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Total variation approximation for random assemblies and a functional limit theorem

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Abstract A class of random weakly logarithmic combinatorial assemblies is explored in the paper. We extend total variation approximations for the distribution of component vector of a random structure. That leads to the probabilistic approach suitable to examine the asymptotic value distribution of additive functions defined on such assemblies with the component sizes restricted to a given set. The results generalize several investigations of random Λ-permutations and their extensions to other structures obtained mainly by the Russian mathematicians. Instead of the most popular approach based upon the Tauber type theorems, we develop a comparative asymptotical analysis of coefficients of two Taylor series. Demonstrating possible applications, we obtain necessary and sufficient conditions for the weak convergence of processes defined via partial sums of an additive function to the Brownian motion.

Keywords Random combinatorial structure · Component size · Additive function · Brownian motion

Mathematics Subject Classification (2000) 60C05 · 05A16 · 60F15

1 Introduction

The present paper is motivated by many recent works on the so-called Λ-permutations. The latest and comprehensive list of references is presented in the book by

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A.L.Yakymiv [21]. To give an impression, we include here the main results of the paper [20].

Let $\Lambda \subset \mathbb{N}$ be a fixed set, $\mathbb{S}_n$ be the symmetric group of permutations of order n, and $\mathbb{T}_n \subset \mathbb{S}_n$ be the subset of permutations which can be decomposed into a product of cycles with lengths in Λ. A $\sigma \in \mathbb{T}_n$ is called an Λ-permutation. Maybe, the simplest instance of the Λ-permutations is the set $\{\sigma \in \mathbb{S}_n : \ \sigma^d = I\}$, where $d \in \mathbb{N}$ and I denotes the identity permutation. For this class of permutations Λ is just the set of natural divisors of d.

Set $\Lambda_n = \{k \in \Lambda : \ k \le n\}$, $l(n) = \sum_{k \in \Lambda_n} 1/k$, and $\nu_n^{(\Lambda)}(\ldots) = |\mathbb{T}_n|^{-1}|\{\sigma \in \mathbb{T}_n : \ \ldots\}|$. By $w^{(\Lambda)}(\sigma)$ we denote the number of cycles of $\sigma \in \mathbb{T}_n$. Let γ and $\Gamma(z)$ stand for the Euler constant and the Euler function. In the sequel, all limits, if not indicated otherwise, are taken as $n \to \infty$.

Theorem (Yakymiv, 89) *If for some a, $0 < a \le 1$,*

(i) $|\Lambda_n|/n \to a$
and, for arbitrary $C_0 > 1$,

(ii) $|\{k \in \Lambda_n, \ m - k \in \Lambda\}|/n \to a^2$ *uniformly in* $m \in [n, C_0 n]$,
then

$$|\mathbb{T}_n| \sim n! n^{a-1} L(n) \mathrm{e}^{-a\gamma} / \Gamma(a), \tag{1}$$

where

$$L(n) = \exp\{l(n) - a \log n\}$$

is slowly oscillating at infinity.
Moreover,

$$\nu_n^{(\Lambda)}\left(w^{(\Lambda)}(\sigma) - l(n) < x\sqrt{a \log n}\right) \to \Phi(x) =: \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-u^2/2} \mathrm{d}u \tag{2}$$

uniformly in $x \in \mathbb{R}$.

The proofs (see [20] or [21]) are based upon some Tauber theorems therefore one can hardly avoid conditions (i) and (ii).

In a similar manner, probabilistic problems on subsets of other combinatorial structures have been examined. For instance, Sachkov [16] (see also [17]) initiated the research on the random mappings of a finite set into itself with some restrictions on the components. The list of references could be further continued. So far, the main feature of such results is the use of some regularity condition on the set of sizes of components comprising these structures. Under this condition, the cardinality of structures of size n, as $|\mathbb{T}_n|$ in the instance above, behaves regularly.

On the other hand, as it is shown in [12], asymptotic formulas like (1) are not necessary for the limit relations analogous to (2). However, the paper [12] does not cover the mentioned works on random Λ-mappings and other similar subsequent studies of structures with component restrictions as well (see bibliography given in [21]). We now refine our approach to explore the value distribution of functions defined on subsets of combinatorial structures even when their cardinalities are irregular.

Let us discuss the general decomposable combinatorial structures called *assemblies* and keep the terminology presented in the book [2]. If $\mathcal{Q}$ is a class of labeled combinatorial structures $\varkappa_i$, $i \geq 1$, with the exponential generating series

$$Q(z) = \sum_{j\geq 1} \frac{q_j}{j!} z^j,$$

where q_j denotes the number of structures in $\mathcal{Q}$ of size j, then an assembly σ (in the instance above, a permutation) is a finite collection of $\varkappa_i$ relabeled in an appropriate way. For Λ-permutations, $\mathcal{Q}$ is just the set of cycles, thus, $q_j = (j-1)!$ if $j \in \Lambda$ and or $q_j = 0$ otherwise. If $\mathcal{A}_n$ denotes the class of assemblies of size n, $\mathcal{A}_0 = \{\emptyset\}$, then all information about enumeration of structures lays in the formal equality

$$\sum_{n=0}^{\infty} \frac{|\mathcal{A}_n|}{n!} z^n = \exp\{Q(z)\}.$$

Each $\sigma \in \mathcal{A}_n$ is characterized by the component vector $\bar{k}(\sigma) = (k_1(\sigma), \ldots, k_n(\sigma))$, where $k_j(\sigma) \geq 0$ is the number of components of size j in σ. Hence $\ell\left(\bar{k}(\sigma)\right) := 1k_1(\sigma) + \cdots + nk_n(\sigma) \equiv n$ for $\sigma \in \mathcal{A}_n$.

