FUNCTIONAL LIMIT THEOREMS IN PROBABILISTIC NUMBER THEORY

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ABSTRACT. This survey is an extended version of the invited talk delivered by the author at the conference "Erdös and his Mathematics", Budapest, July 4–11, 1999. Starting from the invariance principle established by P.Erdös and M.Kac in the forties and more general functional limit theorems for partial sum processes for i.r.vs, we describe the development of a parallel theory dealing with specifically dependent r.vs which appear in probabilistic number theory. The main efforts are paid to survey the results on the weak convergence of processes defined in terms of arithmetical functions. An extensive updated bibliography is given.

1 Introduction

We will consider sequences of real functions $f_n(m,t)$ defined on $\mathbf{N}\times[0,1]$ which express some arithmetic properties of a natural number m. If $m \le n$ is taken at random, $f_n(m,\cdot)$ can be viewed as a random process with paths in some functional space. The goal then is to describe the asymptotic behaviour as $n \to \infty$ of its distribution. That is implemented within the general theory on convergence of probability measures in functional spaces (see, for instance [Bi1]).

Many of the results on functional limit theorems for number-theoretic objects surveyed in this article had been obtained in the seventies and eighties, nevertheless they were not touched upon by P.D.T.E.Elliott [El] nor by the recent book of G.Tenenbaum [Te2]. J.Kubilius [K3] (see Chapter 7) included just his pioniering result [K1] from 1955. So did Yu.V.Linnik devoting Chapter 10 of his book [Li] to a model of Brownian motion defined in terms of the Legendre symbols. S.M.Ermakov also mentioned it in Chapter 8 of his monograph [Er]. This result, obtained in [K-Li], generalizes the one-dimensional limit theorem by H.Davenport and P.Erdös [D-E]. G.J.Babu [B4] in Concluding Remarks just mentioned his result [B2] from 1973. The situation with the survey papers is not better: either they were written in this respect too early as [G], [K4], [Sc]

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or were aimed to present achievements of a particular school as [M1], [La]. Our purposes are fairly broad, nevertheless due to the volume limitations, we stress here the main directions of the development and present more or less final results. Of course, the influence of the personal taste is unavoidable. Having in mind that some references are hardly known to a western reader, we are attempting to include a complete updated list of references.

2 Partial sum processes for i.r.vs

In the forties P.Erdös and M.Kac [E-Ka1], [E-Ka2] took over the initiative of L.Bachelier, P.Lévy and A.N.Kolmogorov to investigate the limit distribution laws of functionals on the partial sums of independent random variables (i.r.vs).

Let $\xi_1, \xi_2, \ldots$ be i.r.vs with mean $\mathbf{E}\xi_j = 0$ and variance $\mathbf{V}\xi_j = 1$, $s_k = \xi_1 + \cdots + \xi_k$, $k \geq 1$. The limit distributions of r.vs

$$\max\{s_1, \ldots, s_n\}, \quad \max\{|s_1|, \ldots, |s_n|\}, \quad |s_1| + \cdots + |s_n|, \quad s_1^2 + \cdots + s_n^2$$

under appropriate normalization in the case of identically distributed X_j were found in [E-Ka1]. The next paper [E-Ka2] was devoted to the following extension of the Lévy's result.

THEOREM 2.1 (E-Ka2)**.** *Let ξ_j be i.r.vs, $\mathbf{E}\xi_j = 0, \mathbf{E}\xi_j^2 = 1$. Denote N_n the number of positive sums among $s_1, \ldots, s_n$. Then*

$$\lim P(N_n < xn) = 2\pi^{-1} \arcsin \sqrt{x} =: As(x), \quad 0 \leq x \leq 1.$$

Here and in what follows the limits, if not indicated otherwise, are taken with respect to $n \to \infty$.

The appearance of the arcsine law in similar problems is in no way the unique phenomenon, for other cases we refer to the recent study [Ho-J]. Apart from the results, the very approach used in [E-Ka1], [E-Ka2] is of great significance. The authors noticed that if the limit distribution of any of the above quantities is found for one particular sequence $\xi_1, \xi_2, \ldots$, then it holds for all sequences of r.vs satisfying the relevant condition, in fact, the Lindeberg condition. That was the birth of the *invariance principle.*

In the new wave of investigations initiated by M.Donsker [Do], Yu.V.Prokhorov [P], A.V.Skorokhod [Sk1] another point of view became dominating. The main concern was the existence and the properties of the limit law for the partial sum process

$$X_n := X_n(t) := \sum_{t_{nk} \leq t} \xi_{nk}$$

where $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}$ is a sequence of arrays of i.r.vs and $0 = t_{n0} < t_{n1} < \cdots < t_{nk_n} = 1$ is a partition of the interval [0,1]. That was even carried out as a particular case of the weak convergence problem of processes having paths in the space of continuous functions with the supremum metric or in the space of functions $\mathbf{D} := \mathbf{D}[0,1]$ endowed with the Skorokhod topology. Here and in what follows we use the widely accepted terminology and notation presented in the P.Billingsley's book [Bi1] but set $Y_n \Rightarrow Y$ to denote the weak convergence of distributions of random elements Y_n to that of Y or weak convergence of nondecreasing bounded functions. If $X_n \Rightarrow X$ is established, then by the very definition, we have

$$\phi(X_n) \Rightarrow \phi(X)$$

for arbitrary bounded and continuous $P \cdot X^{-1}$ - a.e. functional $\phi : \mathbf{D} \to \mathbf{R}$. This opened a new way to deal with the limit distributions of r.vs mentioned above and many others as well. It is worthwhile to quote one of the Prohorov's theorems (see [P], Theorem 3.2 or [Sk3], Theorem 3, Section 8.40), a fairly good patern for raising questions in probabilistic number theory. Set $\xi(\varepsilon) = \xi$ if $|\xi| \le \varepsilon$ and $\xi(\varepsilon) = 0$ if $|\xi| > \varepsilon$. For a stochastic process $X(t)$ with paths in $\mathbf{D}$, we denote $T_X = \{t \in (0,1) : \ P(X(t-0) \ne X(t)) = 0\} \cup \{0,1\}$. If $T_X = [0,1]$, the process $X(t)$ will be called (stochastically) continuous.

