Functional central limit theorems for nearly nonstationary processes and applications for testing epidemic change

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AR(1) process

First order autoregressive $AR(1)$ process is generated according to the scheme

$$y_k = \phi y_{k-1} + \varepsilon_k, \quad k \geq 1, \tag{1}$$

where (ε_k) are the innovations at time k, and ϕ is an unknown parameter.

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where (ε_k) are the innovations at time k, and ϕ is an unknown parameter.

1. $|\phi| < 1$, then (1) is stationary process ;
2. $|\phi| > 1$, then (1) is explosive process ;
3. $\phi = 1$, then (1) is nonstationary process.

ϕ is "close" to 1

1. For $|\phi| \leq 1$ the asymptotics :

$$\left(\sum_{k=1}^{n} y_{k-1}^2 \right)^{1/2} (\widehat{\phi} - \phi) \xrightarrow[n \to \infty]{\mathbb{R}} \mathfrak{N}(0, 1). \quad (2)$$

2. For $\phi = 1$

$$\left(\sum_{k=1}^{n} y_{k-1}^2 \right)^{1/2} (\widehat{\phi} - 1) \xrightarrow[n \to \infty]{\mathbb{R}} \frac{\frac{1}{2}(W^2(1) - 1)}{\left(\int_0^1 W^2(t)\,\mathrm{d}t \right)^{1/2}} \quad (3)$$

3. Because of $\mathbb{P}(\tau \leq 0) = \mathbb{P}(W^2(1) \leq 1) = 0.684$, (2) may not be a satisfactory approximation when ϕ is "close" to 1 and the sample size is moderate. Also (3) could be used. However, neither (2) nor (3) seems to be intuitive approximations.

Nearly nonstationary process

Suppose we have first-order autoregressive process $(y_{n,k})$ given by

$$y_{n,k} = \phi_n y_{n,k-1} + \varepsilon_k, \quad k \geq 1, \quad n \geq 1, \tag{4}$$

where

1. $\phi_n \to 1$, as $n \to \infty$,
2. (ε_k) is a sequence of i.i.d.random variables with $\mathbb{E}\varepsilon_k = 0$ and $\mathbb{E}\varepsilon_k^2 = 1$,
3. $y_{n,1} \dots, y_{n,n}$ are observations and n is a sample size,
4. for simplicity $y_{n,0} = y_0 = 0$.

Parametrisation of ϕ_n

Case 1

$\phi_n = \mathrm{e}^{\gamma/n}$ with constant $\gamma < 0$.
This parametrisation was suggested by Phillips (1987 m.).

Case 2

$\phi_n = 1 - \frac{\gamma_n}{n}$, $\gamma_n \to \infty$ slower than n.
This parametrisation was suggested by Phillips and Giraitis (2006 m.)

We will use LSE estimate

$$\widehat{\phi}_n = \frac{\sum_{k=1}^n y_k y_{k-1}}{\sum_{k=1}^n y_{k-1}^2}$$

based on observations $y_1, \ldots, y_n$ with $y_0 = 0$ for convenience.

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Process built on the y_k's

We focus on polygonal line processes built on the y_k's :

$$S_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} y_{k-1} + (nt - [nt]) y_{[nt]}, \quad t \in [0,1], \quad n \geq 1.$$

and on polygonal line process built on the $\widehat{\varepsilon}_k$'s :

$$\widehat{W}_n^{\mathrm{pl}}(t) := \sum_{k=1}^{[nt]} \widehat{\varepsilon}_k + (nt - [nt]) \widehat{\varepsilon}_{[nt]+1}, \quad t \in [0,1], \quad n \geq 1,$$

where $\widehat{\varepsilon}_k$ are residuals of the process y_k defined by

$$\widehat{\varepsilon}_k = y_k - \widehat{\phi}_n y_{k-1}$$

Function spaces

The polygonal line process S_n^{pl} can be viewed as a random element in Hölder space $\mathrm{H}_\alpha^o[0,1]$. For $\alpha \in (0,1)$

$$\mathrm{H}_\alpha^o[0,1] := \left\{ f \in C[0,1] : \lim_{\delta \to 0} \omega_\alpha(f,\delta) = 0 \right\},$$

endowed with the norm $\|f\|_\alpha := |f(0)| + \omega(f,1)$, where

$$\omega_\alpha(f,\delta) := \sup_{\substack{s,t \in [0,1] \\ 0 < t-s < \delta}} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

is a separable Banach space.