To begin a probabilistic theory, we introduce a weighted measure $\nu_n^{(w)}$ on $\mathcal{A}_n$. For a sequence $w_j \geq 0$, $1 \leq j \leq n$, set

$$w(\sigma) = \prod_{j=1}^{n} w_j^{k_j(\sigma)}, \quad 0^0 := 1, \quad W_n = \sum_{\sigma\in\mathcal{A}_n} w(\sigma),$$

and

$$\nu_n^{(w)}(\{\sigma\}) = w(\sigma) W_n^{-1}, \quad \sigma \in \mathcal{A}_n,$$

and extend it additively on all subsets of $\mathcal{A}_n$. So, taking, for instance, $w_j = 0$ or $w_j = 1$ we define the uniform probability measure on the subset of assemblies with the component size restrictions. In other words, the probabilistic theory of Λ-permutations can be treated as a particular case of the theory on the whole symmetric group with respect to some weighted probability measure.

The component vector $\bar{k}(\sigma)$, $\sigma \in \mathcal{A}_n$, has the following distribution:

$$\nu_n(\bar{k}(\sigma) = \bar{s}) = \mathbf{1}\{\ell(\bar{s}) = n\} \frac{n!}{W_n} \prod_{j=1}^{n} \frac{1}{s_j!} \left(\frac{q_j w_j}{j!}\right)^{s_j},$$

where $\bar{s} = (s_1, \ldots, s_n) \in \mathbb{Z}_+^n$. This leads to the *Conditioning Relation*

$$\nu_n^{(w)}(\bar{k}(\sigma) = \bar{s}) = P\left(\bar{\xi} = \bar{s} | \ell(\bar{\xi}) = n\right), \tag{3}$$

where $\bar{\xi} := (\xi_1, \ldots, \xi_n)$ and ξ_j, $j \geq 1$, are mutually independent Poisson r.vs defined on some probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ with $\mathbf{E}\xi_j = u^j q_j w_j / j!$, $j \geq 1$, where $u > 0$ is an arbitrary number. If $q_j w_j = 0$ the corresponding ξ_j becomes the degenerated at the zero point random variable. Formula (3) generalizes (2.29) in [2] (p 59), where $w_j \equiv w \neq 0$, $j \geq 1$, is assumed; their proofs are the same, however. Simplifying the notation we introduce the following class of assemblies.

Definition Let $\mathcal{A}_n$ be a set of assemblies of size $n \geq 1$, and μ_n be a probability measure on $\mathcal{A}_n$. The pair $(\mathcal{A}_n, \mu_n)$ is called weakly logarithmic if the following conditions are satisfied:

(I) there exists a random vector $\bar{\xi} = (\xi_1, \ldots, \xi_n)$ with mutually independent poissonian coordinates, $\mathbf{E}\xi_j =: \lambda_j \geq 0$, and such that

$$\mu_n(\bar{k}(\sigma) = \bar{s}) = P\left(\bar{\xi} = \bar{s} | \ell(\bar{\xi}) = n\right)$$

for each $\bar{s} \in \mathbb{Z}_+^n$;
and

(II) there exist positive constants Θ, θ_0, θ_1, and N such that

$$d_j := j\lambda_j \leq \Theta \tag{4}$$

for all $1 \leq j \leq n$ and, for $n > N$,

$$\sum_{j \leq u} d_j \geq \theta_0 u \tag{5}$$

uniformly in u, $N \leq u \leq n$, and

$$\theta_1(v) := \min_{0 \leq k \leq vn} \sum_{k<m<n} \frac{d_{m-k} d_{n-m}}{m} \geq \theta_1 > 0 \tag{6}$$

for some fixed $0 < v < 1$.

The class of Λ-permutations satisfying conditions (i) and (ii) above is weakly logarithmic because then $d_j = 1$ if $j \in \Lambda$ and $d_j = 0$ otherwise. Moreover, condition (i) implies (5) and, from (ii), we obtain

$$\begin{aligned}\theta_1(v) &\geq \frac{1}{n} \min_{0 \leq k \leq vn} \sum_{1 \leq j < n-k} d_j d_{n-k-j} \\ &\geq (1-v) \min_{(1-v)n \leq m \leq n} \frac{1}{m} \sum_{1 \leq j < m} d_j d_{m-j} \\ &\geq (1-v) a^2/2 > 0\end{aligned}$$

for each $0 < v < 1$ if n is sufficiently large.

In our notation, the logarithmic class of assemblies investigated in [2] is reckoned by the relation $d_j \sim \Theta_2$ for some constant $\Theta_2 > 0$ as $j \to \infty$. Stark [18,19] has examined assemblies under an analytic condition. He assumed that

$$Q(z) = -\Theta_2 \log(1-z) + Q_1(z),$$

where $Q_1(z)$ is an analytic in the unit disk function having an analytic continuation to some specific region outside it. That also implies asymptotically regular behavior of d_j as $j \to \infty$ and, consequently, regularity of $|\mathcal{A}_n|$. In [12,13], we dealt with the case when $0 < \theta_0 \le d_j \le \Theta$ for all $1 \le j \le n$. So, the Definition is rather general. Moreover, and maybe, that is even more interesting, the conditions in Definition allow us to take parameters d_j, $1 \le j \le n$, depending on n. Consequently, it extends classes of Λ-permutations allowing Λ to vary together with n. Observe that sparse subsets of assemblies such as $\{\sigma = \tau^r : \tau \in \mathbb{S}_n\}$, $r \ge 2$, in the case of permutations, investigated, for instance, in [22], do not satisfy condition (II).

The main result of this paper is a total variation approximation. Let $\mathcal{L}(X)$ be the distribution of a r. v. X with respect to the appropriate probability measure. Afterwards the index r, $1 \le r \le n$, added to the vectors $\bar{k}(\sigma)$ and $\bar{\xi}$ will denote that only the first r coordinates are taken. Let $x_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$ and $\ll$ be an analog of the symbol $\mathrm{O}(\cdot)$. Afterwards, all further involved constants $c, c_1, \ldots, C, C_1, \ldots$, and those hidden in various symbols, if not stated otherwise, depend only on Θ, θ_0, θ_1, v, and N in the Definition.