THEOREM 2.2 (P). *Let $\xi_{n1}, \xi_{n2}, \dots, \xi_{nk_n}$ be a sequence of arrays of i.r.vs and $0 = t_{n0} < t_{n1} < \dots < t_{nk_n} = 1$ be a partition of the interval $[0,1]$ such that*

(i) $\lim \max_i P(|\xi_{ni}| > \varepsilon) = 0$ for any $\varepsilon > 0$;

(ii) $\lim \max_i |t_{ni} - t_{n,i-1}| = 0$;

(iii) $(X_n(t_1), \dots, X_n(t_s)) \Rightarrow (X(t_1), \dots, X(t_s))$ for some continuous process $X(t)$, arbitrary points $0 \le t_1 < \dots < t_s \le 1$, and any $s \ge 1$;

(iv) for some $\varepsilon > 0$,

$$\lim_{h\to 0} \limsup_{n\to\infty} \sup_{0\le t\le 1-h} \sum_{t_{ni}\in[t,t+h]} P(|\xi_{ni}| > \varepsilon) = 0,$$

$$\lim_{h\to 0} \limsup_{n\to\infty} \sup_{0<|t'-t''|\le h} \left| \sum_{t_{ni}\in[t',t'']} \mathbf{E}\xi_{ni}(\varepsilon) \right| = 0,$$

$$\lim_{h\to 0} \limsup_{n\to\infty} \sup_{0\le t\le 1-h} \sum_{t_{ni}\in[t,t+h]} \mathbf{V}\xi_{ni}(\varepsilon) = 0;$$

then $X_n \Rightarrow X$.

If (i), (ii) are given *a fortiori*, the conditions (iii) and (iv) are necessary for the convergence $X_n \Rightarrow X$ provided $X(t)$ is continuous process and has independent increments. Note that (iv) represents a test of very useful form to check the tightness of the sequence of measures $P \cdot X_n^{-1}$. A.V.Skorohod [Sk2] (see also [Gi-Sk], Theorem 1, Chapter 9) noted that for identically distributed i.r.vs ξ_{ni}, $1 \le i \le k_n$, satisfying the condition of infinitesimality (i) and the partition given by $t_{ni} = i/k_n$, $0 \le i \le k_n$, the convergence $X_n \Rightarrow X$ is equivalent to the one-dimensional convergence $X_n(1) \Rightarrow X(1)$. In probabilistic number theory, we deal with nonidentically distributed r.vs therefore we [M4] have extracted another case of Theorem 2.2 complementing this observation.

Let now $\xi_{ni} = \xi_i/\beta(n)$, $1 \le i \le n$ be i.r.vs and $\beta(n)$ ($\beta(n) > 0$, $\beta(n) \to \infty$) be some normalizing sequence, $F_{ni}(x) = P(\xi_{ni} < x)$. We set $u^* = u$ if $|u| < 1$, and $u^* = \operatorname{sgn} u$ otherwise. Denote

$$a_{ni}(\tau) = \int\limits_{|x|<\tau} x d\, F_{ni}(x),$$

where $\tau > 0$. Apart from the condition (i), we will assume the following simplifying condition

$$\lim_{\tau\to 0} \limsup_{n\to\infty} \sum_{i=1}^{n} a_{ni}^2(\tau) = 0. \tag{2.1}$$

Let

$$\psi_{ni}(u) = \int\limits_{-\infty}^{u} {x^*}^2 d\, F_{ni}(x), \quad \psi_n(u) = \sum_{i=1}^{n} \psi_{ni}(u).$$

For $t \in [0, 1]$, we let

$$y(t) = y(t, n) = \max\bigl\{l : \ l \le n, \sum_{i\le l} \Psi_{ni}(+\infty) \le t\Psi_n(+\infty)\bigr\}. \tag{2.2}$$

We define stochastic processes with trajectories in the space $\mathbf{D}$ by setting

$$Y_n = Y_n(t) = \sum_{i\le y(t)} (\xi_{ni} - \gamma_{ni}),$$

where

$$\gamma_{ni} = \int\limits_{-\infty}^{\infty} x^* d\, F_{ni}(x).$$

The processes $X = X(t)$ appearing as limits for Y_n will be continuous and will have independent increments. It is convenient to use the following modification of the Lévy's canonical expression of their characteristic functions

$$\mathbf{E} \exp\{i\lambda X(t))\} = \exp\biggl\{ \int\limits_{-\infty}^{\infty} (e^{i\lambda u} - 1 - i\lambda u^*) {u^*}^{-2} d\, M_t(u) \biggr\}, \quad \lambda \in \mathbf{R}, \tag{2.3}$$

where $M_t(u)$ is a bounded nonnegative function which is continuous in t and nondecreasing in u and t so that $M_t(u) - M_s(u)$ is also nondecreasing in u if $s < t$. For $u = 0$ the integrand in (2.3) is equal to $-\lambda^2/2$.

THEOREM 2.3 (M4)**.** *Let the i.r.vs $\xi_{ni} = \xi_i/\beta(n)$, $1 \le i \le n$, satisfy (i) of Theorem* 2.2*, conditions* (2.1) *and*

$$\psi_n(u) \Rightarrow \Psi(u), \tag{2.4}$$

where $\Psi(u)$ is a nondecreasing bounded function defined on $\bar{\mathbf{R}}$, $0 = \Psi(-\infty) < \Psi(+\infty)$, so that $\Psi_n(+\infty) \to \Psi(+\infty)$. Then $Y_n \Rightarrow X$, where X is continuous process with independent increments which characteristic function has expression (2.3) *with*

$$M_t(u) := M(b(t), u) := \int_{-\infty}^{u/b(t)} (b(t)v)^{*^2} v^{*^{-2}} d\,\Psi(v), \tag{2.5}$$

where $b(t) := \lim \beta(y(t,n))/\beta(n)$ *and is a continuous solution to* $M(b(t), +\infty) = t\Psi(+\infty)$.

The derivation [M4] of the last theorem from Theorem 2.2 relies on the investigation of convergence of

$$M_t^n(u) := \sum_{i \le y(t)} \psi_{ni}(u) = \int_{-\infty}^{u\beta(n)/\beta(y(t))} \left(v\frac{\beta(y)}{\beta(n)} \right)^{*^2} v^{*^{-2}} d\,\psi_{y(t)}(v).$$

The goal is to show that the convergence $M_t^n(+\infty) \to M_t(+\infty)$ is uniform in $t \in [0, 1]$. By virtue of (2.1), this implies the tightness condition (iv).

Since under other restrictions in Theorem 2.3 condition (2.4) is necessary and sufficient for $Y_n(1) \Rightarrow X(1)$, our functional limit result is equivalent to the one-dimensional limit theorem. The reason for this, apparently, is the succesful choice of the time index function (2.2), going back to the paper on probabilistic number theory [T-U1]. As it has been shown by V.M.Kruglov [Kru] in the case of Brownian motion, the use of other choices of this function are also possible.

