Functional central limit theorems for sums of nearly nonstationary processes

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AR(1) process

First order autoregressive $AR(1)$ process is generated according to the scheme

$$y_k = \phi y_{k-1} + \varepsilon_k, \quad k = 1, 2, \dots n, \quad y_0 = 0. \tag{1}$$

where $(\varepsilon_k)_{k \geq 0}$ are the innovations at time k, and ϕ is an unknown parameter.

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1. $|\phi| < 1$ - (1) is stationary process ;
2. $|\phi| > 1$ - (1) is explosive process ;
3. $\phi = 1$ - (1) is nonstationary process.

Asymptotic behaviour of ϕ

The least-squares estimator (LSE) of ϕ is

$$\widehat{\phi} = \frac{\sum_{k=1}^{n} y_k y_{k-1}}{\sum_{k=1}^{n} y_{k-1}^2}.$$

When $|\phi| < 1$ it is well known (see, for example, Mann and Wald (1943) and Anderson (1959)) that the standardized LSE is asymptotically normal :

$$\tau_n := \left(\sum_{k=1}^{n} y_{k-1}^2 \right)^{1/2} (\widehat{\phi} - \phi) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,1).$$

Here $\mathfrak{N}(0,\sigma^2)$ denotes normal distribution with mean 0 and variance σ^2.

Asymptotic behaviour of ϕ

For $\phi > 1$, Anderson (1959) showed that

$$\tau_n := \left(\sum_{k=1}^{n} y_{k-1}^2 \right)^{1/2} (\widehat{\phi} - \phi) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,1)$$

is true when the ε_k's are i.i.d. For general ε_k's he showed that limiting distribution of τ_n may not exists.
The classical convergence in distribution of a sequence of random variables is denoted by $\xrightarrow[n\to\infty]{\mathbb{R}}$.

Asymptotic behaviour of ϕ

However when $\phi = 1$, the limit distribution of the properly standardized sequence of the least-squares estimators is non-normal, and it was shown by White (1958) (see also Rao (1978)) that

$$\left(\sum_{k=1}^{n} y_{k-1}^2\right)^{1/2} (\widehat{\phi} - 1) \xrightarrow[n\to\infty]{\mathbb{R}} \tau := \frac{\frac{1}{2}(W^2(1) - 1)}{\left(\int_0^1 W^2(t)\,\mathrm{d}t\right)^{1/2}}$$

where $(W(t), 0 \le t \le 1)$ is a standard Brownian motion.

ϕ is "close" to 1

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3. Also

$$\left(\sum_{k=1}^{n} y_{k-1}^2 \right)^{1/2} (\widehat{\phi} - 1) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\frac{1}{2}(W^2(1) - 1)}{\left(\int_0^1 W^2(t)\,\mathrm{d}t \right)^{1/2}} \quad (3)$$

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4. However, neither (2) nor (3) seems to be intuitive approximations.

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Nearly nonstationary process

Suppose we have first-order autoregressive process $(y_{n,k} | k = 1, \dots, n; n = 1, 2, \dots)$ given by

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \quad y_{n,0} = y_0 = 0 \tag{4}$$

where

1. $\phi_n \to 1$, as $n \to \infty$,
2. $(\varepsilon_k)_{k \geq 0}$ is a sequence of i.i.d.random variables with $\mathbb{E}\varepsilon_k = 0$ and $\mathbb{E}\varepsilon_k^2 = 1$,
3. n is a sample size.

Parametrisation of ϕ_n

Case 1

$\phi_n = \mathrm{e}^{\gamma/n}$ with constant $\gamma < 0$.
This parametrisation was suggested by Phillips (1987 m.).

Case 2

$\phi_n = 1 - \frac{\gamma_n}{n}$, $\gamma_n \to \infty$ slower than n.
This parametrisation was suggested by Phillips and Giraitis (2006 m.)

Some results (Phillips (1987 m.))