Theorem 1 *Let $(\mathcal{A}_n, \mu_n)$ be weakly logarithmic and $n \ge 1$. There exists a constant c, $0 < c \le 1/2$, such that*

$$\rho_{TV}\left(\mathcal{L}\left(\bar{k}_r(\sigma)\right), \mathcal{L}(\bar{\xi}_r)\right) := \sum_{\bar{s} \in \mathbb{Z}_+^r} \left(\mu_n\left(\bar{k}_r(\sigma) = \bar{s}\right) - P(\bar{\xi}_r = \bar{s})\right)_+ \ll \left(\frac{r}{n}\right)^c \tag{7}$$

uniformly in $1 \le r \le n$.

Theorem 1 deserves the title *Fundamental Lemma* which is used sometimes. It reduces the value distribution problem for an arbitrary mapping defined on $\mathcal{A}_n$ via $\bar{k}_r(\sigma)$, where $r = \mathrm{o}(n)$, to a problem for independent r vs. It is impossible (see [2]) to extend this transfer if the coordinates of $\bar{k}(\sigma)$ with large indexes are involved. Sometimes in such the cases, it suffices to apply inequalities for conditional probabilities. This idea goes back to probabilistic number theory, in particular, to Ruzsa's paper [15]. It proved to be useful for permutations [3,8], for the logarithmic assemblies [14], and their generalizations [13] as well. We now extend the same principle.

Firstly, we introduce some notation taken from the theory of euclidean spaces. For two vectors $\bar{s} = (s_1, \ldots, s_n)$ and $\bar{t} = (t_1, \ldots, t_n)$ from the semi-lattice $\mathbb{Z}_+^n$, we set $\bar{s} \perp \bar{t}$ if $s_1 t_1 + \cdots + s_n t_n = 0$ and write $\bar{s} \le \bar{t}$ if $s_j \le t_j$ for each $j \le n$. Further, we adopt the notation $\bar{s} \parallel \bar{t}$ for the expression "$\bar{s}$ exactly enters $\bar{t}$" which means that $\bar{s} \le \bar{t}$ and $\bar{s} \perp \bar{t} - \bar{s}$. For a subset $U \subset \mathbb{Z}_+^n$, we define its extension

$$\begin{aligned} V = V(U) = \{\bar{s} = \bar{t}^{(1)} + \bar{t}^{(2)} - \bar{t}^{(3)} :\ & \bar{t}^{(1)}, \bar{t}^{(2)}, \bar{t}^{(3)} \in U, \\ & \bar{t}^{(1)} \perp (\bar{t}^{(2)} - \bar{t}^{(3)}),\ \bar{t}^{(3)} \parallel \bar{t}^{(2)}\}. \end{aligned} \tag{8}$$

Set also $\theta = \min\{1, \theta_0\}$.

Theorem 2 *Let $(\mathcal{A}_n, \mu_n)$ be weakly logarithmic, $n \geq 1$, and $\bar{\xi}$ be the poissonian random vector introduced in the Definition. For arbitrary $U \subset \mathbb{Z}_+^n$,*

$$\mu_n\left(\bar{k}(\sigma) \notin V\right) = P\left(\bar{\xi} \notin V \,|\, \ell(\bar{\xi}) = n\right) \ll P^{\theta}(\bar{\xi} \notin U) + \mathbf{1}\{\theta < 1\} n^{-\theta}.$$

Theorems 1 and 2 are the basic tools to prove general limit theorems for additive functions defined on $\mathcal{A}_n$. To save some space, we now recall their general definition. Let $(\mathbb{G}, +)$ be an abelian group and $h_j(s)$, $j, s \geq 1$, be a two-dimensional array in $\mathbb{G}$ and $h_j(0) \equiv 0$. Then an *additive function* $h\colon \mathcal{A}_n \to \mathbb{G}$ is defined by

$$h(\sigma) = \sum_{j \leq n} h_j\left(k_j(\sigma)\right). \tag{9}$$

If $h_j(s) = a_j s$ for some $a_j \in \mathbb{G}$, where $j, s \geq 1$, then the function $h(\sigma)$ is called *completely additive*. So far, several one-dimensional limit theorems for $h(\sigma)$ were proved (see, for instance, [2], Section 8.5 or [12]). A lot of attention has been paid to the weak convergence of random combinatorial processes (see [3–5,9,10], and [2], Section 8.1). Partial cases of Theorems 1 and 2 have also been applied to examine the strong convergence of truncated sums of additive functions and to prove the iterated logarithm laws (see [13] or [14] and the references therein).

We now generalize the weak invariance principle establishing necessary and sufficient conditions. The result extends to the class of weakly logarithmic assemblies the corresponding functional limit theorems obtained for permutations [3] and all mappings of a finite set into itself [9]. Unfortunately, it does not cover theorems presented in Chapter 8 of [2] where stronger types of convergence are explored. Avoiding technical difficulties, we fix the class of weakly logarithmic assemblies, e.g. we assume that the sequences λ_j, d_j, $j \geq 1$, in the Definition do not depend on n. Let $h_j(k) \in \mathbb{R}$, $j, k \geq 1$, be an arbitrary two-dimensional sequence, also not depending on n, $h_j(0) \equiv 0$, and $a_j := h_j(1)$ for $j \geq 1$. Set

