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LAWS OF THE ITERATED LOGARITHM FOR ADDITIVE FUNCTIONS

Eugenijus MANSTAVIČIUS

1. Introduction

The most of investigations in probabilistic number theory is devoted to the convergence of distributions generated by arithmetic functions. The purpose of the present paper is to cal one's attention to analogues of probability theorems formulated in terms of event occurence with probability of one (almost sure occurence). Among the results we can reckon only proposition 6 from P. Erdös' paper [1] and theorem 7.2 J. Kubilius' monograph [2]. More systematic investigations can be found in our papers [3–6] and in short communications [7–11].

Our interest is attached to additive function $h:\mathbb{N}\to\mathbb{R}$ as well as to sequence

$$\Big\{ h_k(m) \stackrel{\text{def}}{=} \sum_{p\le k} h^{(p)}(m); \quad k\ge 2 \Big\}$$

where $h^{(p)}(m)=h(p^\alpha)$ if $p^\alpha \| m$ and $h^{(p)}(m)=0$ otherwise. Here and in the following p stands for a prime number, $\alpha\in\mathbb{N}$. Put $\nu_n(\ldots)=n^{-1}\#\{m\le n;\ldots\}$. Denote $u^*=\operatorname{sgn} u$ if $|u|\ge 1$ and $u^*=u$, if $|u|<1$. Let ζ_p be independent random variables getting values $h(p)$ and 0 with probabilities $\frac{1}{p}$ and $1-\frac{1}{p}$ respectively. The constants in the symbol $\ll$ are considered to be absolute.

All the proofs given below are based on a tuncation procedure and the following lemmas.

Lemma 1. *Let $r=r(n)\to\infty$, $\log r=o(\log n)$ when $n\to\infty$ and $P_i\subset\{p;\,p\le r\}$, $i=1,\ldots,s$. Then uniformly in $\mathbb{B}\subset\mathbb{R}^s$ and $s\ge 1$*

$$\nu_n\Bigg(\Big(\sum_{p|m,\,p\in P_1} h(p),\ \ldots,\ \sum_{p|m,\,p\in P_s} h(p)\Big)\in\mathbb{B}\Bigg) = P\Bigg(\Big(\sum_{p\in P_1}\zeta_p,\ \ldots,\ \sum_{p\in P_s}\zeta_p\Big)\in\mathbb{B}\Bigg)+o(1)$$

as $n\to\infty$.

The proposition represents one of the versions of the Kubilius Fundamental lemma [2].

Lemma 2. *Let* $p_1, \ldots, p_M$ *be any rearrangement of the primes up to* n , $M = \pi(n)$. *For arbitrary constants* $c_1 \geq c_2 \geq \ldots \geq c_M > 0$, $u > 0$ *and* $0 < M_1 \leq M$ *we have*

$$\nu_n\left(\max_{M_1 \leq k \leq M} c_k \left|\sum_{i \leq k} h^{(p_i)}(m) - \sum_{i \leq k} \frac{h(p_i)}{p_i}\right| \geq u\right) \ll$$

$$\ll u^{-2}\left(c_{M_1}^2 \sum_{i \leq M_1} \frac{h^2(p_i)}{p_i} + \sum_{M_1 < i \leq M} \frac{c_i^2 h^2(p_i)}{p_i}\right).$$

The inequality is an easy modification of theorem 4.1 from I. Z. Ruzsa's preprint [12].

2. A new form of the Erdös–Wintner theorem

As we have noted in paper [3] the celebrated Erdös–Wintner theorem [13] has another form.

Theorem 2.1 ([3]). *The following propositions are equivalent:*
(I) for every $\varepsilon > 0$

$$a(\varepsilon) \stackrel{\text{def}}{=} \lim_{n_1 \to \infty} \limsup_{n \to \infty} \nu_n\left(\max_{n_1 \leq k \leq n} |h(m) - h_k(m)| \geq \varepsilon\right) = 0;$$

(II) series

$$\sum_p \frac{h^*(p)}{p}, \qquad \sum_p \frac{{h^*}^2(p)}{p}$$

converge.