If $(y_k)_{k\geq 0}$ is a nearly non stationary process generated by (4) and $\phi_n = \mathrm{e}^{\gamma/n}$, $(\varepsilon_k)_{k\geq 0}$ are i.i.d. random variables with $\mathbb{E}\varepsilon_0 = 0$ and $\mathbb{E}\varepsilon_0^2 = \sigma^2$, then :

$$n^{-1/2} y_{[nt]} \xrightarrow[n\to\infty]{\mathrm{D}[0,1]} \sigma U_\gamma(t),$$

$$n^{-3/2} \sum_{j=1}^{n} y_j \xrightarrow[n\to\infty]{\mathbb{R}} \sigma \int_0^1 U_\gamma(r)\,\mathrm{d}r,$$

$$n^{-2} \sum_{j=1}^{n} y_j^2 \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 \int_0^1 U_\gamma^2(r)\,\mathrm{d}r,$$

$$n^{-1} \sum_{j=1}^{n} y_{j-1}\varepsilon_j \xrightarrow[n\to\infty]{\mathbb{R}} \sigma^2 \int_0^1 U_\gamma(r)\,\mathrm{d}W(r).$$

where

$$U_\gamma(t) = \int_0^t \mathrm{e}^{(t-s)\gamma}\,\mathrm{d}W(s) \quad 0 \leq t \leq 1.$$

Some results (Phillips and Giraitis (2006 m.))

Suppose $(y_k)_{k\geq 0}$ is a nearly non stationary process generated by (4) and $(\varepsilon_k)_{k\geq 0}$ are i.i.d. random variables with $\mathbb{E}\varepsilon_0 = 0$ and $\mathbb{E}\varepsilon_0^2 = \sigma^2$. Under assumption : $n(1-\phi_n) \to \infty$, as $n \to \infty$:

$$\frac{\left(1-\phi_n^2\right)^{1/2}}{n^{1/2}} \sum_{j=1}^{n} \varepsilon_j y_{j-1} \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,\sigma^4),$$

$$\frac{1-\phi_n^2}{n} \sum_{j=1}^{n} y_{j-1}^2 \xrightarrow[n\to\infty]{\mathrm{P}} \sigma^2,$$

$$\frac{(1-\phi_n)}{n^{1/2}} \sum_{j=1}^{n} y_j \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,\sigma^2).$$

Convergence in probability is denoted by $\xrightarrow[n\to\infty]{\mathrm{P}}$.

The asymptotic behaviour of ϕ_n

Phillips (1987 m.) :

$$n(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \frac{\int_0^1 U_\gamma(r)\,\mathrm{d}W(r)}{\int_0^1 U_\gamma^2(r)\,\mathrm{d}r}$$

Phillips and Giraitis (2006 m.) :

$$\frac{n^{1/2}}{(1-\phi_n^2)^{1/2}}(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,1)$$

$$\frac{n^{1/2}}{\left|1-\widehat{\phi}_n^2\right|^{1/2}}(\widehat{\phi}_n - \phi_n) \xrightarrow[n\to\infty]{\mathbb{R}} \mathfrak{N}(0,1)$$

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Process built on the y_k's

We focus on polygonal line processes built on the y_k's :

$$S_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} y_{k-1} + (nt - [nt]) y_{[nt]}, \quad t \in [0,1], \quad n \geq 1.$$

Remark

The definition of the S_n^{pl} is quite unusual with a general term y_{k-1} were one would expect y_k. However, asymptotic results remains true with y_{k-1} replaced by y_k as well.

Function spaces

The polygonal line process S_n^{pl} can be viewed as a random element either in $\mathrm{C}[0,1]$ or in $\mathrm{H}_\alpha^o[0,1]$. Continuous function space $\mathrm{C}[0,1]$ is endowed with the uniform norm

$$\|f\|_\infty = \sup_{0\le t\le 1} |f(t)| \quad f \in \mathrm{C}[0,1].$$

For $\alpha \in (0,1)$ the Hölder space

$$\mathrm{H}_\alpha^o[0,1] := \left\{ f \in C[0,1] : \lim_{\delta\to 0} \omega_\alpha(f,\delta) = 0 \right\},$$

endowed with the norm $\|f\|_\alpha := |f(0)| + \omega(f,1)$, where

$$\omega_\alpha(f,\delta) := \sup_{\substack{s,t\in[0,1] \\ 0<t-s<\delta}} \frac{|f(t)-f(s)|}{|t-s|^\alpha}$$

is a separable Banach space.