We end this section mentioning a recent result [Ba-M1] motivated mainly by number-theoretic applications. The convergence problem in the space $\mathbf{D}$ for processes

$$Z_n(t) := e^{-\alpha(z(t))} \prod_{k \le z(t)} |\xi_k|^{1/\beta(n)} \operatorname{sgn} \xi_k, \tag{2.6}$$

where ξ_k, $k \ge 1$ are i.r.vs, $\alpha(u)$, $\beta(u)$ denote some normalizing sequences, and $z(t) = z_n(t)$ is a time index function, has been investigated. We stress that achieving these purposes we had to modify the definition of the weak convergence of finite dimensional distributions and also needed some refinement of the process $Z_n(t)$ in the neibourhood of the zero point.

3 Additive functions and functionals on them

A mapping $h : \mathbf{N} \to \mathbf{R}$ is called *additive function* if it satisfies the following relation

$$h(mn) = h(m) + h(n)$$

for each pair of coprime numbers $m, n \in \mathbf{N}$. It has the canonical representation

$$h(m) = \sum_p h(p^{\alpha_p(m)}),$$

where p denotes a prime number and $\alpha_p(m) = k$ if $p^k | m$ and $p^{k+1} \not| m$. For brevity, afterwards we confine ourselves to the case of strongly additive functions. By definition they have the expression

$$h(m) = \sum_{p|m} h(p) = \sum_p h(p)\delta_p(m),$$

where $\delta_p(m) = 1$ or 0 according to p divides m or does not. Set $h^r(m)$ to denote its partial sum over $p \le r$, $2 \le r \le n$. Let $\nu_n(\dots)$ be the probability measure on $\mathbf{N}$ ascribing probability $1/n$ for any $m \le n$. Distributions of the vector of "truncated" (strongly) additive functions

$$(h^{x_1}(m), \dots, h^{x_n}(m)), \quad 2 \le x_1 \le \dots \le x_n \le n,$$

or its transforms, especially when the number of points x_j is not fixed, with respect to ν_n are of great interest. By virtue of strong dependence of $\delta_p(m)$ for large primes p, this problem is pretty difficult.

We quote the historically first result of this kind obtained by J.Kubilius [K1]. Set

$$A(u) = \sum_{p \le u} \frac{h(p)}{p}, \quad B^2(n) = \sum_{p \le u} \frac{h(p)^2}{p}, \quad t_{nq} = B^2(q)/B^2(n),$$

where $2 \le u \le n$ and q denotes a prime number. Denote

$$\nu_n(\psi_1, \psi_2) = \nu_n\biggl(\psi_1(t_{nq}) < \frac{h^q(m) - A(q)}{B(n)} < \psi_2(t_{nq}),\ q \le n\biggr),$$

where ψ_1, ψ_2 are continuously differentiable functions, $\psi_1(t) < 0 < \psi_2(t)$, $t \in [0, 1]$. Let $w(x, t)$ be a solution to the equation

$$\frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial^2 w}{\partial x^2} = 0$$

subject to conditions $w(x, 1) = 1$ for $\psi_1(1) < x < \psi_2(1)$ and $w(\psi_1(t), t) = w(\psi_2(t), t) = 0$ for $0 \le t < 1$.

THEOREM 3.1 (K1)**.** *Suppose that $B(n) \to \infty$ and*

$$\max_{p \le n} |f(p)|/B(n) \le \mu_n = o(1). \tag{3.1}$$

Then

$$\nu_n(\psi_1, \psi_2) - w(0, 0) =: R_n = o(1).$$

Let us observe that $w(0, 0)$ equals the probability that the standard Brownian motion process $W(t)$ starting at $t = 0$ does not reach the boundaries $\psi_1(t)$ and $\psi_2(t)$. Later in [K5], it was shown that $R_n \ll \mu_n \log(\mu_n^{-1}) / \log\log(\mu_n^{-1})$. It was also remarked that Theorem 3.1 holds under the Lindeberg condition

$$\frac{1}{B^2(n)} \sum_{\substack{p \le n \\ |h(p)| \ge \varepsilon B(n)}} \frac{h(p)^2}{p} = o(1) \tag{3.2}$$

for each $\varepsilon > 0$, and conditions on the boundaries were relaxed. Actually, linear combinations of additive functions were investigated in [K1] but we leave relevant problems for the next sections.

In 1972 G.J.Babu [B1] derived asymptotical formulas of distribution of the maximum type functionals defined on truncated additive functions.

THEOREM 3.2 (B1). *Assume that $B(n) \to \infty$ and condition (3.2) is satisfied. Then for $x > 0$,*

$$\nu_n\left(\max_{k\le n}\frac{h^k(m)-A(k)}{B(n)} < x\right) = \frac{2}{\sqrt{2\pi}}\int_0^x e^{-u^2/2}\,du + o(1)$$

and

$$\nu_n\left(\max_{k\le n}\left|\frac{h^k(m)-A(k)}{B(n)}\right| < x\right) = \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}(-1)^k\int_{-x}^{x}\exp\left\{-\frac{(u-2kx)^2}{2}\right\}du+o(1).$$

The aformentioned Erdös-Kac formulas for partial sums of i.r.vs were used in the proof. Observe that the limit distributions coinside with that of $\max_{0\le t\le 1} W(t)$ and $\max_{0\le t\le 1}|W(t)|$ for the Brownian motion $W(t)$.

Several years later P.Erdös, remaining faithful to the invariance principle, in 1969 formulated the following analog of Theorem 2.1.

THEOREM 3.3 (E2). *Let $p_1(m) < \cdots < p_w(m)$ be the prime factors of m and $\omega = \omega(m)$ be their number. Set $Ly = \log\max\{e, y\}$. We have*

$$\lim_{n\to\infty}\nu_n\left(\frac{1}{LLm}|\{1\le j\le\omega(m):\ LLp_j(m) < j\}| < x\right) = As(x)$$

uniformly in $0 \le x \le 1$.

The hint to apply Theorem 2.1 was also given. A proof going along this line was furnished by the author in [M7].

To our knowledge, the above listed results were the only attempts to examine the weak convergence of distributions of functionals on the sequence of truncated additive functions without the direct use of limit theorems in functional spaces.

Values of the maximum type functionals on normalized integer-valued arithmetic functions have a particular lattice structure thefore stronger approximations of their distributions are available to obtain. The convergence in total variation distance was established in some cases (cf. [Da-L], [L3], [Kr3], [Kr4]).