Arguments form the proof of theorem 2.2 below can be repeated to show us that always $a(\varepsilon) = 0$ or $a(\varepsilon) = 1$ (the zero–one law). Therefore proposition (I) can be used to determine convergence of $h_k(m)$ to $h(m)$ "for almost all m" when $k \to \infty$. In the following we shall use a more general definition.

Definition. *Let* b_k , $a_{kn} \in \mathbb{R}$, $b_k > 0$. *We say that* $b_k\big(h(m) - h_k(m) - a_{kn}\big)$ *tends to zero almost everywhere if for every* $\varepsilon > 0$

$$\lim_{n_1 \to \infty} \limsup_{n \to \infty} \nu_n\left(\max_{n_1 \leq k \leq n} b_k |h(m) - h_k(m) - a_{kn}| \geq \varepsilon\right) = 0.$$

For the sake of convenience the last statement we express as

$$b_k\big(h(m) - h_k(m) - a_{kn}\big) \Rightarrow 0 \quad \text{a.e.}$$

This new definition enables us to estimate the convergence rate of $h_k(m)$ to $h(m)$ a.e. At first, we point out a fairly simple approach.

Let $b_k \uparrow \infty$ as $k \to \infty$. According to theorem 2.1 convergence of series

$$\sum_p \frac{\big(b_p h(p)\big)^*}{p}, \qquad \sum_p \frac{\big(b_p h(p)\big)^{*2}}{p} \tag{1}$$

yields

$$g_k(m) \stackrel{\text{def}}{=} \sum_{k<p\le n} b_p\, h^{(p)}(m) \Rightarrow 0 \quad \text{a.e.}$$

Now by the Abel summation due to monotonicity of b_k we come to inequality

$$b_k\Big|\sum_{k<p\le n} h^{(p)}(m)\Big| \le 2 \max_{k\le l\le n} |g_l(m)|\,.$$

So from the convergence of series (1) we obtain

$$b_k\big(h(m) - h_k(m)\big) \Rightarrow 0 \quad \text{a.e.} \tag{2}$$

When only the second of series (1) converges one must use centralizing sequences. But such approach does not yield the exact order of the convergence rate. For some classes of functions $h(m)$ it can be found by use of corresponding results of probability theory.

Put for the sake of brevity $Lu = \log\max\{\mathrm{e}, u\}$,

$$A(k) = \sum_{p\le k} \frac{h(p)}{p}, \quad B^2(k) = \sum_{p\le k} \frac{h^2(p)}{p}, \quad U_k^2 = \sum_{p>k} \frac{h^2(p)}{p},$$

$$A_{kn} = A(n) - A(k), \quad v_k^2 = 2LLU_k^2, \quad R_{kn}(m) = U_k^{-1} v_k^{-1}\big(h(m) - h_k(m) - A_{kn}\big)\,.$$

Theorem 2.2 *Let $u_n > 0$ for every $n \in \mathbb{N}$. Suppose that for $n \to \infty$ and some sequence $r = r(n)$, $\log r = o(\log n)$, we have*

$$\rho_n \stackrel{\text{def}}{=} \sum_{r<p\le n} \frac{h^2(p)}{p\,U_p^2} = o(v_r^2)$$

and $U_n \to 0$. If $h(p) = o\big(U_p v_p^{-1}\big)$ as $p \to \infty$ then for every $\varepsilon > 0$

$$\lim_{n_1\to\infty} \limsup_{n\to\infty} \nu_n\Big(\max_{n_1\le k\le n} |R_{kn}(m)| \ge 1 + \varepsilon \Big) = 0 \tag{3}$$

and for every $a \in [-1, 1]$

$$\lim_{n_1\to\infty} \liminf_{n\to\infty} \nu_n\Big(\min_{n_1\le k\le n} |R_{kn}(m) - a| \le \varepsilon \Big) = 1\,. \tag{4}$$

Consequently, under the conditions of theorem 2.2 we have the exact order of the convergence rate as well as the cluster set of sequence $\{R_{kn}(m)\}$. We can say that the last one coinsides with interval [-1,1] a.e.