$$B^2(j) = \sum_{i \leq j} a_i^2 \lambda_i, \quad \tau_{nj} = \frac{B^2(j)}{B^2(n)}, \quad \tau_{n0} = 0, \qquad a_{nj} = \frac{a_j}{B(n)},$$
$$\widehat{H}_n := \widehat{H}_n(\sigma, t) = \sum_{j, \tau_{nj} \leq t} a_{nj}\left(k_j(\sigma) - \lambda_j\right),$$
$$\widetilde{H}_n := \widetilde{H}_n(\sigma, t) = \frac{1}{B(n)} \sum_{j, \tau_{nj} \leq t} \left(h_j(k_j(\sigma)) - a_j \lambda_j\right),$$

and

$$X_n := X_n(t) = \sum_{j, \tau_{nj} \leq t} a_{nj}(\xi_j - \lambda_j),$$

where $0 \leq t \leq 1$. Let H_n be either of the processes $\widehat{H}_n$ or $\widetilde{H}_n$. All just defined trajectories belong to the $\mathbb{D}[0, 1]$ space (see [6]). We consider the weak convergence (denoted afterwards by $\Rightarrow$) of the process H_n to the standard Brownian motion, e.g. the weak convergence of distributions $\mu_n \circ H_n^{-1}$ to the Wiener measure W concentrated on the space $\mathbb{C}[0,1]$. In the case of such limiting measure, it is natural to use the uniform metric in $\mathbb{D}[0,1]$. Of course, taking the linearized versions of the processes, one could equivalently examine convergence in $\mathbb{C}[0,1]$.

Theorem 3 *Let $(\mathcal{A}_n, \mu_n)$ satisfy the conditions in Definition and, in addition, let $\theta_1(v) \to \infty$ as $v \to 0$. Assume that $B(n) \to \infty$. The following assertions are equivalent*:

(I_1) $\mu_n \circ H_n^{-1} \Rightarrow W$;
(I_2) $P \circ X_n^{-1} \Rightarrow W$;
(I_3)

$$\sum_{j\le n} \mathbf{1}\{|a_{nj}| \ge \varepsilon\} a_{nj}^2 \lambda_j = \mathrm{o}(1)$$

for every $\varepsilon > 0$.

It is worth to recall [4] that, in general, (I_3) is not necessary for the one-dimensional convergence of $\mu_n \circ H_n^{-1}(\cdot, 1)$ to the standard normal distribution.

2 Proof of Theorem 1

The first lemma reduces the problem to a one-dimensional case. For $\bar{s} = (s_1, \ldots, s_n)$, set $\ell_{ij}(\bar{s}) = (i+1)s_{i+1} + \cdots + js_j$ if $0 \le i < j \le n$. Moreover, let $\ell_r(\bar{s}) := \ell_{0r}(\bar{s})$, where $1 \le r \le n$. Then $\ell_n(\bar{s}) = \ell(\bar{s})$.

Lemma 1 *We have*

$$\begin{aligned}\rho_{TV}\left(\mathcal{L}\left(\bar{k}_r(\sigma)\right), \mathcal{L}(\bar{\xi}_r)\right) &= \rho_{TV}\left(\mathcal{L}\left(\bar{\xi}_r | \ell(\bar{\xi}) = n\right), \mathcal{L}\left(\bar{\xi}_r\right)\right) \\ &= \sum_{m\in\mathbb{Z}_+} P\left(\ell_r(\bar{\xi}) = m\right)\left(1 - \frac{P\left(\ell_{rn}(\bar{\xi}) = n-m\right)}{P\left(\ell(\bar{\xi}) = n\right)}\right)_+ \end{aligned} \tag{10}$$

Proof See [1] or [2], p. 69.

Since

$$P\left(\ell_{rn}(\bar{\xi}) = m\right) = \frac{1}{2\pi i} \int\limits_{|z|=1} \frac{1}{z^m} \exp\left\{\sum_{r<j\le n} \lambda_j (z^j - 1)\right\} \mathrm{d}z, \tag{11}$$

we can apply the comparative analysis of the Taylor coefficients developed in [11,13].

For an arbitrary sequence $d_j = d_j(n) \geq 0, 1 \leq j \leq n$, and $0 \leq r \leq n$, we introduce the following functions

$$D(z) := \exp\left\{\sum_{j\leq n} \frac{d_j}{j} z^j\right\} =: \sum_{s=0}^{\infty} D_s z^s$$

and

$$F(z) := \exp\left\{\sum_{r<j\leq n} \frac{d_j}{j} z^j\right\} =: \sum_{s=0}^{\infty} F_s z^s.$$

Set

$$e_r = \exp\left\{-\sum_{j\leq r} \frac{d_j}{j}\right\}.$$

Proposition *Assume that condition* (II) *in the Definition is satisfied. There exist* $n_0 \geq N, 0 < \delta_0 \leq 1/2$, *and* $c_1 > 0$ *such that, for every* $1/n \leq \delta \leq \delta_0, n \geq n_0$, *and* $0 \leq \eta \leq 1/2$,

$$F_m/(e_r D_n) - 1 \ll (\eta + (r/n)\mathbf{1}\{r \geq 1\})\,\delta^{-1} + \delta^{c_1}$$

uniformly in

$$0 \leq r \leq \delta n, \quad n(1-\eta) \leq m \leq n. \tag{12}$$

The constant in $\ll$ *depends only on* n_0, δ_0, *and* c_1.

The Proposition will be proved in the next section. Using it, we now afford the first task.

Proof of Theorem 1 If $\lambda_j = d_j/j$, then, by (11) and our notation,

$$P\left(\ell_{rn}(\bar{\xi}) = n-m\right) = \frac{F_{n-m}}{e_r D(1)}, \quad P\left(\ell(\bar{\xi}) = n\right) = \frac{D_n}{D(1)}. \tag{13}$$

Let n_0, δ_0 and c_1 be as in the Proposition. For $n \geq n_0$ and $1 \leq r \leq \delta_0^{2(1+c_1)} n =: c_2 n$, we can choose

$$\eta = \sqrt{r/n}, \quad \delta = (r/n)^{1/2(1+c_1)}$$

to obtain

$$\frac{P\left(\ell_{rn}(\bar{\xi}) = n-m\right)}{P\left(\ell(\bar{\xi}) = n\right)} - 1 \ll \left(\frac{r}{n}\right)^{c_1/2(1+c_1)} =: \left(\frac{r}{n}\right)^{c_3}$$

uniformly in $0 \leq m \leq \eta n = \sqrt{rn}$.

