Section 5.3 Permutations and Combinations

Permutations (order counts)

- A *permutation* of a set is an arrangement of the objects from a set.
- There are $n!$ permutations of an n-element set, where $0! = 1$ and $n! = n \cdot (n - 1)!$
- The number of permuations of length r chosen from an n-element set (where $r \leq n$) is

$$P(n,r) = \frac{n!}{(n-r)!}.$$

Example/Quiz. Let $S = \{a, b, c\}$. Calculate each expresssion and list the corresponding permutations of S: $P(3, 3)$, $P(3, 2)$, and $P(3, 1)$.

Solution.

$P(3, 3) = 3!/0! = 6$ with permutations *abc, acb, bac, bca, cab, cba*.
$P(3, 2) = 3!/1! = 6$ with permutations *ab, ac, ba, bc,ca, cb*.
$P(3, 1) = 3!/2! = 3$ with permutations *a, b*, and *c*.

Quiz. Find the number of permutations of the letters in the word *radon*.

Answer. $5! = 120$.

Bag Permutations

The number of permutations of an n-element bag with k distinct elements, where the ith distinct element is repeated m_i times is

$$\frac{n!}{m_1!\cdots m_k!}.$$

The idea behind the formula
The idea is easy to see from an example. Suppose the bag is [*a, a, b, b, b*]. Then we can think of the letters as distinct elements of a set by placing subscripts on the repeated elements to obtain the set $\{a_1, a_2, b_1, b_2, b_3\}$. There are 5! permutations of this set. But we don't want to count permutations that are repeated if we drop the subscripts. For example, don't want to count $a_1a_2b_1b_2b_3$ and $a_2a_1b_1b_2b_3$ as different. So we need to divide 5! by the number of permutations of each subscripted element: 2! for $\{a_1, a_2\}$ and 3! for $\{b_1, b_2, b_3\}$. This gives 5!/(2!3!) = 10 permutations of the bag [*a, a, b, b, b*].

Example. Calculate the number of permutations of [*a, a, b, b*] and list each permutation.

Answer. 4!/(2!2!) = 6 with permutations *aabb, abab, abba, bbaa, baba, baab*.

Quiz. Find the number of permutations of the letters in the word *babbage*.

Answer. 7!/(2!3!) = 420.

Quiz (5 minutes). Find the smallest size n for strings of length n over $\{a, b, c\}$ that can be used as distinct codes for 27 people, where a is repeated k times, b is repeated l times, c is repeated m times, and $k + l + m = n$.

Answer. Use trial and error to solve the following inequality for the smallest n that satisfies the given conditions.

$$\frac{n!}{k!l!m!} \geq 27$$

The solution is $n = 5$ with, for example, $k = 2$, $l = 2$, and $m = 1$.

Combinations (order does not count)

The number of *combinations of n things taken k at a time* is the number of k-element subsets of an n-element set, and is given by

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The idea: Take the number of permutations of n things taken k at a time. This count includes all the permutations of k elements. So divide by $k!$ to get the count without regard to order.

Example/Quiz. Let $S = \{a, b, c\}$. Calculate each expression and list the corresponding subsets of S. $C(3, 3)$, $C(3, 2)$, $C(3, 1)$, $C(3, 0)$.

Solution. $C(3, 3) = 1$ with the subset $\{a, b, c\}$. $C(3, 2) = 3$ with subsets $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. $C(3, 1) = 3$ with subsets $\{a\}$, $\{b\}$, $\{c\}$. $C(3, 0) = 1$ with subset $\varnothing$.

Quiz. Suppose there is a set of 5 cans of soda $\{a, b, c, d, e\}$. Find the number of combinations of 5 cans of soda taken 3 at a time and list each combination.

Answer. $\quad C(5,3) = \binom{5}{3} = \dfrac{5!}{3!\,2!} = 10.$

The 3-element subsets are $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, e\}$, $\{a, c, d\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{b, c, d\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{c, d, e\}$.

Binomial Theorem. $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. *Note*: $\binom{n}{k}$ is called a *binomial coefficient*.

Example.

$$(x+y)^3 = \sum_{k=0}^{3} \binom{n}{k} x^{3-k} y^k = \binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} x y^2 + \binom{3}{3} y^3 = x^3 + 3x^2 y + 3xy^2 + y^3.$$

Bag Combinations

The number of k-element bags over an n-element set (with k and n positive) is given by

$$\binom{n+k-1}{k}.$$

The idea behind the formula

There is a bijection between the k-element bags over $\{1, 2, \ldots, n\}$ and the k-element subsets of $\{1, 2, \ldots, n, n+1, \ldots, n+(k-1)\}$. The bijection associates each k-element bag $[x_1, x_2, \ldots, x_k]$ where $x_i \le x_{i+1}$, with the k-element subset $\{x_1, x_2+1, x_3+2, \ldots, x_k+(k-1)\}\}$, and the number of these k-element subsets is given by desired formula.

Example/Quiz. Let S = $\{a, b, c\}$. Calculate the number of 3-element bags over S and list each bag.

Solution. $\binom{3+3-1}{3} = \binom{5}{3} = \frac{5!}{3!\,2!} = 10.$

The 3-element bags are $[a, b, c]$, $[a, b, b]$, $[a, c, c]$, $[a, a, b]$, $[a, a, c]$, $[a, a, a]$, $[b, c, c]$, $[b, b, c]$, $[b, b, b]$, $[c, c, c]$.

Quiz. Find the number of ways that 5 cans of soda can be chosen from a machine that dispenses 4 kinds of soda $\{a, b, c, d\}$.

Solution. $\binom{4+5-1}{5} = \binom{8}{5} = \frac{8!}{3!\,5!} = 56.$ For example, $[a, a, b, b, d]$, and so on.