4 Additive functions and Brownian motion

The first published remark on limit theorems for processes defined in terms of additive functions is the abstract announced by P.Billingsley [Bi2]. Basing upon [Bi3] and the W.Philipp's comments [Ph] mentioning also an unpublished manuscript, we can imagine that the weak convergence of processes

$$H_n := H_n(m,t) = \frac{1}{B(n)}\sum_{p|m,\, p\le x(t)} h(p)\big(\delta_p(m) - \frac{1}{p}\big),$$

where $x(t) := x_n(t) = \max\{u : B^2(u) \le tB^2(n)\}$, under the conditions $B(n) \to \infty$ and $h(p) \ll 1$, was established. To be more exact, the result was proved for sequences of additive functions. In the sequel we will stress the dependence on n by attaching the additional lower index n only when proofs in this more general case require new ideas. We now present a result of G.J.Babu [B2], [B3], and W.Philipp [Ph] proved in different ways.

THEOREM 4.1 (B2, Ph)**.** *Assume that $B(n) \to \infty$ and the Lindeberg condition* (3.2) *is satisfied. Then $H_n \Rightarrow W$ in the space $\mathbf{D}$.*

Since the paths of $H_n(m,t)$ can be easily drawn, it is often called a *model* of the Brownian motion. Either of the proofs use the Kubilius' method going back to [K2] (see also [K3]). To get an impression, we give a few details.

Fundamental Lemma (K3)**.** *Let $2 \le r \le n^\varepsilon$ for each $\varepsilon > 0$ and $2, 3, \dots, p_s$ be all prime numbers not exceeding r. Then there exists a probability space $\{\mathbf{N}, \mathcal{F}, P\}$ and i.r.vs ξ_p, $p \le r$, defined on it by*

$$P(\xi_p = 1) = 1 - P(\xi_p = 0) = 1/p$$

such that

$$\sum_{\bar\delta \in \{0,1\}^s} \left| \nu_n(\delta_p(m) = \delta_p,\ p \le r) - \prod_{p \le r} P(\xi_p = \delta_p) \right| = o(1). \tag{4.1}$$

Here the sum is taken over all choices of vectors $\bar\delta := (\delta_2, \dots, \delta_{p_s}) \in \{0,1\}^s$.

Denote

$$W_n(t) = \frac{1}{B(n)} \sum_{p \le x(t)} h(p)(\xi_p - \frac{1}{p}).$$

The total variance distance estimate (4.1) gives at once

$$\sup_{B \subset \mathbf{D}} \left| \nu_n(H_n^r(m,\cdot) \in B) - P(W_n^r(\cdot) \in B) \right| = o(1), \tag{4.2}$$

where H_n^r and W_n^r are obtained from $H_n(m,t)$ and $W_n(t)$ by substituting $\min\{x(t), r\}$ for $x(t)$. The processes X_n^r are treated using Prokhorov's Theorem 2.2. Thus, condition (3.2) is necessary and sufficient for $X_n^r \Rightarrow W$, and by virtue of (4.2), the same holds for for $H_n^r \Rightarrow W$. This condition implies also the estimate

$$P(\varepsilon) := P(\sup_{t \in [0,1]} |X_n(t) - X_n^r(t)| \ge \varepsilon) = o(1)$$

for any $\varepsilon > 0$. To complete his proof, G.J.Babu [B2] obtains a similar estimate for the difference of $H_n(m,t) - H_n^r(m,t)$. W.Philipp [Ph] uses Fundamental Lemma to prove convergence of finite-dimensional distributions of H_n but fairly complicately deals with the tightness of the sequence of measures $\nu_n \cdot H_n^{-1}$

repeating much of the traditional argumentation used in the case of i.r.vs. The author [M3] observed that, from the I.Z.Ruzsa's [R1] inequality, it follows

$$\nu_n\Big(\sup_{t\in[0,1]} |H_n(m,t) - H_n^r(m,t)| \geq \varepsilon\Big) \ll P(\varepsilon/3). \tag{4.3}$$

This makes either of the proofs shorter.

Though H_n asymptotically models Brownian motion and is algorithmically simple, it is hard to expect that it is much useful for numerical imitation problems. This scepticism is based on the slow convergence rates. To illiustrate that consider a linearized version of the process H_n with paths in the space of continuous functions $\mathbf{C} := \mathbf{C}[0,1]$.

Let us take $h := h_n$, a sequence of strongly arithmetic functions normalized so that

$$\sum_{p\leq n} \sigma_{np}^2 = 1, \tag{4.4}$$

where $\sigma_{np}^2 = h^2(p)p^{-1}(1-p^{-1})$. Set

$$t_{np} = \sum_{q\leq p} \sigma_{nq}^2, \quad \alpha_s = \alpha_{sn} = \sum_{p\leq n} |h(p)|^s p^{-1}.$$

Define the polygonal line $\widehat{H}_n := \widehat{H}_n(m,t)$ with vertices

$$\Big(t_{np}, \sum_{q\leq p} h(q)(\delta_q(m) - 1/q)\Big),$$

where $p \leq n$, starting at the point $(0,0)$. Thus, in the space $\mathbf{C}$ provided with the supremum norm, an arithmetic process is defined. Let further, $\mathcal{C}$ be the σ-algebra of Borel sets of the space $\mathbf{C}$, $B \in \mathcal{C}$, and B^ε be the ε- neighbourhood of the set B. In the space $\mathfrak{M}(\mathbf{C})$ of probability measures defined on $\mathcal{C}$, we study the Lévy-Prokhorov distance $\rho(\widehat{H}_n, W)$ of the measure corresponding to the process $\widehat{H}_n$ from the Wiener measure. We recall that for measures $P, Q \in \mathfrak{M}(\mathbf{C})$ the distance cited above is defined as $\inf \varepsilon$, for which $P(B) \leq Q(B^\varepsilon) + \varepsilon$ and $Q(B) \leq P(B^\varepsilon) + \varepsilon$. In the symbol $\rho(\cdot,\cdot)$ we indicate the random elements with values in $\mathbf{C}$ and not their distributions.

THEOREM 4.2 (M3)**.** *For a sequence of strongly additive functions h_n satisfying* (4.4) *and the inequality $\alpha_s \leq 1/3$ for $2 < s \leq 3$, one has the estimate*

$$\rho(\widehat{H}_n, W) \leq C(s)\alpha_s^{1/(s+1)} \left(\ln\ln \frac{1}{\alpha_s}\right)^{(s-2)/(2(s+1))}. \tag{4.5}$$

Comparing this estimate with the unimprovable A.A.Borovkov's result [Bo] obtained for i.r.vs, we see that (4.5) has the only extra iterated logarithm factor. Under condition (4.4) for $s = 3$, by Cauchy's inequality we have $\alpha_3 \gg (LLn)^{-1/2}$. Thus the convergence rate in terms of α_3 in (4.5) is not better than $(LLn)^{-1/8}(LLLLn)^{1/8}$. Dropping of the second factor would be of little help for numerical applications.

