otimes 2}) - \frac{\mathrm{tr}^2(\hat{\gamma})}{n-k} \right]. \end{aligned} \tag{5.50}$$

THEOREM 5.10

1. The k samples (5.23) are with $\eta_1(t), \eta_2(t), \cdots, \eta_k(t) \in \mathcal{L}^2(\mathcal{T})$ and $\text{tr}(\gamma) < \infty$.

3. As $n \to \infty$, the k sample sizes satisfy $n_i/n \to \tau_i$, $i = 1, 2, \cdots, k$ such that $\tau_1, \tau_2, \cdots, \tau_k \in (0, 1)$.
4. The subject-effect functions $v_{ij}(t) = y_{ij}(t) - \eta_i(t), j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k$ are i.i.d..
5. The subject-effect function $v_{11}(t)$ satisfies $\mathrm{E}\|v_{11}\|^4 < \infty$.
6. The maximum variance $\rho = \max_{t \in \mathcal{T}} \gamma(t, t) < \infty$.
7. The expectation $\mathrm{E}[v_{11}^2(s) v_{11}^2(t)]$ is uniformly bounded.

Theorem 5.10 *Under Assumptions KS1 and KS3 through KS7, as $n \to \infty$, we have $tr(\hat{\gamma}) \xrightarrow{p} tr(\gamma)$ and $tr(\hat{\gamma}^{\otimes 2}) \xrightarrow{p} tr(\gamma^{\otimes 2})$. Furthermore, as $n \to \infty$, we have*

$$\hat{\beta} \xrightarrow{p} \beta, \quad \hat{\kappa} \xrightarrow{p} \kappa,$$

where $\hat{\beta}$ and $\hat{\kappa}$ are the naive or bias-reduced estimators of β and κ.

EXAMPLE

Table 5.6 *Traces of the pooled sample covariance functions* $\hat{\gamma}(s,t)$ *and its cross-square function* $\hat{\gamma}^{\otimes 2}(s,t)$ *of the Canadian temperature data, calculated with resolution* $M = 1,000$ *over various seasons.*

	Spring	Summer	Fall	Winter	Whole year
[a,b]	$[60, 151]$	$[152, 243]$	$[244, 334]$	$[335, 365]\&[1, 59]$	$[1, 365]$
$\mathrm{tr}(\hat{\gamma})$	2,481	859	1,342	4,572	9,255
$\mathrm{tr}(\hat{\gamma}^{\otimes 2})$	5,081,346	686,818	1,519,203	20,318,974	58,152,749