Process built on the ε_k's

The polygonal line process built on i.i.d. random variables (ε_j) is

$$W_n^{\mathrm{pl}}(t) = \sum_{j=1}^{[nt]} \varepsilon_j + (nr - [nt])\varepsilon_{[nt]+1}, \quad t \in [0,1].$$

By classical Donsker-Prohorov invariance principle

$$n^{-1/2} W_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W.$$

In what follows $\xrightarrow[n\to\infty]{\mathrm{E}}$ signifies convergence in distribution in the metric space E.

Invariance principle in Hölder space

By the classical Levy's result on the modulus of continuity of W, $W \in \mathrm{H}_\alpha^o[0,1]$ with probability one for every $0 \leq \alpha < 1/2$. Račkauskas and Suquet (2004) proved that for $0 < \alpha < 1/2$ the convergence

$$n^{-1/2}\sigma^{-1}W_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^0[0,1]} W$$

holds if and only if

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)}P(|\varepsilon_1| \geq t) = 0. \qquad (5)$$

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Convergence in $\mathrm{C}[0,1]$ and $\mathrm{H}_\alpha^o[0,1]$ spaces

Theorem

Suppose that (y_k) is generated by (4), $\phi_n = \mathrm{e}^{\gamma/n}$ with $\gamma < 0$ and that the sequence of polygonal lines $n^{-1/2} W_n^{\mathrm{pl}}$ converges weakly to the standard Brownian motion W either in $\mathrm{C}[0,1]$ or in $\mathrm{H}_\alpha^o[0,1]$ for some $0 < \alpha < 1/2$. Then $n^{-3/2} S_n^{\mathrm{pl}}$ converges weakly in the space under consideration to the integrated Ornstein-Uhlenbeck process J defined by :

$$J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s, \quad 0 \le t \le 1, \tag{6}$$

where $U_\gamma(s) = \int_0^s \mathrm{e}^{\gamma(s-r)}\,\mathrm{d}W(r)$.

Convergence with i.i.d. innoovations

Taking into account the classical Donsker-Prohorov invariance principle and the functional central limit theorem proved by R&S (2004) we have the following corollary.

Corollary

When the ε_k's are i.i.d. and centered, the weak convergence of $n^{-3/2}S_n^{\mathrm{pl}}$ to J holds

- *in* $\mathrm{C}[0,1]$ *provided that* $\mathbb{E}\varepsilon_1^2 < \infty$ *;*
- *in* $\mathrm{H}_\alpha^o[0,1]$ *for* $0 < \alpha < 1/2$ *under condition* (5)*.*

$y_{n,0} \neq 0$

Remark

For the sake of simplicity, we assumed the initialization $y_{n,0} = y_0 = 0$. Nevertheless if $y_{n,0} \neq 0$, then

$$y_k = \sum_{j=1}^{k} \mathrm{e}^{(k-j)\gamma/n} \varepsilon_j + \mathrm{e}^{k\gamma/n} y_{n,0}$$

and because of the normalization $n^{-3/2}$ the proof of the theorem holds, provided that $y_{n,0} = o_{\mathrm{P}}(n^{3/2})$.

Convergence $\mathrm{C}[0,1]$ space

Theorem

Suppose $(y_k)_{k\geq 0}$ *is generated by* (4) *and* $\phi_n = 1 - \gamma_n/n$, *where* (γ_n) *is a sequence of non negative numbers and tends to infinity slower than* n. *Assume also that the innovations* $(\varepsilon_k)_{k\geq 0}$ *are i.i.d. with* $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = 1$. *Then the following convergence holds.*