Observe that taking the sets $B \in \mathcal{C}$ with boundary δB, satisfying the condition $P(W \in (\delta B)^{\varepsilon}) \le K_B \varepsilon$ for any $\varepsilon > 0$, we get from (4.5)

$$|\nu_n(H_n(m,\cdot) \in B) - P(W \in B)| \le (K_B+1)C(s)\alpha_s^{1/(s+1)}(\ln\ln 1/\alpha_s)^{(s-2)/(2(s+1))}.$$

This estimate yields convergence rates for distributions of functionals defined on $\widehat{H}_n$.

Convergence of processes H_n yields the one-dimensional relation $H_n(\cdot,1) \Rightarrow W(1)$, the assertion known since 1956 (see [K2], [K3] or [El]). Fascinatingly, but the last relation does not imply the Lindeberg condition (3.2), as N.M.Timofeev [T] has proven by a counterexample. As we have seen, (3.2) makes the influence of large prime factors negligible, nevertheless the sum of dependent summands $h(p)(\delta_p(m) - 1/p)/B(n)$, $n^{\varepsilon} \le p \le n$, in its turn can yield the limiting normal distribution. Convergence of the processes H_n is more restrictive. Recall that the Brownian motion has independent increments therefore one could expect that these summands should be negligible in the functional limit theorem. That motivates the necessity of condition (3.2) which was proved by N.M.Timofeev and Kh.Kh.Usmanov [T-U3]. Here we would like to ask two questions.

PROBLEMS. *Is it true that* $H_n(\cdot,1) \Rightarrow W(1)$ *and* $\max_{0\le t\le 1} H_n(\cdot,t) \Rightarrow \max_{0\le t\le 1} W(t)$ *imply* (3.2)*? For which functionals* $\phi : \mathbf{D} \to \mathbf{R}$ *the joint convergence* $H_n(\cdot,1) \Rightarrow W(1)$ *and* $\phi(H_n) \Rightarrow \phi(W)$ *implies* (3.2)*?*

We now formulate a result for a more general model than H_n. Set

$$G_n := G_n(m,t) = \frac{1}{\beta(n)} \sum_{p|m,\, p\le z(t)} h(p) - \alpha(n, z(t)),$$

where $\beta(n) > 0$ is arbitrary,

$$z(t) := z_n(t) = \max\{u :\ B^2(n,u) \le tB^2(n,n)\}, \quad t \in [0,1],$$

$$B^2(n,u) = \sum_{p\le u} \left(\frac{h(p)}{\beta(n)}\right)^{*^2} \frac{1}{p}, \quad \alpha(n,u) = \sum_{p\le u} \left(\frac{h(p)}{\beta(n)}\right)^{*} \frac{1}{p}.$$

THEOREM 4.3 (T-U3)**.** *Assume that* $h : \mathbf{N} \to \mathbf{R}$ *is a fixed function,* $\beta(n) \to \infty$*. For convergence* $G_n \Rightarrow W$*, the condition*

$$\Psi_n(u) := \sum_{\substack{p\le n\\ h(p)<u\beta(n)}} \left(\frac{h(p)}{\beta(n)}\right)^{*^2} \frac{1}{p} \to \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u < 0 \end{cases}$$

is necessary and sufficient.

One of the ways how to derive Theorem 4.1 from this result is indicated in [U5]. Note that it is easy to prove the sufficiency in Theorem 4.3 for sequences of functions h_n. The necessity in this more involved case follows from the author's result [M6].

5 Models of other processes with independent increments

Let us return to the arithmetic processes H_n. Can they model stochastic processes other than the Brownian motion? The positive answer to this question was given by P.Billingsley [Bi4]. Considering sequences of functions h_n, he succeeded to model the Poisson process and any other process with independent increments which distributions have finite variances. The last limitation comes from the use of standard normalizations $A(n)$ and $B(n)$ in the definition of H_n. To verify this, one needs just to apply the Kubilius' inequality ([K3], Lemma 3.1). The models G_n do not have this limitation. The very idea to study them comes from the paper [T-U1] by N.M.Timofeev and Kh.Kh.Usmanov who presented fairly general sufficient conditions assuring the convergence $G_n \Rightarrow X$ for a fixed additive function h. It appeared in the eighties that the conditions used by them were necessary in the case of limit processes with independent increments. In the previous section we have discussed the case of the Brownian motion. The necessity of their conditions for a stable limit processes was proved by the author in [M2], [M3].

THEOREM 5.1 (M3). *Assume that $h : \mathbf{N} \to \mathbf{R}$ is a fixed function, $\beta(n) \to \infty$. Let X be a homogenuous stable process with characteristic function*

$$M \exp\{i\lambda X(t)\} = \exp\left\{ta_1 \int_{-\infty}^{0} (e^{i\lambda u} - 1 - i\lambda u^*)\, d(|u|^{-\alpha}) - \right.$$
$$\left. -ta_2 \int_{0}^{\infty} (e^{i\lambda u} - 1 - i\lambda u^*)\, d(u^{-\alpha}) \right\},$$
$$a_1, a_2 \geq 0, \quad a_1 + a_2 > 0, \quad 0 < \alpha < 2, \quad 0 \leq t \leq 1.$$

In order that $G_n \Rightarrow X$, it is necessary and sufficient that for any $u > 0$,

$$\sum_{\substack{p \leq n \\ h(p) < -u\beta(n)}} p^{-1} \to a_1 u^{-\alpha}, \qquad \sum_{\substack{p \leq n \\ h(p) > u\beta(n)}} p^{-1} \to a_2 u^{-\alpha},$$

and

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \beta^{-2}(n) \sum_{\substack{p \leq n, \\ |h(p)| < \varepsilon\beta(n)}} h^2(p) p^{-1} = 0.$$

Using an idea going back to the P.D.T.A.Elliott's book [El], Chapter 16, we easily construct an example of the function satisfying the conditions of Theorem 5.1. One can take

$$h(m) = \sum_{\substack{p|m \\ p \equiv 1(4)}} \left(\frac{2a_1}{\{p\sqrt{2}\}}\right)^{1/\alpha} - \sum_{\substack{p|m \\ p \equiv 3(4)}} \left(\frac{2a_2}{\{p\sqrt{2}\}}\right)^{1/\alpha}, \quad 0 < \alpha < 2,$$

where now $\{u\}$ denotes the fractional part of u, $\beta(n) = (\log\log n)^{1/\alpha}$, and $z(t) =$ $= \exp(\log^t n)$.