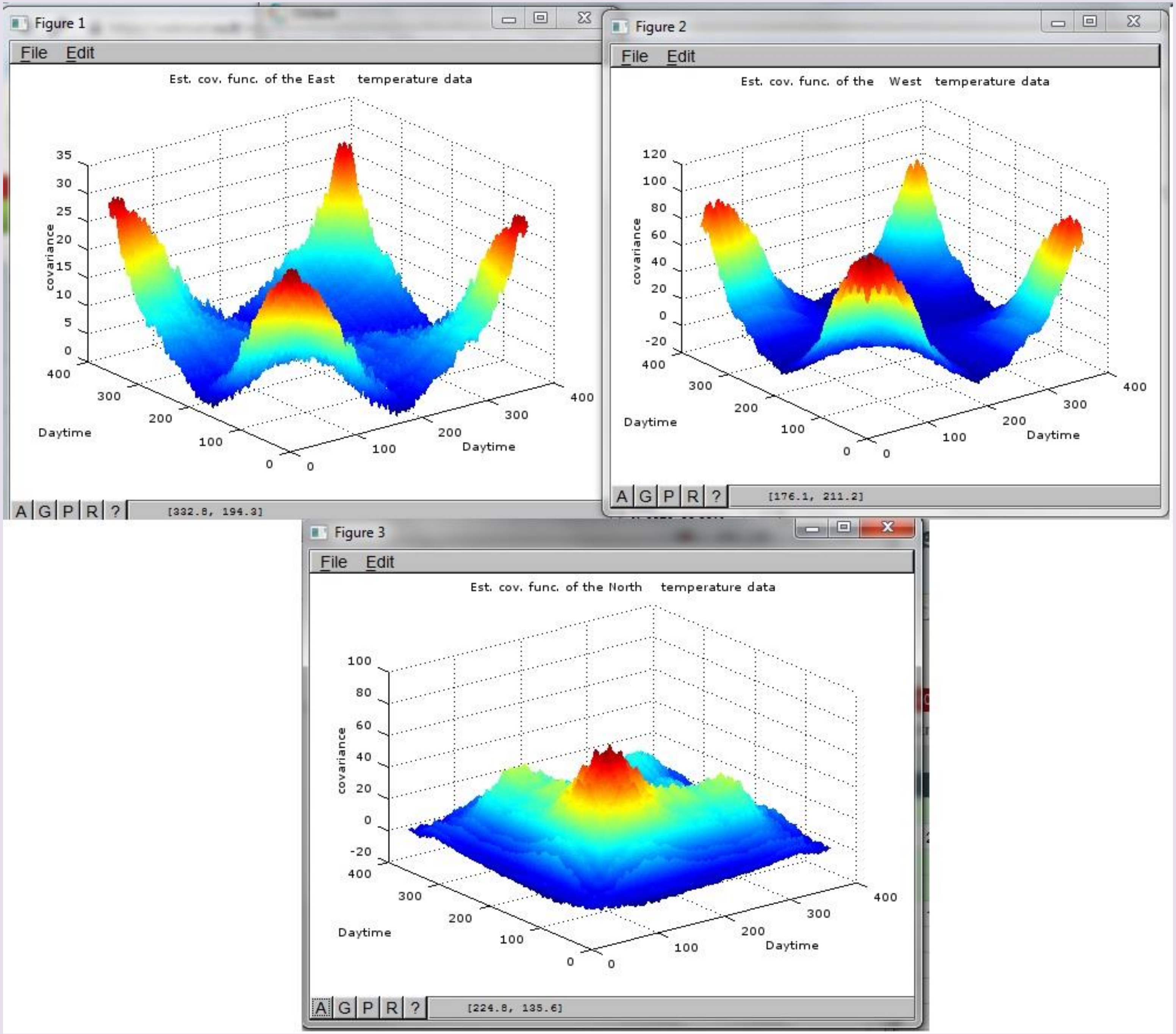


Figure 1
File Edit
Est. cov. func. of the East temperature data
covariance
35
30
25
20
15
10
5
0
400
300
200
100
0
Daytime
0
100
200
300
400
Daytime
A G P R ?
[332.8, 194.3]
Figure 2
File Edit
Est. cov. func. of the West temperature data
covariance
120
100
80
60
40
20
0
-20
400
300
200
100
0
Daytime
0
100
200
300
400
Daytime
A G P R ?
[176.1, 211.2]
Figure 3
File Edit
Est. cov. func. of the North temperature data
covariance
100
80
60
40
20
0
-20
400
300
200
100
0
Daytime
0
100
200
300
400
Daytime
A G P R ?
[224.8, 135.6]

EXAMPLE

Table 5.7 *The L^2-norm-based test for the one-way ANOVA problem (5.25) for the Canadian temperature data with resolution $M = 1,000$.*

Method	Time period	T_n	$\hat{\beta}$	$\hat{d}$	P-value
Naive	Spring	8.58e4	2.05e3	2.42	$1.67e-9$
	Summer	1.87e4	7.99e2	2.15	$1.01e-5$
	Fall	7.60e4	1.13e3	2.37	$5.44e-15$
	Winter	1.22e5	4.44e3	2.06	$1.25e-6$
	Whole year	3.02e5	6.28e3	2.95	$1.86e-10$
Bias-reduced	Spring	8.58e4	1.91e3	2.47	$4.24e-10$
	Summer	1.87e4	7.50e2	2.17	$4.86e-6$
	Fall	7.60e4	1.06e3	2.41	$5.55e-16$
	Winter	1.22e5	4.18e3	2.06	$5.29e-7$
	Whole year	3.02e5	5.82e3	3.05	$3.43e-11$

Note: The P-values by the naive method are generally comparable with those by the bias-reduced method although the former are generally larger than the latter.