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Convergence in $\mathrm{H}_\alpha^o[0,1]$ spaces

Theorem

Suppose that (y_k) is generated by (4), $\phi_n = \mathrm{e}^{\gamma/n}$ with $\gamma < 0$ and that the sequence of polygonal lines $n^{-1/2}W_n^{\mathrm{pl}}$ converges weakly to the standard Brownian motion W in $\mathrm{H}_\alpha^o[0,1]$ for some $0 < \alpha < 1/2$. Then $n^{-3/2}S_n^{\mathrm{pl}}$ converges weakly in the space under consideration to the integrated Ornstein-Uhlenbeck process J defined by :

$$J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s, \quad 0 \le t \le 1, \tag{5}$$

where $U_\gamma(s) = \int_0^s \mathrm{e}^{\gamma(s-r)}\,\mathrm{d}W(r)$.

Convergence in $\mathrm{H}^0_\beta[0,1]$ space

Theorem

Suppose (y_k) *is generated by* (4) *and* $\phi_n = 1 - \gamma_n/n$, *where* (γ_n) *is a sequence of non negative numbers,* $\gamma_n \to \infty$ *and* $\gamma_n/n \to 0$ *as* $n \to \infty$. *Assume also that the innovations* (ε_k) *are i.i.d. and satisfy condition* $\lim_{t\to\infty} t^p P(|\varepsilon_0| > t) = 0$ *for some* $p > 2$. *Put* $\alpha = \frac{1}{2} - \frac{1}{p}$. *Then for* $0 < \beta < \alpha$,

$$n^{-1/2}(1-\phi_n)S_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}^0_\beta[0,1]} W,$$

provided that

$$\liminf_{n\to\infty} \gamma_n n^{-\frac{\beta}{\alpha}} > 0.$$

Convergence in $\mathrm{H}_\alpha^o[0,1]$

Theorem

Let $\alpha \in (0, 1/2)$*. Suppose that* (y_k) *is generated by* (4)*,* $\phi_n = \mathrm{e}^{\gamma/n}$*. Also* (ε_k) *are independent identically distributed random variables with* $\mathbb{E}\varepsilon_0 = 0$*. Then*

$$n^{-1/2}\widehat{W}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\alpha^o[0,1]} W - A^{-1}B'J, \tag{6}$$

if and only if condition

$$\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| \geq t) = 0. \tag{7}$$

holds.
Here $B' = \int_0^1 U_\gamma(r)dW(r)$*,* $A = \int_0^1 U_\gamma(r)^2 dr$ *and* $J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s$*.*

Convergence in $\mathrm{H}_\alpha^o[0,1]$

Theorem

Suppose (y_k) *is generated by* (4) *and* $\phi_n = 1 - \gamma_n/n$*, where* γ_n *is a sequence of non negative constants,* $\gamma_n \to \infty$ *and* $\gamma_n/n \to 0$ *as* $n \to \infty$*. Assume also that the innovations* (ε_k) *are i.i.d. and satisfy condition*

$$\lim_{t\to\infty} t^p \mathbb{P}(|\varepsilon_0| > t) = 0 \tag{8}$$

for some $p > 2$*. Put* $\alpha = \frac{1}{2} - \frac{1}{p}$*. Then for* $0 < \beta \le \alpha$*,*

$$n^{-1/2}\widehat{W}_n^{\mathrm{pl}} \xrightarrow[n\to\infty]{\mathrm{H}_\beta^0[0,1]} W, \tag{9}$$

where W *is a standard Wiener process if*

$$\liminf_{n\to\infty} \gamma_n n^{-\frac{2\beta}{1+2\alpha}} > 0. \tag{10}$$

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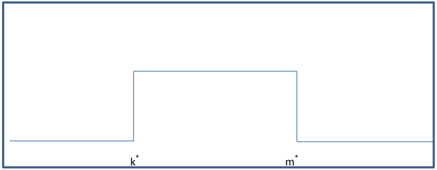
Epidemic change in mean