The most difficult step in the proof was to show that the truncation condition $B(n,n) - B(n,n^{\varepsilon}) = o(1)$ for each $\varepsilon > 0$ is given implicitely by the conditions of Theorem 5.1 and also to derive it in the necessity part. Later [M4] an extension of the approach gave necessary and sufficient conditions for a larger class of limit processes including processes X for which the values $X(1) - X(t)$ in some neigbourhood of $t = 1$ had absolutely continuous with respect to Lebesgue measure distributions. Generalizing Theorem 5.1, N.M.Timofeev and Kh.Kh.Usmanov [T-U4] reached a remarkable milestone. In the previous notation we have

THEOREM 5.2 (T-U4)**.** *Assume that* $h : \mathbf{N} \to \mathbf{R}$ *is a fixed function,* $\beta(n) \to \infty$*. In order that* $G_n \Rightarrow X$*, where* X *is a process in* $\mathbf{D}$ *with independent increments, it is necessary and sufficient that the following two conditions are satisfied*

(i) there exists a nondecreasing function $\Psi(u)$ *defined on* $\bar{\mathbf{R}}$ *such that* $\Psi_n(u) \Rightarrow \Psi(u)$ *including* $\Psi_n(\pm\infty) \to \Psi(\pm\infty)$*;*

(ii) $B(n,n) - B(n,n^{\varepsilon}) = o(1)$ *for each* $0 < \varepsilon < 1$*. The characteristic function of the limit process is of the form given in Theorem 2.3.*

Actually, the authors were using the superfluos condition $\Psi(-1) \neq \Psi(1)$ in the sufficiency part. Moreover, some arguments should have been added in the necessity part if the value $X(1)$ has the distribution degenerated at one point. By a result of A.Hildebrand [Hi], under the conditions of Theorem 5.2, the value $X(1)$ can have arbitrary distribution from the class of selfdecomposable laws. Formulas (2.3), (2.4), and (2.5) give then the characterization of the distributions of $X(t)$ for each $t \in [0,1]$.

Remark that having in advance some additional information on the values $h(p)$, say, if they are not very large in average, one can strengthen the necessity part of Theorem 5.2. The requirement on the increments of X may be ommited in such a case (cf. [T-U5] for the details). The same can be achieved under *a fortiori* condition on normalizing sequences $\beta(n) = L(\log n) \to \infty$ with some slowly varying function $L(u)$ in the Karamata sence. Moreover, Theorem 5.2 remains to be true if (ii) is changed by the last requirement on $\beta(n)$.

Desiring to extend Theorem 5.2, one can ask when do the arithmetic processes defined using arrays of additive functions h_n instead of $h/\beta(n)$ converge to a continuous process with independent increments which characteristic function is given by (2.3). An answer was given by the author in [M6].

Let $v(\cdot) := v_n(\cdot) : [0,1] \to \{1, \dots, n\}$ be a monotonically increasing mapping with the range $v_n([0,1]) = \{1 = k_{n1} < \cdots < k_{nj_n} = n\}$. We will assume that

$$\max_{1 \le j \le j_n} (k_{n,j+1} - k_{nj}) = o(n^{\varepsilon}) \tag{5.1}$$

for each $\varepsilon > 0$ and

$$\max_{1 \le j \le j_n} meas\ v^{-1}(k_{nj}) = o(1). \tag{5.2}$$

As earlier, confying ourselves to strongly additive functions h_n, set

$$V_n := V_n(m,t) = \sum_{\substack{p|m \\ p \le v(t)}} h_n(p) - \sum_{p \le v(t)} \frac{h_n^*(p)}{p}$$

and

$$\tilde{B}^2(n,u) = \sum_{p\le u} \frac{h_n^{*^2}(p)}{p}, \quad M_t^n(u) = \sum_{\substack{p\le v(t)\\ h_n(p)<u}} \frac{h_n^{*^2}(p)}{p}.$$

THEOREM 5.3 (M6). *Assume that $h_n(p) \to 0$ for each fixed prime number p and conditions* (5.1) *and* (5.2) *are satisfied. In order that $V_n \Rightarrow X$, where X is a continuous process in* **D** *with independent increments having the characteristic function* (2.3), *it is necessary and sufficient that the following two conditions are satisfied*

(i) for any $t \in [0,1]$, $M_t^n(u) \Rightarrow M_t(u)$ including $M_t^n(\pm\infty) \to M_t(\pm\infty)$;

(ii) $\tilde{B}(n,n) - \tilde{B}(n,n^{\varepsilon}) = o(1)$ for each $0 < \varepsilon < 1$.

It is easy to believe that independency of increments of X should exclude influence of strongly dependent summands with the large prime indeces. That is the case and, as in the theorems above, (ii) expresses this phenomenon.

Further, one can ask: *Is it possible to obtain models of processes having dependent increments*? The answer was given in [T-U2].

THEOREM 5.4 (T-U2). *Assume that an additive function h satisfies the condition (i) of Theorem* 5.2,

$$\sum_{p\le y} \left(\frac{|h(p)|}{\beta(n)}\right)^{*} \frac{1}{p} \ll \left(\frac{\log y}{\log n}\right)^{\gamma}$$

for some $\gamma > 0$ and all $2 \le y \le n$;

$$\frac{1}{\log n} \sum_{\substack{p\le n\\ h(p)<x\beta(n)}} \frac{\log p}{p} \Rightarrow U(x) \ne \begin{cases} 1 & \text{if } x > 0,\\ 0 & \text{if } x < 0 \end{cases}$$

for some nondecreasing function $U(x)$. Then $\beta(n) = (\log n)^{\rho} L(\log n)$ where $L(u)$ is a slowly varying function, $\rho > 0$, and the processes G_n with $z(t) = \exp\{t^{1/\rho}\log n\}$ weakly converge in the space **D**.

The limit process has dependent increments, his pretty complicated characteristic function is given in [T-U2]. The strongly additive function defined by $h(p) = (\log p)^{\rho}$, $\rho > 0$, is a typical example illiustrating Theorem 5.4. The interest to this very function was raised by P.Erdös in the paper [E1].