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W.$$

Convergence in $\mathrm{H}_\alpha^o[0,1]$ space

Theorem

Suppose $(y_k)_{k\geq 0}$ *is generated by* (4) *and* $\phi_n = 1 - \gamma_n/n$, *where* (γ_n) *is a sequence of non negative numbers and tends to infinity slower than n. Assume also that the innovations* $(\varepsilon_k)_{k\geq 0}$ *are i.i.d. and satisfy condition* $\lim_{t\to\infty} t^p P(|\varepsilon_0| > t) = 0$ *for some* $p > 2$. *Put* $\alpha = \frac{1}{2} - \frac{1}{p}$. *Then for* $0 < \beta < \alpha$,

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\beta^0[0,1]} W,$$

provided that

$$\liminf_{n\to\infty} \gamma_n n^{-\frac{\beta}{\alpha}} > 0.$$

Essential lemma

A key point in all the following limit theorems is to keep a good control on the asymptotic behaviour of $\max_{1\le k\le n}|y_k|$.

Lemma

Let $p \ge 2$. Assume that the innovations $(\varepsilon_k)_{k\ge 0}$ satisfy

$$\begin{array}{ll} \lim_{t\to\infty} t^p P(|\varepsilon_0| > t) = 0, & \text{if } p > 2 \\ \mathbb{E}\varepsilon_0^2 < \infty & \text{if } p = 2 \end{array}$$

For $p \ge 2$, put $\alpha = 1/2 - 1/p$. Then

$$n^{-1/2}\gamma_n^{\alpha} \max_{1\le k\le n} |y_k| \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

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Residuals process

In further research we focus on processes built from $\widehat{\varepsilon}_k$:

$$\widehat{Z}_n^{\mathrm{pl}}(t) = \sum_{j=1}^{[nt]} \widehat{\varepsilon}_j + (nr - [nt])\widehat{\varepsilon}_{[nt]+1} \tag{7}$$

where $\widehat{\varepsilon}_k$ are residuals of the process y_k defined by

$$\widehat{\varepsilon}_k = y_k - \widehat{\phi}_n y_{k-1}$$

Convergence in case 1

Theorem

Let $\alpha \in (0, 1/2)$. *Suppose that* y_k *is generated by* (4), $\phi_n = \mathrm{e}^{\gamma/n}$. *Also* (ε_k) *are independent identically distributed random variables with* $\mathbb{E}\varepsilon_0 = 0$ *and* $\mathbb{E}\varepsilon_0^2 = 1$. *Then*

$$n^{-1/2}\widehat{Z}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^{\mathrm{o}}[0,1]} W - A^{-1}BJ, \tag{8}$$

if and only if condition (5) *holds.*
Here $B = \int_0^1 U_\gamma(r)dW(r)$, $A = \int_0^1 U_\gamma(r)^2 dr$ *and* $J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s$.

Convergence in case 2 in $\mathrm{C}[0,1]$

Theorem

Suppose y_k is generated by (4) and $\phi_n = 1 - \gamma_n/n$, where γ_n is non negative and goes to infinity slower than n. Assume also that the innovations (ε_k) are i.i.d. with $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = 1$. Then the following convergences hold :

$$n^{-1/2}\widehat{Z}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{C}[0,1]} W. \qquad (9)$$

Convergence in case 2 in $\mathrm{H}_\alpha^o[0,1]$

Theorem

Suppose y_k is generated by (4) and $\phi_n = 1 - \gamma_n/n$, where γ_n is non negative and goes to infinity slower than n. Assume also that the innovations $(\varepsilon_k)_{k\geq 1}$ are i.i.d. and satisfy condition

$$\lim_{t\to\infty} t^p \mathbb{P}(|\varepsilon_0| > t) = 0 \tag{10}$$

for some $p > 2$. Put $\alpha = \frac{1}{2} - \frac{1}{p}$. Then for $0 < \beta \leq \alpha$,

$$n^{-1/2}\widehat{Z}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\beta^0[0,1]} W, \tag{11}$$

where W is a standard Wiener process if

$$\liminf_{n\to\infty} \gamma_n n^{-\frac{2\beta}{1+2\alpha}} > 0. \tag{12}$$

Possible appilcations

1. Epidemic change in mean of innovations ;
2. Epidemic change in variance of innovations ;
3. Epidemic change in parameter γ ;
4. Epidemic change in coefficient ϕ_n between stationary and nonstationary state.

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