Proof of Theorem 2.2. As in convergence (2) values $h(p^\alpha)$ for $\alpha \geq 2$ have no influence on the final result. In fact, for every $\delta > 0$

$$\nu_n\left(\max_{n_1\leq k\leq n} \sum_{\substack{k<p\leq n,\,\alpha\geq 2\\ p^\alpha\|m}} \Big(\big|h(p^\alpha)\big| + |h(p)|\Big) \geq \delta\right) \leq$$

$$\leq \nu_n\big(m;\ \exists\, p^\alpha\|m,\ \alpha\geq 2,\ n_1\leq p\leq n\big) \ll \sum_{n_1\leq p\leq n} p^{-2} = o(1)$$

when $n\to\infty$ and $n_1\to\infty$. Therefore in the following we consider only the case when $h(p^\alpha) = h(p)$ for every $\alpha \geq 1$ and p.

Let ν_1 and ν_2 be frequences in equalities (3) and (4) respectively. From lemma 2 with reverse rearragement of primes $n_1 < p \leq n$ we have

$$\Delta_n(\delta) \stackrel{\text{def}}{=} \nu_n\Big(\max_{r<k\leq n} |R_{kn}(m)| \geq \delta\Big) \ll \frac{\rho_n}{\delta^2 v_r^2} = o(1)$$

as $n\to\infty$. Whence

$$\begin{aligned}
&\nu_1 \leq \nu_n\Big(\max_{n_1\leq k\leq r} |R_{kn}(m)| \geq 1+\varepsilon\Big) + \Delta_n(1+\varepsilon) \leq \\
(5)\qquad &\leq \nu_n\Big(\max_{n_1\leq k\leq r} |h_r(m) - h_k(m) - A_{kr}|U_k^{-1}v_k^{-1} \geq 1+\frac{\varepsilon}{2}\Big) + \\
&+ \Big(\Delta_n\Big(\frac{\varepsilon}{2}\Big) + o(1)\Big) \stackrel{\text{def}}{=} v_1' + o(1), \quad n\to\infty\,.
\end{aligned}$$

Setting $S_k = \sum\limits_{p\geq k} \bar{\zeta}_p$ from lemma 1 we obtain

$$\begin{aligned}
v_1' &= P\Big(\max_{n_1\leq k\leq r} |S_k - S_r|U_k^{-1}v_k^{-1} \geq 1+\frac{\varepsilon}{2}\Big) + o(1) \leq \\
&\leq P\Big(\sup_{k\geq n_1} |\,S_k\,|U_k^{-1}v_k^{-1} \geq 1+\frac{\varepsilon}{4}\Big) + \\
&+ \bigg(P\Big(|\,S_r\,| \geq \frac{\varepsilon\, U_r v_r}{4}\Big) + o(1)\bigg) \stackrel{\text{def}}{=} P_{n_1} + o(1), \quad n\to\infty.
\end{aligned}$$

Now by use of theorem 2 [14] we conclude that $P_{n_1} = o(1)$ when $n_1\to\infty$. This, via inequality (5), proves the first part of our theorem.

Similarly, according to lemma 1 and the theorem used

$$\nu_2 \geq \nu_n\Big(\min_{n_1\leq k\leq r}|R_{kn}(m)-a|\leq\varepsilon\Big)\geq$$

$$\geq \nu_n\bigg(\min_{n_1\leq k\leq r}\bigg|\frac{h_r(m)-h_k(m)-A_{kr}}{U_k v_k}-a\bigg|\leq\frac{\varepsilon}{2}\bigg)-\Delta_n\Big(\frac{\varepsilon}{2}\Big)=$$

$$=P\Big(\min_{n_1\leq k\leq r}\big|(S_k-S_r)U_k^{-1}v_k^{-1}-a\big|\leq\frac{\varepsilon}{2}\Big)+o(1)\geq$$

$$\geq P\Big(\inf_{k\geq n_1}\big|S_kU_k^{-1}v_k^{-1}-a\big|\leq\frac{\varepsilon}{4}\Big)+o(1)$$

when $n\to\infty$ for every fixed $n_1\geq 2$. Theorem 2.2 is proved.