F TYPE TEST

TEST DEFINITION

F-Type Test When the k samples (5.23) are Gaussian, we can conduct an F-type test for the main-effect test (5.25). The F-type test statistic is defined as

$$F_n = \frac{\int_{\mathcal{T}} \mathrm{SSH}_n(t)dt/(k-1)}{\int_{\mathcal{T}} \mathrm{SSE}_n(t)dt/(n-k)}.$$

UNDER H_0

Under the null hypothesis (5.25) and by Theorems 5.8, we have

$$F_n \stackrel{d}{=} \frac{\sum_{r=1}^{m} \lambda_r A_r/(k-1)}{\sum_{r=1}^{m} \lambda_r E_r/(n-k)}, \qquad (5.53)$$

where $A_r \stackrel{i.i.d.}{\sim} \chi^2_{k-1}, E_r \stackrel{i.i.d.}{\sim} \chi^2_{n-k}$ and they are all independent; $\lambda_1, \lambda_2, \cdots, \lambda_m$ are all the positive eigenvalues of $\gamma(s,t)$. It follows that the null distribution of F_n can be approximated by the two-cumulant matched F-approximation method described in Section 4.4 of Chapter 4. By that method, we have

$$F_n \sim F_{(k-1)\hat{\kappa},(n-k)\hat{\kappa}} \text{ approximately,} \qquad (5.54)$$

EXAMPLE

Table 5.8 *The F-type test for the one-way ANOVA problem (5.25) for the Canadian temperature data with resolution $M = 1,000$.*

Method	Time period	F_n	$\hat{d}_1$	$\hat{d}_2$	P-value
Naive	Spring	17.30	2.42	38.76	$3.09e-4$
	Summer	10.91	2.15	34.41	$5.12e-3$
	Fall	28.30	2.37	37.99	$1.29e-6$
	Winter	13.31	2.06	32.91	$1.39e-3$
	Whole year	16.33	2.95	47.13	$9.15e-4$
Bias-reduced	Spring	17.30	2.47	39.46	$3.27e-4$
	Summer	10.91	2.17	34.65	$5.21e-3$
	Fall	28.30	2.41	38.61	$1.37e-6$
	Winter	13.31	2.06	30.01	$1.40e-3$
	Whole year	16.33	3.05	48.85	$10.29e-4$

Note: The P-values by the naive method are generally comparable with those by the bias-reduced method.

BOOTSTRAP TEST

IDEA

When the sample sizes $n_1, n_2, \cdots, n_k$ are large, one can apply some parametric bootstrap (PB) methods for testing the main-effect test (5.27). From (5.146) in the proof of Theorem 5.9, we can see that under the null hypothesis, as $n \to \infty$, we have

$$T_n = \int_{\mathcal{T}} \mathrm{SSH}_n(t)dt \xrightarrow{d} \sum_{i=1}^{k-1} \int_{\mathcal{T}} w_i^2(t)dt,$$

where $w_i(t), i = 1, 2, \cdots, k-1$ are the $k-1$ components of $\mathbf{w}(t) \sim \mathrm{GP}_{k-1}(\mathbf{0}, \gamma \mathbf{I}_{k-1})$. That is, $w_i(t), i = 1, \cdots, k-1 \overset{i.i.d.}{\sim} \mathrm{GP}(0, \gamma)$ which are known except $\gamma(s,t)$. The unbiased estimator $\hat{\gamma}(s,t)$ of $\gamma(s,t)$ is given in (5.32).