Hypothesis for y_k's

$z_{n,k} = a_n \mathbf{1}_{\{k^* < k \le k^* + l^*\}} + y_{n,k}$
$H_0 : a_n = 0\,;$
$H_A : a_n \neq 0.$

Hypothesis for ε_k's

$H_0 : \mathbb{E}\varepsilon_k = 0\,;$
$H_A : \mathbb{E}\varepsilon_k = a\mathbf{1}_{\{k^* < k \le k^* + l^*\}}.$



Test statistics with y_k's

We construct uniform increment statistics :

$$UI_S(n,\alpha) := \sup_{k^*,l^*} \frac{\left| S_n^{\mathrm{st}}(\mathbb{I}_n^*) - \frac{l^*}{n} S_n^{\mathrm{st}}(n) \right|}{\left| \frac{l^*}{n}\left(1 - \frac{l^*}{n}\right) \right|^{\alpha}}$$

where

$$\mathbb{I}_n^* = \{k^*+1, \dots, k^*+l^*\}$$
$$S_n^{\mathrm{st}}(\mathbb{I}_n^*) = \sum_{k\in\mathbb{I}_n^*} y_{n,k-1}$$
$$S_n^{\mathrm{st}}(n) = \sum_{k=1}^{n} y_{n,k-1}$$

Behaviour under null hypothesis in case 1

Theorem

Suppose that (y_k) *is generated by* (4), $\phi_n = \mathrm{e}^{\gamma/n}$ *with* $\gamma < 0$ *and that the sequence* (ε_k) *are i.i.d. random variables with mean* 0 *and satisfy condition* $\lim_{t\to\infty} t^{1/(1/2-\alpha)} P(|\varepsilon_1| \geq t) = 0$*. Then under hypothesis* H_0

$$n^{-3/2} UI_S(n,\alpha) \xrightarrow[n\to\infty]{\mathbb{R}} UI_{S,1}(\alpha). \tag{11}$$

Here

$$UI_{S,1}(\alpha) = \sup_{t,s} \frac{|J(t) - J(s) - (t-s)J(1)|}{|(t-s)(1-(t-s))|^{\alpha}}$$

where $J(t) = \int_0^t U_\gamma(r)\,\mathrm{d}r$ *and* $U_\gamma(r) = \int_0^s \mathrm{e}^{\gamma(r-v)}\,\mathrm{d}W(v)$.

Consistency of the test in case 1

Theorem

Let $0 < \alpha < 1/2$. Suppose (y_k) is generated by (4) and $\phi_n = \mathrm{e}^{\gamma/n}$, where $\gamma < 0$ is a constant. Under alternative H_A, if

$$\lim_{n\to\infty} |a_n| \, h_n^{1-\alpha} n^{-1/2} = \infty, \quad \textit{where} \quad h_n := \frac{l^*}{n}\left(1 - \frac{l^*}{n}\right)$$

then

$$n^{-3/2} UI_S(n, \alpha) \xrightarrow[n\to\infty]{\mathrm{P}} \infty.$$

Behaviour under null hypothesis in case 2

Theorem

Suppose (y_k) *is generated by* (4) *and* $\phi_n = 1 - \gamma_n/n$, *where* (γ_n) *is a sequence of non negative numbers,* $\gamma_n \to \infty$ *and* $\gamma_n/n \to 0$, *as* $n \to \infty$. *Assume also that the innovations* (ε_k) *are i.i.d. and satisfy condition* $\lim_{t\to\infty} t^p P(|\varepsilon_0| > t) = 0$ *for some* $p > 2$. *Put* $\alpha = \frac{1}{2} - \frac{1}{p}$. *Also for* $0 < \beta < \alpha$ *condition* $\liminf_{n\to\infty} \gamma_n n^{-\frac{\beta}{\alpha}} > 0$ *holds. Then under hypothesis* H_0

$$n^{-1/2}(1-\phi_n)UI_S(n,\alpha) \xrightarrow[n\to\infty]{\mathbb{R}} UI_2(\alpha),$$