Equality (4) can be strengthenden to

$$\lim_{n\to\infty}\nu_n\Big(\min_{n_1\leq k\leq n}|R_{kn}(m)-a|\leq\varepsilon\Big)=1$$

for every $n_1\geq 2$ as we have seen for strongly additive functions. Is it possible to prove proposition (3) with $\varepsilon=0$?

3. An analogue of the Feller theorem

Now we shall consider the exact growth order of the sequence $\{h_k(m)-A(k);\ k\geq 2\}$ a.e. At first for the comparison we quote an analogue of the Kolmogorov law of the iterated logarithm in the so–called Strassen's formulattion.

Theorem 3.1 ([4]). *Let* $\beta_n\stackrel{\text{def}}{=}B(n)\sqrt{2LLB(n)}\to\infty$ *and*

$$\max_{p\leq n}|h(p)|=O\Big(\frac{B(n)}{\sqrt{LLB(n)}}\Big)$$

as $n\to\infty$. *Then for every* $\varepsilon>0$

$$\lim_{n_1\to\infty}\limsup_{n\to\infty}\nu_n\Big(\max_{n_1\leq k\leq n}\beta_k^{-1}|h_k(m)-A(k)|\geq 1+\varepsilon\Big)=0$$

and for every $a\in[-1,1]$

$$\lim_{n_1\to\infty}\liminf_{n\to\infty}\nu_n\Big(\min_{n_1\leq k\leq n}\big|\beta_k^{-1}\big(h_k(m)-A(k)\big)-a\big|\leq\varepsilon\Big)=1\,.$$

So in this case we can say that infinitely often (i.o.) for almost all (a.e.) m inequalities

$$(1-\varepsilon)\beta_k\leq h_k(m)-A(k)\leq(1+\varepsilon)\beta_k$$

are satisfied. Term $\varepsilon\,\beta_k$ can be improved by use of W. Feller's paper [15]. Not trying to change its style we use normalizations

$$b_k \stackrel{\text{def}}{=} d_k \left(\sum_{p\le k} E\,\bar\zeta_p^{\,2} \right)^{1/2} = d_k \left(\sum_{p\le k} \frac{h^2(p)}{p} \Bigl(1-\frac{1}{p}\Bigr) \right)^{1/2} \stackrel{\text{def}}{=} d_k\, D(k)$$

in the place of β_k. Here d_k stands for a monotonically increasing to infinity sequence which would be determined. Moreover, avoiding too cumbersome conditions in this paragraph we confine ourselves to strongly additive functions satisfying

$$|h(p)| \le \lambda_p D(p) \uparrow \infty \tag{6}$$

with $\lambda_p \downarrow 0$ when $p \to \infty$.

Let us define some quantities frequently used in the large deviation theorems. Define, in a formal way, coefficients Γ_{pl} by

$$\log \left[\prod_{q\le p} \Bigl(1 + \frac{\exp\{th(q)\}-1}{q} \Bigr) \exp\Bigl\{ -\frac{th(q)}{q} \Bigr\} \right] = \sum_{l=2}^{\infty} \frac{\Gamma_{pl}\, t^l}{l!}\,.$$

Here q denotes a prime number, $t \in \mathbb{R}$. For $|\,x\,|\lambda_p \le \frac{1}{12}$ under condition (6) equation

$$\sum_{l=2} \frac{\Gamma_{pl}\, t^{l-1}}{(l-1)!} = x\, D(p)$$

has a unique solution, say $t = t(x)$. Let $Q_p(x)$ be the function defined by

$$x^2 + \bigl(1 + Q_p(x)\bigr) = 2 \sum_{l=2}^{\infty} \Gamma_{pl}\, \frac{l-1}{l!}\, t^l\,.$$

