Section 5.5 Solving Recurrences

Any recursively defined function f with domain $\mathbf{N}$ that computes numbers is called a *recurrence* or *recurrence relation*. We often denote $f(n)$ by f_n. The goal is to find a closed formula for the general term f_n.

Simple Recurrences

It's easy to solve simple recurrences of the following form, where a_i and b_i are expressions that do not contain r.

$$r_0 = b_0$$
$$r_n = a_n\, r_{n-1} + b_n$$

Substitution Method

Start with the general equation $r_n = a_n\, r_{n-1} + b_n$ and use the definition to keep substituting for r on the right side until we discover a general pattern that allows us to skip ahead and substitute for the basis $r_0 = b_0$.

Example. Solve the recurrence

$$a_0 = 1$$
$$a_n = a_{n-1} + 2n.$$

Solution.
$$
\begin{aligned}
a_n &= a_{n-1} + 2n \\
&= a_{n-2} + 2(n-1) + 2n \\
&= a_{n-3} + 2(n-2) + 2(n-1) + 2n \\
&\quad \dots \\
&= a_0 + 2 + 2{\cdot}2 + \dots + 2(n-2) + 2(n-1) + 2n \\
&= 1 + 2 + 2{\cdot}2 + \dots + 2(n-2) + 2(n-1) + 2n \\
&= 1 + 2(1 + 2 + \dots + (n-2) + (n-1) + n) = 1 + n(n+1).
\end{aligned}
$$

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Cancellation Method

Start with the general equation $r_n = a_n r_{n-1} + b_n$. Then write a new equation whose left side is the term involving r on the right side of the given equation. Continue this process by writing a new equation whose left side is the term involving r on the right side of the previous equation. Do this until we discover a general pattern that allows us to skip ahead and write the last equation that contains r_0 on the right side. Then add up the equations and cancel the like terms to obtain an expression for r_n.

Example. Solve the recurrence

$$a_0 = 1$$
$$a_n = a_{n-1} + 2n.$$

Solution.
$$a_n \;\; = a_{n-1} + 2n$$
$$a_{n-1} = a_{n-2} + 2(n-1)$$
$$a_{n-2} = a_{n-3} + 2(n-2)$$
$$\ldots$$
$$a_1 \;\; = a_0 + 2(1)$$

Now add up the equations and cancel the like terms on either side to obtain the following equation for a_n:

$$a_n \;\; = a_0 + 2(1) + \ldots + 2(n-2) + 2(n-1) + 2n$$
$$= 1 + 2[1 + \ldots + (n-2) + (n-1) + n]$$
$$= 1 + n(n+1).$$

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Divide and Conquer Algorithms. Many divide and conquer algorithms are special cases of the following general form. To solve a problem with input of size n split it into s smaller problems each with input of size n/b. To simplify things we assume that $n = b^k$ for two positive integers b and k. Assume that for inputs of size n it takes tn operations to split up the problem and to do other tasks such as assembling the solution. Let a_n be the number of operations to solve the problem with input size n. Then we have the recurrence

$$a_1 = t$$
$$a_n = sa_{n/b} + tn.$$

Solution (by cancellation):

$$a_n = sa_{n/b} + tn$$
$$sa_{n/b} = s^2 a_{n/b^2} + (s/b)tn$$
$$s^2 a_{n/b^2} = s^3 a_{n/b^3} + (s/b)^2 tn$$
$$\vdots$$
$$s^{k-1} a_{n/b^{k-1}} = s^k a_{n/b^k} + (s/b)^{k-1} tn$$

Since $n = b^k$, we stop with the last equation.
Now add the equations and cancel to obtain

$$a_n = s^k a_1 + tn \sum_{i=0}^{k-1} (s/b)^i.$$

For example, the closed forms for the cases $s = b$ and $s \neq b$ are as follows:

$$(s = b): \ a_n = b^{\log_b n} a_1 + \sum_{i=0}^{k-1} tn = tn + ktn = tn + (\log_b n)tn = tn(1 + \log_b n).$$

$$(s \neq b): \ a_n = s^k a_1 + tn \sum_{i=0}^{k-1} (s/b)^i = ts^{\log_b n} + tn \left(\frac{(s/b)^{\log_b n} - 1}{(s/b) - 1} \right).$$

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Generating Functions

The *generating function* for the infinite sequence $a_0, a_1, \ldots, a_n, \ldots$ is given by the following expression, where x is an indeterminate symbol.

$$\sum_{n=0}^{\infty} a_n x^n.$$

Other names are *formal power series* and infinite polynomial. They can be added, multiplied, divided, and equated just like polynomials.