The summands over $m > \sqrt{rn}$ in (10) contribute not more than

$$(rn)^{-1/2}\mathbf{E}\ell_r(\bar{\xi}) = (rn)^{-1/2}\sum_{j\le r} j\lambda_j \le \Theta(r/n)^{1/2}.$$

Hence, by (10), we obtain

$$\rho_{TV}\left(\mathcal{L}\left(\bar{k}_r(\sigma)\right), \mathcal{L}(\bar{\xi}_r)\right) \ll (r/n)^c,$$

where $c := \min\{1/2, c_3\} = c_3$ and $1 \le r \le c_2 n$. Since the claim of Theorem 1 is trivial for $n \le n_0$ or $c_2 n \le r \le n$, we complete the proof.

3 Proof of Proposition

We can follow the path drawn up in our papers [11,13]. In the sequel some of the steps are repeated, however, due to the weaker conditions than that used previously, the calculations become more cumbersome. As in [13], we introduce the following notation. Let $K = K(n)$, $1 \le \max\{\delta n, N\} < K \le n$, to be chosen later. For a fixed $0 < \alpha < 1$, we set

$$G(z) := \sum_{r<j\le n} d_j z^{j-1}, \quad G_1(z) = \exp\left\{\alpha \sum_{r<j\le K} \frac{d_j}{j} z^j\right\},$$
$$G_2(z) = \exp\left\{-\alpha \sum_{K<j\le n} \frac{d_j}{j} z^j\right\}, \quad G_3(z) = F^\alpha(z) - G_1(z).$$

Denote by $[z^k]V(z)$ the kth Taylor coefficient of a function $V(z)$ analytic at zero and observe that

$$[z^k]G_3(z) \le [z^k]F^\alpha(z) \tag{14}$$

for each $k \ge 0$.

Set $T = C(\delta n)^{-1}$, where $C > 0$ is a constant to be chosen later,

$$\Delta = \{z = \mathrm{e}^{it} : T < |t| \le \pi\}, \quad \Delta_0 = \{z = \mathrm{e}^{it} : |t| \le T\}.$$

We start from the following identity

$$
\begin{aligned}
F_m &= \frac{1}{2\pi i m} \int\limits_{|z|=1} \frac{F'(z)}{z^m}\,\mathrm{d}z \\
&= \frac{1}{2\pi i m} \left(\int\limits_{\Delta_0} + \int\limits_{\Delta} \right) \frac{F'(z)\,(1-G_2(z))}{z^m}\,\mathrm{d}z \\
&\quad + \frac{1}{2\pi i m} \int\limits_{|z|=1} \frac{F'(z)G_2(z)}{z^m}\,\mathrm{d}z =: J_0 + J_1 + J_2.
\end{aligned} \tag{15}
$$

In the proof of Proposition, there are two milestones. Firstly, we have to estimate the integrals J_1 and J_2. Secondly, changing the integrand, we have to reduce J_0 to the main term of an asymptotical formula for $e_r D_n$. The technical details are hidden in a few lemmas.

Lemma 2 *If conditions* (4) *and* (6) *are satisfied, then there exist* $n_0 \geq 1$ *and* $c_4 > 0$ *such that*

$$
c_4 D(1) \leq n D_n \leq \Theta D(1) \tag{16}
$$

for all $n \geq n_0$.

Proof Differentiating the function $D(z)$ we obtain

$$
s D_s = \sum_{m=0}^{s-1} D_m d_{s-m}, \quad s \geq 1. \tag{17}
$$

Consequently, the second inequality in (16) follows from (4). Similarly, by condition (6), using repeatedly (17), we obtain

$$
\begin{aligned}
n D_n &= d_n + \sum_{m=1}^{n-1} d_{n-m} \left(\frac{1}{m} \sum_{k=0}^{m-1} D_k d_{m-k} \right) \\
&= d_n + \sum_{k=0}^{n-2} D_k \sum_{m=k+1}^{n-1} \frac{d_{n-m} d_{m-k}}{m} \\
&\geq \theta_1(v) \sum_{k \leq vn} D_k \geq \theta_1 \sum_{k \leq vn} D_k.
\end{aligned} \tag{18}
$$

To substitute $D(1)$ for the last sum, we take an arbitrary $0 < x < 1$. Then

$$\begin{aligned} D(x) &\le \sum_{k\le vn} D_k + \frac{xD'(x)}{vn}\,\frac{D(x)}{D(x)} \\ &= \sum_{k\le vn} D_k + \frac{D(x)}{vn}\sum_{j\le n} d_j x^j \\ &\le \sum_{k\le vn} D_k + \frac{\Theta D(x)}{(1-x)vn}. \end{aligned}$$

Hence

$$\sum_{k\le vn} D_k \ge D(x)\left(1 - \frac{\Theta}{(1-x)vn}\right)$$

Now we choose $x = \mathrm{e}^{-2\Theta/vn}$. If $n \ge \max\{4\Theta/v, 1\} = n_0$, applying the inequalities $t\mathrm{e}^{-t} \le 1 - \mathrm{e}^{-t} \le t$ if $t \ge 0$, we obtain

$$\begin{aligned} \sum_{k\le vn} D_k &\ge D(\mathrm{e}^{-2\Theta/vn})\left(1 - \frac{1}{2}\mathrm{e}^{2\Theta/vn}\right) \\ &\ge \left(1 - \frac{\sqrt{\mathrm{e}}}{2}\right)\exp\left\{\sum_{j\le n}\frac{d_j}{j}\left(1 + (\mathrm{e}^{-2\Theta j/vn} - 1)\right)\right. \\ &\ge (1 - \sqrt{\mathrm{e}}/2)D(1)\exp\{-2\Theta^2 v^{-1}\} = c_5 D(1). \end{aligned}$$

Recalling (18), we see that the left-hand inequality in (16) holds with $c_4 = \theta_1 c_5$.
The lemma is proved.