Function $Q_p(x)$ is analytic for $|\,x\,|\lambda_p < \frac{1}{12}$ and for the coefficients of the expansion

$$Q_p(x) = \sum_{l=1}^{\infty} a_{pl}\, x^l = \frac{1}{3D^3(p)} \sum_{q\le p} \frac{h^3(q)}{q} \Bigl(1 - \frac{3}{q} + \frac{2}{q^2}\Bigr) x + \dots$$

estimates

$$\frac{|a_{pl}| \le \bigl(12\lambda_p\bigr)^l}{7}\,, \qquad l = 1, 2, \dots$$

are known (s.f. [15]).

Theorem 3.2. *Suppose condition (6) is satisfied, $d_p \geq 2$, $d_p \uparrow \infty$ when $p \to \infty$ and*

$$\lambda_p \leq \frac{1}{200} d_p \,. \tag{7}$$

The following propositions are true:

(I) if series

$$\sum_p \frac{h^2(p)\, d_p}{p\, D^2(p)} \exp\Big\{ -\frac{d_p^2}{2}\big(1 + Q_p(d_p)\big)\Big\} \tag{8}$$

converges, then

$$\lim_{n_1\to\infty} \limsup_{n\to\infty} \nu_{n_1,n} \stackrel{\text{def}}{=} \lim_{n_1\to\infty} \limsup_{n\to\infty} \nu_n\Big(\max_{n_1\leq k\leq n} b_k^{-1}|h_k(m) - A(k)| \geq 1\Big) = 0\,;$$

(II) if $\lim_{n_1\to\infty} \liminf_{n\to\infty} \nu_{n_1,n} = 0$, then series (8) converges;

(III) if series (8) diverges, then for every $n_1 \geq 2$

$$\lim_{n\to\infty} \nu_n\Big(\max_{n_1\leq k\leq n} b_k^{-1}|h_k(m) - A(k)| \geq 1\Big) = 1\,;$$

(IV) if $\lim_{n_1\to\infty} \limsup_{n\to\infty} \nu_{n_1,n} = 1$, then series (8) diverges.

Corollary ([10]). *If in the place of condition (7) estimate*

$$\lambda_p = O\big(d_p^{-3}\big)$$

is satisfied, then the propositions of the theorem are true after the change of series (8) by

$$\sum_p \frac{h^2(p)\, d_p}{p\, D^2(p)} \exp\Big\{ -\frac{d_p^2}{2}\Big\}\,.$$

Proof of Theorem 3.2. Choose $r = \max\big\{\log n, \exp\{d_n^{-1}\log n\}\big\}$. Then conditions (6) and (7) yield

$$D^2(n) - D^2(r) \ll D^2(n) d_n^{-2} \log\frac{\log n}{\log r} \leq D^2(n)\, d_n^{-2} \log d_n\,.$$

Therefore $2D^2(r) \geq D^2(n)$, when n is large enough. Now from lemma 2 we have

$$\begin{aligned}\Delta_n &\stackrel{\text{def}}{=} \nu_n\big(\max_{r\leq k\leq n} b_k^{-1}|h_k(m) - A(k)| \geq 1\big) \ll \\ &\ll d_r^{-2} + \sum_{r<p\leq n} \frac{h^2(p)}{p b_p^2} \ll d_r^{-2} = o(1), \quad n \to \infty.\end{aligned} \tag{9}$$

Whence setting $\eta_k = \sum_{p\le k} \bar\zeta_p$ and using lemma 1 for $n\to\infty$ we obtain

$$\nu_{n_1,n} \le \nu_n\Big(\max_{n_1\le k\le r} b_k^{-1}|h_k(m)-A(k)|\ge 1\Big)+\Delta_n = \tag{10}$$
$$= P\Big(\max_{n_1\le k\le r} b_k^{-1}|\eta_k|\ge 1\Big)+o(1) = W_{n_1}+o(1)$$

and

$$\nu_{n_1,n} \ge \nu_n\Big(\max_{n_1\le k\le r} b_k^{-1}|h_k(m)-A(k)|\ge 1\Big) = W_{n_1}+o(1) \tag{11}$$

where $W_{n_1} = P\big(\sup_{k\ge n_1} b_k^{-1}|\eta_k|\ge 1\big)$.