ALGORITHM

PB Algorithm for One-Way ANOVA (I)

1. Compute $\hat{\gamma}(s,t)$ using (5.32) based on the k samples (5.23).
2. Re-sample the Gaussian processes $w_i^*(t), i = 1, 2, \cdots, k-1$ from $\text{GP}(0, \hat{\gamma})$.
3. Compute $T_B^* = \sum_{i=1}^{k-1} \int_{\mathcal{T}} [w_i^*(t)]^2 dt$.
4. Repeat Steps 2 and 3 a large number of times to obtain a sequence of T_B^* whose sample percentiles can be used to approximate the percentiles of T_n.

$$y_{i1}(t), \cdots, y_{in_i}(t),\ i = 1, \cdots, k. \tag{5.23}$$

$$\begin{aligned} \hat{\eta}_i(t) &= \bar{y}_{i\cdot}(t) = n_i^{-1} \sum_{j=1}^{n_i} y_{ij}(t), \quad i = 1, 2, \cdots, k, \\ \hat{\gamma}(s,t) &= (n-k)^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} [y_{ij}(s) - \bar{y}_{i\cdot}(s)][y_{ij}(t) - \bar{y}_{i\cdot}(t)], \end{aligned} \tag{5.32}$$

ALGORITHM

PB Algorithm for One-Way ANOVA (II)

1. Compute $\hat{\gamma}(s,t)$ using (5.32) based on the k samples (5.23).
2. Compute the positive eigenvalues $\hat{\lambda}_r, r = 1, 2, \cdots, \hat{m}$ of $\hat{\gamma}(s,t)$.
3. Re-sample $A_r, i = 1, 2, \cdots, k-1$ from χ^2_{k-1}.
4. Compute $T_B^* = \sum_{r=1}^{\hat{m}} \hat{\lambda}_r A_r$.
5. Repeat Steps 3 and 4 a large number of times to obtain a sequence of T_B^* whose sample percentiles can be used to approximate the percentiles of T_n.

IDEA

$$V_n = \sum_{1 \le i < j \le k} n_i \int_{\mathcal{T}} [\bar{y}_{i\cdot}(t) - \bar{y}_{j\cdot}(t)]^2 dt. \tag{5.55}$$

They imposed Assumption KS3, that is, as $n \to \infty$,

$$\frac{n_i}{n} \to \tau_i \in (0,1), i = 1, 2, \cdots, k. \tag{5.56}$$

Under the above condition and under the null hypothesis (5.25), they showed that

$$V_n \xrightarrow{d} \sum_{1 \le i < j \le k} \int_{\mathcal{T}} [w_i(t) - \sqrt{\tau_i/\tau_j} w_j(t)]^2 dt, \tag{5.57}$$

where $w_i(t), i = 1, 2, \cdots, k \overset{i.i.d.}{\sim} \text{GP}(0, \gamma)$. Cuevas, Febrero, and Fraiman (2004) computed the P-value or the empirical critical value of V_n by re-sampling $w_i(t), i = 1, 2, \cdots, k$ from $\text{GP}(0, \hat{\gamma})$ a large number of times, where

$\hat{\gamma}(s,t)$ is the pooled sample covariance function given in (5.32). In summary, their PB algorithm can be described as follows:

ALGORITHM

PB Algorithm for One-Way ANOVA (III)

1. Compute $\hat{\gamma}(s,t)$ using (5.32) based on the k samples (5.23).
2. Re-sample the Gaussian processes $w_i^*(t), i = 1, 2, \cdots, k$ from $\mathrm{GP}(0, \hat{\gamma})$.
3. Compute $V_B^* = \sum_{1 \le i < j \le k} \int_{\mathcal{T}} [w_i^*(t) - \sqrt{\tau_i/\tau_j} w_j^*(t)]^2 dt$, where $\tau_i = n_i/n, i = 1, 2, \cdots, k$.
4. Repeat Steps 2 and 3 a large number of times to obtain a sequence of V_B^* whose sample percentiles can be used to approximate the percentiles of V_n.