Lemma 3 *Let $T = C(\delta n)^{-1} \le 1$. If conditions (4) and (5) are satisfied, then there exist sufficiently small positive constants c_6 and δ_0, and sufficiently large C such that*

$$\min_{T\le t\le\pi}\sum_{\delta n<j\le n}\frac{d_j(1-\cos tj)}{j} \ge c_6\log\frac{1}{\delta}$$

and

$$\max_{T\le t\le\pi}\left|F(\mathrm{e}^{it})\right| \ll e_r D(1)\delta^{c_6}$$

for $\delta \le \delta_0$.

Proof Let $0 < \varepsilon < 1$ be arbitrary but sufficiently small so that $a := a(\varepsilon) = \arccos(1-\varepsilon) \le 1/2$, $y_1(k) = (2\pi k - a)/t$, and $y_2(k) = (2\pi k + a)/t$. If $N = [(nt + a)/(2\pi)]$

and $M = \lceil (\delta n t - a)/(2\pi) \rceil$, then

$$\begin{aligned} S_{\delta n}(t) &:= \sum_{\delta n < j \le n} \frac{d_j (1 - \cos tj)}{j} \ge \varepsilon \sum_{k=M}^{N-1} \sum_{y_2(k) < j \le y_1(k+1)} \frac{d_j}{j} \\ &= \varepsilon \sum_{y_1(M) < j \le y_2(N)} \frac{d_j}{j} - \varepsilon \sum_{k=M}^{N} \sum_{y_1(k) < j \le y_2(k)} \frac{d_j}{j} \\ &=: \varepsilon \Sigma_1 - \varepsilon \Sigma_2. \end{aligned}$$

By virtue of conditions (4) and (5), for arbitrary $N \le x < y$,

$$\begin{aligned} \Sigma(x, y) &:= \sum_{x < j \le y} \frac{d_j}{j} = \int_x^y \frac{1}{u} \, \mathrm{d} \left(\sum_{j \le u} d_j \right) \\ &= \frac{1}{y} \sum_{j \le y} d_j - \frac{1}{x} \sum_{j \le x} d_j + \int_x^y \left(\sum_{j \le u} d_j \right) \frac{\mathrm{d}u}{u^2} \\ &\ge \theta_0 \log \frac{y}{x} - \Theta. \end{aligned} \tag{19}$$

Hence

$$\Sigma_1 = \Sigma(y_1(M), y_2(N)) \ge \theta_0 \log \frac{1}{\delta} - C_1$$

provided that $y_1(M) \ge N$. This can be assured by a choice of C.

Further, exploiting the well-known four terms expansion for the harmonic numbers, we obtain

$$\begin{aligned} \Sigma_2 &\le \Theta \sum_{k=M}^{N} \sum_{y_1(k) < j \le y_2(k)} \frac{1}{j} \\ &\le \Theta \sum_{k=M}^{N} \left(\log \frac{y_2(k)}{y_1(k)} + \mathrm{O}\left(\frac{1}{y_1^2(k)} \right) \right) \\ &= \Theta \sum_{k=M}^{N} \left(\log \left(1 + \frac{2a}{2\pi k - a} \right) + \mathrm{O}\left(\frac{1}{k^2} \right) \right) \\ &= \frac{\Theta a}{\pi} \log \frac{N}{M} + C_2 \le \frac{\Theta a}{\pi} \log \frac{1}{\delta} + C_3. \end{aligned}$$

Returning to the initial sum, we have

$$S_{\delta n}(t) \ge \varepsilon \left(\theta_0 - \frac{\Theta a}{\pi} \right) \log \frac{1}{\delta} - C_4 \ge \frac{\varepsilon \theta_0}{4} \log \frac{1}{\delta}$$

uniformly in $T \le t \le \pi$ provided that ε is chosen so small that $a = a(\varepsilon) \le \theta_0\pi/2\Theta$ and $\delta \le \delta_0$, where $\log(1/\delta_0) \ge 4C_4/(\varepsilon\theta_0)$.

To prove the second claim of the lemma, it suffices to apply the relation

$$\frac{|F(\mathrm{e}^{it})|}{D(1)} = e_r \exp\left\{\sum_{r<j\le n} \frac{d_j(\cos tj - 1)}{j}\right\} \le e_r \exp\left\{-S_{\delta n}(t)\right\}.$$

The lemma is proved.

Lemma 4 *Assume that conditions* (4) *and* (6) *are satisfied. If* $0 < \alpha < 1$, $1 \le \delta n < K < n$, *and* $n \ge n_0$, *then*

$$J_2 \ll D_n e_r (K/n)^{\alpha\theta_0}$$

uniformly in $0 \le r \le \delta n$ *and* $n/2 \le m \le n$.

Proof For brevity, let

$$u_s := [z^s]G_1(z), \quad v_l := [z^l]F^{1-\alpha}(z), \quad s, l \ge 0.$$

Since

$$F'(z)G_2(z) = G(z)G_1(z)F^{1-\alpha}(z),$$

from Cauchy's formula, we have

$$\begin{aligned} J_2 &= \frac{1}{2\pi i m} \int_{|z|=1} G(z)G_1(z)F^{1-\alpha}(z)\frac{\mathrm{d}z}{z^m} \\ &= \frac{1}{m} \sum_{r<j\le m} d_j \sum_{s+l=m-j} u_s v_l. \end{aligned}$$

Hence, by condition (4),

$$\begin{aligned} |J_2| &\le \frac{2\Theta}{n} \sum_{s\le n} u_s \sum_{l\le n} v_l \\ &\le \frac{2\Theta}{n} F^{1-\alpha}(1)G_1(1) = \frac{2\Theta F(1)}{n} \exp\left\{-\alpha \sum_{K<j\le n} \frac{d_j}{j}\right\} \\ &\le \frac{2\Theta D(1) e_r}{n} \exp\{-\alpha\Sigma(K, n)\} \ll D_n e_r (K/n)^{\alpha\theta_0} \end{aligned}$$

by (19) and Lemma 2.