If series (8) converges then according to the Feller theorem $W_{n_1}=o(1)$ as $n_1\to\infty$. This via estimate (10) proves the first proposition of our theorem. Similarly, the second one follows from estimate (11).

When series (8) diverges then by the Feller theorem $P\big(b_k^{-1}|\eta_k|\ge 1 \text{ i.o.}\big)=1$. Consequently, $W_{n_1}=1$ for every $n_1\ge 2$ and estimate (11) yields statement (III) of our theorem.

The condition of the last proposition of theorem 3.2 and estimate (10) yield asymptotic $W_{n_1}=1+o(1)$ as $n_1\to\infty$. This due to the quoted probabilistic theorem proves the divergence of series (8). Theorem 3.2 is proved.

For example, we note that for every $s\ge 4$ and $\delta>0$

$$\Big(2L_2k\big(L_4k+\frac{3}{2}L_5k+L_6k+\ldots+(1-\delta)L_sk\big)\Big)^{\frac12} \le \sum_{p\le k,\, p|m} 1 - L_2k \le$$
$$\le \Big(2L_2k\big(L_4k+\frac{3}{2}L_5k+L_6k+\ldots+(1+\delta)L_sk\big)\Big)^{\frac12} \quad \text{i.o. a.e.}$$

Here $L_{k+1}U = L(L_kU)$ for $k\ge 1$.

It is easy to see that theorems 3.1 and 3.2 contain the zero–one law for the possible iterated limits of the frequences considered. The following problem arises: to prove an analogue of the zero–one law without any preliminary condition on additive functions and normalizations.

4. A theorem of the Chung type

Let us consider the sequence

$$\Big\{f_k(m) \overset{\text{def}}{=} \max_{l\le k}|h_l(m)-A(l)|;\quad k\ge 2\Big\}.$$

Using the monotonicity of β_k from theorem 2.1 one can obtain information expressed by

$$1-\varepsilon \le \limsup_{k\to\infty} \beta_k^{-1} f_k(m) \le 1+\varepsilon \quad \text{a.e.}$$

However, fairly frequently $f_k(m)$ takes much smaller values than β_k. The following theorem represents an analogue of the Chung law of the iterated logarithm.

Put for sake of brevity $\gamma_k = \frac{\pi B(k)}{\sqrt{8LLB(k)}}$.

Theorem 4.1. *Let $h(p) = o(\gamma_p)$ and $B(p) \to \infty$ as $p \to \infty$. Then for every $\varepsilon > 0$*

$$\lim_{n_1\to\infty} \limsup_{n\to\infty} \nu_n\Big(\min_{n_1 \le k \le n} \frac{f_k(m)}{\gamma_k} \le 1-\varepsilon \Big) = 0 \tag{12}$$

but

$$\lim_{n_1\to\infty} \liminf_{n\to\infty} \nu_n\Big(\min_{n_1 \le k \le n} \frac{f_k(m)}{\gamma_k} \le 1+\varepsilon \Big) = 1 . \tag{13}$$

Proof. For every $K > 0$ and $\delta > 0$ we have

$$\delta_n \overset{\text{def}}{=} \nu_n\Bigg(\max_{n_1 \le k \le n} \gamma_k^{-1} \sum_{\substack{p^\alpha \| m \\ p \le k,\, \alpha \ge 2}} \big(|h(p^\alpha)| + |h(p)| \big) \ge \delta \Bigg) \le$$

$$\le \nu_n\Bigg(\sum_{p^\alpha \| m,\, p^\alpha \le K} \big(|h(p^\alpha)| + |h(p)| \big) \ge \delta\gamma_{n_1} \Bigg) +$$