BOOTSTRAP FOR NON GAUSSIAN SAMPLE

Let $v^*_{ij}(t), j = 1, 2, \cdots, n_i; i = 1, \cdots, k$, be k bootstrap samples randomly generated from the estimated subject-effect functions $\hat{v}_{ij}(t) = y_{ij}(t) - \hat{\eta}_i(t), j = 1, 2, \cdots, n_{ij}; i = 1, 2, \cdots, k$. Set

$$y^*_{ij}(t) = \hat{\eta}_i(t) + v^*_{ij}(t), j = 1, 2, \cdots, n_{ij}; i = 1, 2, \cdots, k. \qquad (5.61)$$

Then we can compute the k sample group mean functions $\bar{y}^*_{1.}(t), \cdots, \bar{y}^*_{k.}(t)$, the sample grand mean function $\bar{y}^*_{..}(t)$, and the pooled sample covariance function $\hat{\gamma}^*(s, t)$ as in (5.32) but based on the k bootstrap samples (5.61). Then we can compute

$$\begin{aligned} \text{SSH}^*_n(t) &= \textstyle\sum_{i=1}^k n_i \left\{[\bar{y}^*_{i.}(t) - \bar{y}^*_{..}(t)] - [\bar{y}_{i.}(t) - \bar{y}_{..}(t)]\right\}^2, \\ \text{SSE}^*_n(t) &= (n-k)\hat{\gamma}^*(t, t). \end{aligned}$$

BOOTSTRAP FOR NON GAUSSIAN SAMPLE

For the L^2-norm-based bootstrap test or the F-type bootstrap test, we compute

$$T_n^* = \int_{\mathcal{T}} \mathrm{SSH}_n^*(t)dt, \text{ or } F_n^* = \frac{\int_{\mathcal{T}} \mathrm{SSH}_n^*(t)dt/(k-1)}{\int_{\mathcal{T}} \mathrm{SSE}_n^*(t)dt/(n-k)}.$$

Repeat this process a large number of times to obtain a bootstrap sample of T_n^* or F_n^* that can be used to estimate the $100(1-\alpha)$-percentile of T_n or F_n. The L^2-norm-based bootstrap test or the F-type bootstrap test can then be conducted accordingly.

EXAMPLE

Table 5.9 *The L^2-norm-based and F-type bootstrap tests for the one-way ANOVA problem (5.25) with the Canadian temperature data with resolution $M = 1,000$.*

	L^2-norm-based bootstrap test		F-type bootstrap test	
Time period	T_n	P-value	F_n	P-value
Spring	85,815	0	17.30	$3e-4$
Summer	18,748	0	10.91	$2.93e-2$
Fall	76,007	0	28.30	0
Winter	121,670	0	13.31	$6e-4$
Whole year	302,240	0	16.33	0

Note: The number of bootstrap replicates is $N = 10,000$. The effect of the number of bootstrap replicates $N = 10,000$ on the P-values of the L^2-norm-based and F-type bootstrap tests is noted.

TEST OF LINEAR HYPOTHESES

IDEA

In the previous subsection we presented some methods for the main-effect test (5.27). In this subsection, we study how to test the post hoc test (5.28) and the contrast test (5.30) in a unified framework. That is, given the k samples (5.23), we want to test the following general linear hypothesis testing (GLHT) problem:

$$H_0: \ \mathbf{C}\boldsymbol{\eta}(t) \equiv \mathbf{c}(t), \ t \in \mathcal{T}, \quad \text{versus} \quad H_1: \ \mathbf{C}\boldsymbol{\eta}(t) \neq \mathbf{c}(t), t \in \mathcal{T}, \tag{5.62}$$

where $\mathbf{C} : q \times k$ is a known coefficient matrix with $\text{rank}(\mathbf{C}) = q$, and $\mathbf{c}(t) : q \times 1$ is a known constant function, often specified as $\mathbf{0}$. In fact, the post hoc test