The lemma is proved.

Lemma 5 *Assume that conditions* (4) *and* (5) *are satisfied. Let* $0 < \alpha < 1$ *be arbitrary,* δ_0 *be as in Lemma 3,* $1 \le \delta n < K < N$, $\delta \le \delta_0$, *and* $n \ge n_0$. *Then*

$$J_1 \ll \frac{e_r n D_n}{K} \delta^{c_6(1-\alpha)}$$

uniformly in r, $0 \le r \le \delta n$, *and* m, $n/2 \le m \le n$.

Proof Recalling the previous notation we can rewrite

$$J_1 = \frac{1}{2\pi i m} \int_{\Delta} G(z) F^{1-\alpha}(z) G_3(z) \frac{\mathrm{d}z}{z^m}.$$

Hence, by Lemma 3, if $\delta \le \delta_0$,

$$\begin{aligned} J_1 &\ll n^{-1} \max_{z\in\Delta} |F(z)|^{1-\alpha} \int_{|z|=1} |G(z)|\,|G_3(z)|\,|\mathrm{d}z| \\ &\ll n^{-1} \left(e_r D(1) \delta^{c_6}\right)^{1-\alpha} \left(\int_{|z|=1} |G(z)|^2\,|\mathrm{d}z| \right)^{1/2} \\ &\quad \times \left(\int_{|z|=1} |G_3(z)|^2 |\mathrm{d}z| \right)^{1/2}. \end{aligned}$$

By Parseval's equality,

$$\int_{|z|=1} |G(z)|^2\,|\mathrm{d}z| = 2\pi \sum_{r<j\le n} d_j^2 \le 2\pi \Theta^2 n$$

and, recalling (14),

$$\begin{aligned} \int_{|z|=1} |G_3(z)|^2 |\mathrm{d}z| &\le 2\pi \sum_{l>K} \left([z^l] G_3(z)\right)^2 \\ &\le \frac{2\pi}{K^2} \sum_{l=1}^{\infty} l^2 \left([z^l] F^{\alpha}(z)\right)^2 \ll \frac{1}{K^2} \int_{|z|=1} \left|(F^{\alpha}(z))'\right|^2 |\mathrm{d}z| \\ &\ll \frac{(e_r D(1))^{2\alpha}}{K^2} \int_{|z|=1} |G(z)|^2 |\mathrm{d}z| \ll \frac{(e_r D(1))^{2\alpha} n}{K^2}. \end{aligned}$$

Collecting the last three estimates by Lemma 2 we obtain the desired claim.
Lemma 5 is proved.

We now arrive at the first milestone.

Lemma 6 *Let condition* (II) *in the Definition be satisfied and* δ_0 *be as in Lemma 3. If* $\delta \le \delta_0$ *and* $n \ge n_0$*, then there exists a positive constant* c_7 *such that*

$$F_m = J_0 + \mathrm{O}\left(e_r D_n \delta^{c_7}\right) \tag{20}$$

uniformly in $0 \le r \le \delta n$ *and* $n/2 \le m \le n$*. Moreover,*

$$D_n = \frac{1}{2\pi i n} \int\limits_{\Delta_0} D'(z) \frac{\mathrm{d}z}{z^n} + \mathrm{O}\left(D_n \delta^{c_7}\right). \tag{21}$$

Proof It suffices to apply Lemmas 4 and 5 with $K = \delta^{c(\alpha)} n$, where

$$c(\alpha) := \min\{1,\ c_6(1-\alpha)/(\alpha\theta_0 + 1)\}.$$

Then

$$F_m = J_0 + \mathrm{O}\left(e_r D_n \delta^{\theta_0 \alpha c(\alpha)}\right)$$

with the needed uniformity. The constant $c_7 := \sup\{\theta_0 \alpha c(\alpha) :\ 0 < \alpha < 1\}$ would be the best choice.

To obtain (21), use (20) with $r = 0$ and $m = n$.

Lemma 6 is proved.

Lemma 7 *Assume that condition* (II) *in the Definition is satisfied,* δ_0 *is defined in Lemma 3,* $1/n \le \delta \le \delta_0$*, and* $n \ge n_0$*. If* $0 \le \eta \le 1/2$ *is arbitrary, then*

$$J_0 = e_r D_n \left(1 + \mathrm{O}\left(\left(\eta + (r/n)\mathbf{1}\{r \ge 1\}\right) \delta^{-1} + \delta^{c_7}\right)\right)$$

uniformly in $n(1-\eta) \le m \le n$ *and* $0 \le r \le \delta n$*.*

Proof If $z \in \Delta_0$ and $r \ge 1$, then, by condition (4),

$$\begin{aligned} F'(z) &= e_r D(z) \exp\left\{-\sum_{j \le r} \frac{d_j}{j}(z^j - 1)\right\} G(z) \\ &= e_r D(z) \left(1 + \mathrm{O}\left(\frac{r}{\delta n}\right)\right) \left(\sum_{j \le n} - \sum_{j \le r}\right) d_j z^{j-1} \\ &= e_r D'(z) \left(1 + \mathrm{O}\left(r/\delta n\right)\right) + \mathrm{O}\left(r e_r D(1)\right) \end{aligned}$$

and

$$z^{-m} = z^{-n}\left(1 + \mathrm{O}(\eta\delta^{-1})\right).$$

Consequently, by virtue of $m^{-1} = n^{-1}(1+\mathrm{O}(\eta))$, from Lemma 2, and (18), we obtain

$$J_0 = \frac{e_r}{2\pi i n}\left(1+\mathrm{O}\left(\left(\frac{r}{n}+\eta\right)\frac{1}{\delta}\right)\right)\int_{\Delta_0} D'(z)\frac{\mathrm{d}z}{z^n} + \mathrm{O}\left(e_r D_n \frac{r}{\delta n}\right)$$
$$= e_r D_n\left(1+\mathrm{O}\left((r/n+\eta)\delta^{-1}+\delta^{c_7}\right)\right).$$

If $r < 1$, the terms with the fraction r/n do not appear.
The lemma is proved.