Finally, we mention some generalizations obtained so far. They concern the arithmetic models of stochastic processes with values in multidimensional spaces. The first result in this direction belongs to Kh.Kh.Usmanov [U3] who proved a theorem for additive functions taking values in $\mathbf{R}^d$ with $d \ge 1$. B.Grigelionis [Gr] obtained an extension of Theorem 5.3 above to a separable Hilbert space. The results of I.Z.Ruzsa [R2] deserve a special attention. They are stated as limit theorems for sequences of additive functions h_n taking values in a general commutative topological group. Though, roughly speaking, the function space **D** is not a topological group, this paper opens a broad perspective to derive general functional limit theorems.

6 Additive functions on sparce sequences

Functional limit theorems for additive functions on sparce sequences $\{R(m)\}$ such as integer-valued polynomials, the sequence $\{p_m + 1\}$, where p_m runs the set of all prime numbers, and other subsequences of natural numbers have some peculiarities. Consider the asymptotic behaviour of

$$Z_n := Z_n(m,t) = \frac{1}{\beta(n)} \sum_{p|R(m),\, p\le z(t)} h(p) - \alpha(n, z(t)),$$

where $\beta(n) > 0$ is arbitrary, $z(t)$ and $\alpha(n,u)$ are defined as in Section 5. The maximum type functionals in the case of polynomials $R(m)$ were examined by G.J.Babu in 1972 (cf. [B1]). The following question is of great interest: *for what subsequences $R(m)$ of natural numbers we can prove weak convergence of Z_n to a limit process?*

At first, we need some analog of Fundamental lemma quoted in Section 4. As it is described in Chapter 3, [El], the Selberg sieve or many other sieve versions are capable to give it if $R(m)$ has a good in some sense distribution in arithmetic progressions. The paper [U1] and short announcements [U2], [U3] (the details are given in [U4]) contain rather long list of the conditions on $R(m)$ required for that. Some Lévy type estimates, though weaker than (4.3), are used in [U1], [U4] to show that the errors in the truncation procedure are negligible. Characteristically, the condition

$$\frac{1}{n} \sum_{p\ge n^\delta} \left| \frac{h(p)}{\beta(n)} \right|^* \sum_{m\le n, p|R(m)} 1 = o(1)$$

for some $\delta > 0$ is exploited in either of the steps. The dependence of summands in Z_n indexed by large primes is more influential than in the case $R(m) = m$ therefore establishing of necessity of conditions seems to be rather difficult. Despite to this, investigation of the case $R(m) = p_m + 1$ was more successful.

THEOREM 6.1 (U6)**.** *Assume that Z_n is the process above with $R(m) = p_m + 1$ and arbitrary $\beta(n) \to \infty$. In order that $Z_n \Rightarrow X$, where X is a process in $\mathbf{D}$ with independent increments and $X(1)$ is nondegenerate, it is necessary and sufficient that the conditions of Theorem* 5.2 *are satisfied.*

Kh.Kh.Usmanov [U6] dealt also with sums of several additive functions with shifted prime arguments. Nevertheless, in the necessity part, he has left a gap. The relevant asymptotic formula of the increment of the process in the neighbourhood of the point one is not correct but does help in the case of Theorem 6.1. This attempt to deal with sums of functions with shifted natural arguments was not the first one. Say, Theorem 3.1 in [K3] is presented in more general than stated in this paper context. Extending this result, under certain extra conditions, the weak convergence of such arithmetic models to limit processes was also established (see [M4], [M5], [U2], [U3]). The proofs are mainly based upon some analogs of Fundamental lemma. The examination of necessity of these conditions seems to be problematic ([M5]).

The same difficulties arise if we examine the asymptotic distribution of arithmetic processes with respect to the short interval probabilities, e.g. instead of $\nu_n(...)$ we take frequences

$$(x-y)^{-1}|\{m \in (y,x] : \ ...\}|, \quad y < x, \ \ y,x \in \mathbf{N}$$

where $y = y(x)$ and $x \to \infty$. The experience (cf. [I-M], [Ba-I1]) shows that the smaller difference $x - y$ more difficult the problem. The lower bounds for this difference are not known.

Z.Kryžius [Kr1], [Kr2] observed that sums of increasing number of additive functions with shifted arguments can also be used to model random processes. Since definition of the arithmetic processes is different from those considered earlier, we present some sketch.

Let h be an additive function belonging to the Kubilius's $\mathbf{H}$ class, e.g. $B(n) \to \infty$ and $B(n) = L(\log n)$ where L is a slowly varying function on $\mathbf{R}^+$. For a positive sequence $s(n) \to \infty$, define stepwise functions

$$K_n := K_n(m,t) = \frac{1}{B(n)\sqrt{s(n)}} \sum_{j \le ts(n)} \big(h(m+j) - A(n)\big), \quad t \in [0,1].$$

Under an appropriate analog of the Lindeberg condition and a fairly long list of conditions on $s(n)$ convergence $K_n \Rightarrow W$ was established in [Kr2]. The prime divisor function $\omega(m)$ and any sequence $s(n) = o(\log\log / \log\log\log n)$ satisfy these conditions (cf. [Kr1]). Similar models using increasing number of multiplicative functions with shifted arguments appeared even in the fifties. We discuss them in the next section.

7 The use of multiplicative functions

In 1952 H.Davenport and P.Erdös [D-E] proved a one-dimensional limit theorem for sums of the Legendre symbols with shifted arguments. Extending it J.Kubilius and Yu.V.Linnik [K-Li] defined a sequence of arithmetic process and examined convergence of its finite dimensional distributions. In 1973 N.N.Lyashenko [L1] and W.Philipp [Ph] independently proved weak convergence in $\mathbf{D}$.

Let $\left(\frac{a}{b}\right)$ denote the Jacobi symbol, where b is an odd number.

THEOREM 7.1 (K-Li, L1, Ph)**.** *Let Q run through any infinite increasing sequence of odd square free numbers such that for every fixed $c \ge 0$*

$$\prod_{p|Q} \left(1 - \frac{c}{p}\right) \to 1$$

as $Q \to \infty$. Let $v = v(Q) \to \infty$ so that $\log v = o(\log Q)$*. Then the process*

$$L_n := L_n(m,t) := \frac{1}{\sqrt{v}} \sum_{k \le vt} \left(\frac{m+k}{Q}\right)$$

converges weakly to the standard Brownian motion W.

The method of moments was used in the proof. The Dirichlet characters, instead of the Jacobi symbols, can also be used. N.N.Lyashenko [L2] and Z.Kryžius [Kr4] attempted to use more general multiplicative functions. By definition such a function $g : \mathbf{R} \to \mathbf{R}$ satisfies the relation $g(mn) = g(m)g(n)$ for each pair of coprime numbers m, n. The following chalenging hypothesis for the Möbius function μ seems to remain unreachable in the nearest future.