Notice that we have $\mathrm{E}\left[\mathbf{C}\hat{\boldsymbol{\eta}}(t) - \mathbf{c}(t)\right] = \mathbf{C}\boldsymbol{\eta}(t) - \mathbf{c}(t)$ and

$$\operatorname{Cov}\left[\mathbf{C}\hat{\boldsymbol{\eta}}(s) - \mathbf{c}(s), \mathbf{C}\hat{\boldsymbol{\eta}}(t) - \mathbf{c}(t)\right] = \gamma(s,t)\mathbf{C}\mathbf{D}\mathbf{C}^T,$$

where $\mathbf{D} = \operatorname{diag}(\frac{1}{n_1}, \frac{1}{n_2}, \cdots, \frac{1}{n_k})$ as defined earlier. As $\mathbf{C}\mathbf{D}\mathbf{C}^T$ is a square matrix of full rank, we then arrive at the following pivotal test function:

$$\mathbf{z}(t) = \left(\mathbf{C}\mathbf{D}\mathbf{C}^T\right)^{-1/2}\left[\mathbf{C}\hat{\boldsymbol{\eta}}(t) - \mathbf{c}(t)\right]. \tag{5.64}$$

It is easy to see that

$$\mathbf{z}(t) \sim \mathrm{SP}_q(\boldsymbol{\eta}_z, \gamma\mathbf{I}_q), \tag{5.65}$$

where

$$\boldsymbol{\eta}_z(t) = (\mathbf{C}\mathbf{D}\mathbf{C}^T)^{-1/2}\left[\mathbf{C}\boldsymbol{\eta}(t) - \mathbf{c}(t)\right]. \tag{5.66}$$

Under the null hypothesis in (5.62), $\boldsymbol{\eta}_z(t) \equiv 0, t \in \mathcal{T}$. The squared L^2-norm $\|\mathbf{z}(t)\|^2$ of $\mathbf{z}(t)$ at $t \in \mathcal{T}$ can then be used as the pointwise sum of squares due to hypothesis:

$$\mathrm{SSH}_n(t) = \left[\mathbf{C}\hat{\boldsymbol{\eta}}(t) - \mathbf{c}(t)\right]^T \left(\mathbf{C}\mathbf{D}\mathbf{C}^T\right)^{-1}\left[\mathbf{C}\hat{\boldsymbol{\eta}}(t) - \mathbf{c}(t)\right], \tag{5.67}$$

which, together with $\mathrm{SSE}_n(t) = (n-k)\hat{\gamma}(t,t)$, the pointwise sum of squares due to errors, will be used to define various tests for the GLHT problem (5.62).

THEOREM 5.11

1. The k samples (5.23) are with $\eta_1(t), \eta_2(t), \cdots, \eta_k(t) \in \mathcal{L}^2(\mathcal{T})$ and $\text{tr}(\gamma) < \infty$.
2. The k samples (5.23) are Gaussian.

Theorem 5.11 *Under Assumptions KS1 and KS2 and the null hypothesis in (5.62), we have*

$$\int_{\mathcal{T}} SSH_n(t)dt \stackrel{d}{=} \sum_{r=1}^{m} \lambda_r A_r, \quad \int_{\mathcal{T}} SSE_n(t)dt \stackrel{d}{=} \sum_{r=1}^{m} \lambda_r E_r,$$

where $A_r, r = 1, 2, \cdots, m \stackrel{i.i.d.}{\sim} \chi_q^2$ *and* $E_r, r = 1, 2, \cdots, m \stackrel{i.i.d.}{\sim} \chi_{n-k}^2$ *are independent, and* $\lambda_1, \cdots, \lambda_m$ *are all the positive eigenvalues of* $\gamma(s, t)$.

THEOREM 5.12

1. The k samples (5.23) are with $\eta_1(t), \eta_2(t), \cdots, \eta_k(t) \in \mathcal{L}^2(\mathcal{T})$ and $\text{tr}(\gamma) < \infty$.

3. As $n \to \infty$, the k sample sizes satisfy $n_i/n \to \tau_i,\ i = 1, 2, \cdots, k$ such that $\tau_1, \tau_2, \cdots, \tau_k \in (0, 1)$.
4. The subject-effect functions $v_{ij}(t) = y_{ij}(t) - \eta_i(t), j = 1, 2, \cdots, n_i; i = 1, 2, \cdots, k$ are i.i.d..