$$+ \nu_n\big(m;\, \exists\, p^\alpha \| m,\ p^\alpha > K,\ \alpha \ge 2\big) \ll o(1) + \sum_{p > \sqrt{K}} p^{-2}$$

when $n \to \infty$ and $n_1 \to \infty$. Consequently, using the standard way we can confine ourselves to strongly additive functions. Now we repeat the truncation procedure. Let

$$r = \max\Big\{ \log n,\ \exp\Big\{ \frac{\log n}{LLB(n)} \Big\} \Big\},$$

then $B(r) = B(n)\big(1+o(1)\big)$ as $n \to \infty$. By $\mu^-_{n_1,n}$ and $\mu^+_{n_1,n}$ we denote the frequences in equalities (12) and (13) respectively. Let η_k be random variables defined in paragraph 3. By use of lemma 1 we have

$$\begin{aligned} \mu^-_{n_1,n} &\le \nu_n\Big(\min_{n_1 \le k \le r} \frac{f_k(m)}{\gamma_k} \le 1-\varepsilon \Big) + \nu_n\Big(\min_{r \le k \le n} \frac{f_k(m)}{\gamma_k} \le 1-\varepsilon \Big) \le \\ &\le P\Big(\min_{n_1 \le k \le r} \big(\gamma_k^{-1} \max_{l \le k} |\eta_l| \big) \le 1-\varepsilon \Big) + o(1) + \\ &+ \nu_n\Big(\max_{l \le r} |h_l(m) - A(l)| \le (1-\varepsilon)\gamma_n \Big) = P\Big(\inf_{k \ge n_1} \big(\gamma_k^{-1} \max_{l \le k} |\eta_l| \big) \le 1-\varepsilon \Big) + \\ &+ P\big(B^{-1}(r) \max_{l \le r} |\eta_l| \le (1-\varepsilon)\gamma_n B^{-1}(r) \big) + o(1), \quad n \to \infty . \end{aligned} \tag{14}$$

According to A. I. Martikainen's theorem [16] the first probability in the right side tends to zero when $n \to \infty$. The second one by the well–known functional limit theorem can be approximated with $o(1)$ error by

$$\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{-1^k}{2k+1}\exp\Big\{-\frac{\pi^2(2k+1)^2}{8(1-\varepsilon)}\,\frac{B^2(r)}{\gamma_n^2}\Big\}.$$

The last quantity due to $\frac{\gamma_n}{B(r)}=o(1)$ tends to zero when $n \to \infty$. So from estimates (14) we obtain the first statement of the theorem.

For $\mu^+_{n_1,n}$ we have

$$\mu^+_{n_1,n}\ge \nu_n\Big(\min_{n_1\le k\le r}\frac{f_k(m)}{\gamma_k}\le 1+\varepsilon\Big)=P\Big(\inf_{k\ge n_1}(\gamma_k^{-1}\max_{l\le k}|\eta_l|)\le 1+\varepsilon\Big)+o(1)$$

as $n \to \infty$. But even

$$P\Big(\liminf_{n_1\to\infty}\big(\gamma_k^{-1}\max_{l\le k}|\eta_l|\big)\le 1+\varepsilon\Big)=1$$

(s.f. [16]), therefore for strongly additive function $h(m)$

$$\lim_{n\to\infty}\mu^+_{n_1,n}=1 \tag{15}$$

when $n_1 \ge 2$ is fixed. Remembering the estimate of δ_n we obtain equality (13). Theorem 4.1 is proved.

The proposition (13) can be strengthend to (15) without the extra hypothesis of strong additivity. This can be done using a more general version of the Fundamental lemma [2]. Some other form can be given to the conditions of theorem 4.1 (s.f. [16]).

In [5] we have laws of the iterated logarithm for additive functions not necessarilly belonging to the H class of Kubilius [2]. Nevertheless, investigation of sequences

$$\Big\{\sum_{p|m,\,p\le k}\log^c p;\ k\ge 2\Big\}$$

with $c > 0$ remains problematic.

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