Here

$$UI_2(\alpha) = \sup_{t,s} \frac{|B(t)-B(s)|}{|(t-s)(1-(t-s))|^{\alpha}}$$

where $B(t) = W(t) - tW(1)$ *is Brownian bridge.*

Consistency of the test in case 2

Theorem

Let $0 < \alpha < 1/2$. Suppose (y_k) is generated by (4) and $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of non negative numbers, $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. Under alternative H_A, if

$$\lim_{n\to\infty} |a_n|\, h_n^{1-\alpha} n^{-1/2}\gamma_n = \infty, \quad \text{where} \quad h_n := \frac{l^*}{n}\left(1 - \frac{l^*}{n}\right)$$

then

$$(1-\phi_n)n^{-1/2} UI_S(n,\alpha) \xrightarrow[n\to\infty]{\mathrm{P}} \infty.$$

Test statistics with $\widehat{\varepsilon}_k$'s

We construct uniform increment statistics :

$$UI_W(n,\alpha) := \sup_{k^*,l^*} \frac{\left|\widehat{W}_n^{\mathrm{st}}(\mathbb{I}_n^*) - \frac{l^*}{n}\widehat{W}_n^{\mathrm{st}}(n)\right|}{\left|\frac{l^*}{n}\left(1-\frac{l^*}{n}\right)\right|^{\alpha}}$$

where

$$\mathbb{I}_n^* = \{k^*+1,\dots,k^*+l^*\}$$

$$\widehat{W}_n^{\mathrm{st}}(\mathbb{I}_n^*) = \sum_{k\in\mathbb{I}_n^*}\widehat{\varepsilon}_k$$

$$\widehat{W}_n^{\mathrm{st}}(n) = \sum_{k=1}^{n}\widehat{\varepsilon}_k$$

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Theorem

Suppose that (y_k) is generated by (4), $\phi_n = \mathrm{e}^{\gamma/n}$ with $\gamma < 0$ and that the sequence (ε_k) are i.i.d. random variables with mean 0 and variance 1. Coefficient $\widehat{\phi}_n$ is estimated by LSE. Then under hypothesis H_0

$$n^{-1/2} UI_W(n, \alpha) \xrightarrow[n\to\infty]{\mathbb{R}} UI_{W,1}(\alpha). \tag{12}$$

Here

$$UI_{W,1}(\alpha) = \sup_{t,s} \frac{\left|B(t) - B(s) - A^{-1}B'(J(t) - J(s) - (t-s)J(1))\right|}{\left|(t-s)(1-(t-s))\right|^{\alpha}}$$

Here $B' = \int_0^1 U_\gamma(r)dW(r)$, $A = \int_0^1 U_\gamma(r)^2 dr$ and $J(t) := \int_0^t U_\gamma(s)\,\mathrm{d}s$, $B(t)$ is a Brownian bridge.

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$$\lim_{n\to\infty} |a|\, h_n^{1-\alpha} n^{1/2} = \infty, \quad \text{where} \quad h_n := \frac{l^*}{n}\left(1 - \frac{l^*}{n}\right)$$

then

$$n^{-1/2} UI_W(n, \alpha) \xrightarrow[n\to\infty]{\mathrm{P}} \infty.$$

Behaviour under null hypothesis in case 2

Theorem

Suppose (y_k) is generated by (4) and $\phi_n = 1 - \gamma_n/n$, where (γ_n) is a sequence of non negative numbers, $\gamma_n \to \infty$ and $\gamma_n/n \to 0$, as $n \to \infty$. Assume also that the innovations (ε_k) are i.i.d. and satisfy condition $\lim_{t\to\infty} t^p P(|\varepsilon_0| > t) = 0$ for some $p > 2$. Put $\alpha = \frac{1}{2} - \frac{1}{p}$. Also for $0 < \beta \le \alpha$ condition $\liminf_{n\to\infty} \gamma_n n^{-\frac{2\beta}{1+2\alpha}} > 0$ holds. Then under hypothesis H_0

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Here

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