Theorem 5.12 *Under Assumptions KS1, KS3, KS4, and the null hypothesis in (5.62), as $n \to \infty$, we have*

$$\int_{\mathcal{T}} SSH_n(t)dt \xrightarrow{d} \sum_{r=1}^{m} \lambda_r A_r,$$

where $A_r, r = 1, 2, \cdots, m \overset{i.i.d.}{\sim} \chi_q^2$ and $\lambda_1, \cdots, \lambda_m$ are all the positive eigenvalues of $\gamma(s, t)$.

Pointwise Tests We describe a pointwise F-test and a pointwise χ^2-test here. The test statistic of the pointwise F-test is defined as

$$F_n(t) = \frac{\mathrm{SSH}_n(t)/q}{\mathrm{SSE}_n(t)/(n-k)}. \tag{5.69}$$

When the k samples (5.23) are Gaussian, under the null hypothesis in (5.62), we have

$$F_n(t) \sim F_{q,n-k}, t \in \mathcal{T}.$$

The pointwise F-test can be conducted accordingly. When the Gaussian assumption is not valid, for large samples, one may use the pointwise χ^2-test. For large samples, that is, under Assumptions KS3 and KS4, it is standard to show that

$$F_n(t) \xrightarrow{d} \chi_q^2/q, \quad t \in \mathcal{T}.$$

The pointwise χ^2-test can be conducted accordingly.

When the k samples (5.23) are not Gaussian and $n_1, \cdots, n_k$ are small, the above pointwise F and χ^2-tests are not preferred. In this case, one may resort to some bootstrap approaches as described at the end of this subsection.

L^2-Norm-Based Test For the GLHT problem (5.62), the L^2-norm-based test uses the following test statistic

$$T_n = \int_{\mathcal{T}} \mathrm{SSH}_n(t)dt.$$

Under the null hypothesis in (5.62) and under the conditions of Theorem 5.11 or under the conditions of Theorem 5.12, we have or approximately have

$$T_n \stackrel{d}{=} \sum_{r=1}^{m} \lambda_r A_r, \; A_r \stackrel{i.i.d.}{\sim} \chi_q^2,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_m$ are all the positive eigenvalues of $\gamma(s,t)$. Then the null distribution of T_n can be approximated by the Welch-Satterthwaite χ^2-approximation method using the methods described in Section 4.3 of Chapter 4. In fact, by this method, we have

$$T_n \sim \hat{\beta}\chi^2_{q\hat{\kappa}} \text{ approximately}$$

where by the naive method, $\hat{\boldsymbol{\beta}}$ and $\hat{\kappa}$ are given in (5.48) and by the bias-reduced method, they are given in (5.49). The L^2-norm-based test can then be conducted accordingly.

F-Type Test As for the main-effect testing problem (5.25), for Gaussian data, we can also conduct an F-type test for the GLHT problem (5.62) using the following F-type test statistic:

$$F_n = \frac{\int_{\mathcal{T}} \mathrm{SSH}_n(t)dt/q}{\int_{\mathcal{T}} \mathrm{SSE}_n(t)dt/(n-k)}.$$

By Theorem 5.11 and under the null hypothesis in (5.62), we have

$$F_n \stackrel{d}{=} \frac{\sum_{r=1}^{m} \lambda_r A_r/q}{\sum_{r=1}^{m} \lambda_r E_r/(n-k)},$$

where $A_r, r = 1, \cdots, m \stackrel{i.i.d.}{\sim} \chi_q^2$ and $E_r, r = 1, \cdots, m \stackrel{i.i.d.}{\sim} \chi_{n-k}^2$ are independent, and $\lambda_1, \lambda_2, \cdots, \lambda_m$ are all the positive eigenvalues of $\gamma(s,t)$. That is, under the null hypothesis, F_n is an F-type mixture. It follows that the null distribution of F_n can be approximated by the two-cumulant matched F-approximation method described in Section 4.4 of Chapter 4. In fact, by that method, we have

$$F_n \sim F_{q\hat{\kappa},(n-k)\hat{\kappa}} \text{ approximately,}$$

where by the naive method, $\hat{\kappa}$ is given in (5.48) and by the bias-reduced method, $\hat{\kappa}$ is given in (5.49). The F-type test can then be conducted accordingly.

NEXT TIME

TWO-WAY ANOVA