CONJECTURE (Kubilius-Linnik, 1959)**.** *There exists a sequence* $v = v(n) \to \infty$ *such that the process*

$$\frac{1}{\sqrt{v}} \sum_{k \le vt} \mu(m+k)$$

weakly converges to the Brownian motion at least for some subsequence $n = n' \to \infty$.

The attempts made by N.M.Lyashenko [L3], [L4] and Z.Kryžius [Kr4] still remain pretty far from the complete solution. They prove the convergence just for processes defined in terms of truncated multiplicative functions. The short annoncement [L5] of a proof of Conjecture should have a gap. As it was observed in the book [Li], the negative solution of the hypohesis would be more desirable.

Having the patern of processes (2.6), we jointly with G.Bareikis [Ba-M2] investigated the modified weak convergence of

$$e^{-\alpha(z(t)} \prod_{p|m,\, p \le z(t)} |g(p)|^{1/\beta(n)} \operatorname{sgn} g(p), \quad t \in [0,1].$$

Here $z(t) = z_n(t)$ and $\alpha(.), \beta(n)$ are appropriately chosen normalizing sequences. By analogy, models with multiplicative functions on shifted primes were studied (cf. [Ba-I2]).

8 Natural divisors and stochastic processes

Properties of natural divisors are closely related to that of prime divisors. Investigating the statistical value distribution of the function $\tau(m)$, the number of natural divisors of $m \in \mathbf{N}$, one can compair it with the function $2^{\Omega(m)}$, where $\Omega(m)$ denotes the number of all prime divisors of m. Motivated by this relation and Erdös' Arcsine law (Theorem 3.3), we have proved the following result.

THEOREM 8.1 (M9)**.** *Let* $1 = d_1(m) < \cdots < d_\tau(m) = m$ *be the natural divisors of* m, $\tau := \tau(m)$, *and* I^+ *be the characteristic function of the set of positive numbers. Then*

$$\lim \nu_n \left(\sum_{j \le \tau} \frac{1}{j} I^+\big(\log_2 j - LLd_j(m)\big) < (L2)uLLn \right) = As(u)$$

uniformly in $0 \le u \le 1$.

Limit theorems for other functionals involving natural divisors can be obtained from the next result. They could comprise some addendum of the interesting study by R.Hall and G.Tenenbaum [H-Te].

THEOREM 8.2 (M9)**.** *Let $\tau(m,u)$ be the number of natural divisors not exceeding u and*

$$T_n := T_n(m,t) := \frac{1}{\sqrt{LLn}}\bigl(\log_2 \tau(m, \exp\{(Ln)^t\}) - tLLn\bigr), \quad 1 \le t \le 1.$$

The sequence of processes T_n weakly converges to the standard Brownian motion.

One of the obstacles in the proof of this theorem was the fact that truncated sums over natural divisors are no longer multiplicative functions.

J.-M.Deshouillers, F.Dress, and G.Tenenbaum [De-Dr-Te] and G.Tenenbaum [Te1] proved two chalenging theorems on the value distribution of the ratio

$$D_n := D_n(m,t) := \frac{\tau(m, n^t)}{\tau(m)}, \quad t \in [0,1].$$

THEOREM 8.3 (De-Dr-Te)**.** *We have*

$$\frac{1}{n}\sum_{m=1}^{n} D_n(m,t) = As(t) + o(1)$$

uniformly in $t \in [0,1]$.

THEOREM 8.4 (Te1)**.** *We have*

$$\nu_n\bigl(D_n(m,t) - D_n(m,s) < x\bigr) \Rightarrow F_{st}(x)$$

uniformly for $0 \le s \le t \le 1$. Here F_{st} is some purely discrete distribution function.

Evidently, the previous theorem deals with the mean value of some arithmetic process, while the last one considers its increments. Thus the problem, raised by the author in [M9], to investigate the weak convergence of the processes D_n in the space $\mathbf{D}$ was a very natural extension of the presented assertions. It was recently solved with some generalization. For a real-valued multiplicative function f, we now set

$$F(m,v) = \sum_{d|m,\, d\le v} f(d), \quad F(m,m) = F(m), \quad S_n := S_n(m,t) = F(m,n^t)/F(m).$$

THEOREM 8.5 (M-T)**.** *If $f(p) = \kappa > 0$ and $f(p^k) \ge 0$ for all prime numbers p and $k \ge 2$, then S_n weakly converges to some process $S := S(t)$ in $\mathbf{D}$.*

Extending this result, G.Tenenbaum [Te3] showed that the support of the measure corresponding to the limit process S is the space of continuous functions on $[0,1]$. As in [De-Dr-Te], we now obtain that the expectation of $S(t)$ is the beta-distribution function, e.g.

$$\mathbf{E}S(t) = \frac{1}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_0^t v^{\gamma_1 - 1}(1-v)^{\gamma_2 - 1}\, dv,$$

where $\gamma_1 = \kappa/(1+\kappa)$, $\gamma_2 = 1/(1+\kappa)$, and Γ denotes the Euler gamma-function. For $\kappa = 1$, the last formula yields the arcsine law.

9 Concluding remarks

Apart from the multiplicative expression of natural numbers by a product of the prime ones, there exist a great variety of additive decompositions. Some of the ideas developed proving the functional limit theorems just discussed can also be adopted in this scheme. The author has shown [M10] that the q-additive sum-of-digit function plays also Brownian motion. Considering other numeration systems, one can face more strongly dependent random variables and thus more difficult obstacles.

A lot of combinatorial structures have similar decompositions into components. So a permutation or a mapping of a finite set into itself have respectively a unique product expression into cycles or into components defined by connected parts of its functional graps. Taking a permutation or a mapping with some probability, we arive into a new less investigated branch of discrete probability theory. The limit distributions of appropriately defined additive functions and processes are of great interest. In this way, the author jointly with G.J.Babu [B-M1], [B-M2], [B-M3], [B-M4] developed an analogue theory for random permutations. Several functional limit theorems for random mappings are also established (cf. [M11], [M12]) so far.

The weak convergence of processes can be strengthend to stronger type of convergence, say, to convergence with probability one. The obstacle that our objects are defined on a sequence of probability spaces can be overcome. This was shown by the author [M8] deriving the Strassen law of iterated logarithm for the arithmetic processes. W.Philipp during the Erdös conference personally communicated to the author that one can construct a sufficiently rich product probability space and find in it strong approximations of additive functions by sums independent gaussian or more general r.vs. This would lead to better understanding of the functional results.

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