Some generating functions have closed forms. For example, the generating function for the sequence $1, 1, \ldots, 1, \ldots$ has the following closed form, which can be verified by multiplying both sides by $1 - x$.

(*Geometric Series Generating Function*) $\quad \sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}.$

Generating functions with closed forms can be used to solve recurrences.

Example. Suppose someone tells us that the generating function for a_n satisfies the equation

$$\sum_{n=0}^{\infty} a_n x^n = \dfrac{3}{2x+1}.$$

Notice that $\quad \displaystyle\sum_{n=0}^{\infty} a_n x^n = \dfrac{3}{2x+1} = 3\left(\dfrac{1}{1-(-2x)}\right) = 3\sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty} 3(-2)^n x^n.$

We can equate coefficients to obtain the solution $a_n = 3(-2)^n$. 4

Example. Solve the following recurrence.

$$a_0 = 3$$
$$a_1 = 5$$
$$a_n = 2a_{n-1} + 3a_{n-2}.$$

Solution. **Step 1** tells us to use the recurrence to find an algebraic equation in terms of the generating function $A(x)$.

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$= 3 + 5x + \sum_{n=2}^{\infty} (2a_{n-1} + 3a_{n-2}) x^n$$

$$= 3 + 5x + \sum_{n=2}^{\infty} 2a_{n-1} x^n + \sum_{n=2}^{\infty} 3a_{n-2} x^n$$

$$= 3 + 5x + 2x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 3x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$= 3 + 5x + 2x \sum_{n=1}^{\infty} a_n x^n + 3x^2 \sum_{n=0}^{\infty} a_n x^n$$

$$= 3 + 5x + 2x(A(x) - a_0) + 3x^2 A(x)$$

$$= 3 + 5x + 2x(A(x) - 3) + 3x^2 A(x).$$

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Step 2 tells us to solve the equation for $A(x)$ and try to transform the resulting expression using known generating functions.

$$A(x) = 3 + 5x + 2x(A(x) - 3) + 3x^2 A(x)$$

$$= 3 - x + 2xA(x) + 3x^2 A(x).$$

Collect terms to obtain $A(x)(1 - 2x - 3x^2) = 3 - x.$

Now solve for $A(x)$: $\quad A(x) = \dfrac{3 - x}{1 - 2x - 3x^2} = \dfrac{3 - x}{(1 + x)(1 - 3x)} = \dfrac{1}{1 + x} + \dfrac{2}{1 - 3x}$

$$= \dfrac{1}{1 - (-x)} + 2 \cdot \dfrac{1}{1 - 3x}$$

$$= \sum_{n=0}^{\infty} (-x)^n + 2 \sum_{n=0}^{\infty} (3x)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n + 2 \sum_{n=0}^{\infty} 3^n x^n$$

$$= \sum_{n=0}^{\infty} ((-1)^n + 2 \cdot 3^n) x^n.$$

Step 3 tells us to equate coefficients to get $\qquad a_n = (-1)^n + 2 \cdot 3^n.$

Step 4 tells us to check the answer. (*Quiz*)

Answer. a_0 and a_1 check out OK. Assume $n > 1$ and a_k is OK for $k < n$. Show a_n is OK.

$$a_n = 2a_{n-1} + 3a_{n-2} = 2((-1)^{n-1} + 2 \cdot 3^{n-1}) + 3((-1)^{n-2} + 2 \cdot 3^{n-2}) = ((-1)^n + 2 \cdot 3^n).$$

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