Proof of Proposition Apply (20) and the last lemma.

4 Proof of Theorem 2 and Corollaries

Set $\mathbb{Z}_+^n(m) = \{\bar{s} \in \mathbb{Z}_+^n : \ell(\bar{s}) = m\}$ where $0 \le m \le n$. For arbitrary distributions $p_j(k)$, $1 \le j \le n$, on $\mathbb{Z}_+$ we define the product measure on $\mathbb{Z}_+^n$ by

$$P(\{\bar{k}\}) = \prod_{j \le n} p_j(k_j), \quad \bar{k} = (k_1, \ldots, k_s) \in \mathbb{Z}_+^n.$$

Denote for brevity $P_n = P(\mathbb{Z}_+^n(n))$. Let $V = V(U)$ be the extension of an arbitrary subset $U \subset \mathbb{Z}_+^n$ defined in (8) and $\overline{U} := \mathbb{Z}_+^n(m)\setminus U$.

Lemma 8 *Suppose $n \ge 1$ and there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that*

(i) $p_j(0) \ge \alpha_1$ *for all* $1 \le j \le n$;
(ii) $P\left(\mathbb{Z}_+^n(m)\right) \le \beta_1 \left(\frac{n}{m+1}\right)^{1-\theta} P_n$ *for* $0 \le m \le n-1$ *and for some* $0 < \theta \le 1$;
(iii) $P_n \ge \alpha_2 n^{-1}$;
(iv) *for* $1 \le m \le n$,

$$\sum_{kj=m} \frac{p_j(k)}{p_j(0)} \le \frac{\beta_2}{m}.$$

Then

$$P\left(\overline{V}\,|\mathbb{Z}_+^n(m)\right) \ll P^\theta(\overline{U}) + n^{-\theta}\mathbf{1}\{\theta < 1\},$$

where the constant in $\ll$ depends on the constants given in the conditions only.

Proof See [3], Appendix.

Proof of Theorem 2 It suffices to verify that condition (II) in the Definition implies relations (i)–(iv) of the last lemma for the poissonian probabilities $p_j(k)$ with parameters λ_j. By virtue of (4), requirements (i) and (iv) are trivial. Further, we find

$$\begin{aligned} P\left(\mathbb{Z}_+^n(m)\right) &= P\left(\sum_{j=1}^m j\xi_j = m, \xi_{m+1} = 0, \ldots, \xi_n = 0\right) \\ &= \exp\left\{-\sum_{j=1}^n \lambda_j\right\} \sum_{\ell_m(\bar{k})=m} \prod_{j=1}^m \frac{\lambda_j^{k_j}}{k_j!} \\ &= \exp\left\{-\sum_{j=1}^n \lambda_j\right\} [z^m] \exp\left\{\sum_{j\le m} \lambda_j z^j\right\}, \quad 0 \le m \le n. \end{aligned}$$

Let $a \asymp b$ mean $a \ll b \ll a$. If $\theta_0 \le 1$, applying Lemma 2 with m instead of n and (19), we obtain

$$P\left(\mathbb{Z}_+^n(m)\right) \asymp \frac{1}{m+1} \exp\left\{-\sum_{m<j\le n} \frac{d_j}{j}\right\} \asymp \frac{1}{m+1}\left(\frac{m+1}{n}\right)^{\theta_0}$$

for $0 \le m \le n$. For $m = n$, this gives $P_n \asymp n^{-1}$ and, further, relation (ii) with $\theta = \min\{1, \theta_0\}$ as well.

The theorem is proved.

The following corollaries of Theorem 2 are of independent interest.

Corollary 1 *Let $(\mathbb{G}, +)$ be an abelian group and $h\colon \mathcal{A}_n \to \mathbb{G}$ be an additive function. Uniformly in $A \subset \mathbb{G}$,*

$$\mu_n\left(h(\sigma) \notin A + A - A\right) \ll P^\theta\left(\sum_{j\le n} h_j(\xi_j) \notin A\right) + \mathbf{1}\{\theta < 1\}n^{-\theta}.$$

Proof Apply Theorem 2 for

$$U = \left\{\bar{t} \in \mathbb{Z}_+^n :\ H(\bar{t}) \in A\right\},$$

where $H(\bar{t}) := \sum_{j\le n} h_j(t_j)$, $\bar{t} = (t_1, \ldots, t_n)$, and check that

$$V(U) \subset \left\{\bar{s} \in \mathbb{Z}_+^n :\ H(\bar{s}) \in A + A - A\right\}.$$

Now

$$\begin{aligned} \mu_n\left(h(\sigma) \notin A + A - A\right) &= P\left(H(\bar{\xi}) \notin A + A - A \mid \ell(\bar{\xi}) = n\right) \\ &\le P\left(\bar{\xi} \notin V(U) \mid \ell(\bar{\xi}) = n\right) \\ &\ll P^\theta\left(\bar{\xi} \notin U\right) + \mathbf{1}\{\theta < 1\}n^{-\theta} \\ &= P^\theta\left(H(\bar{\xi}) \notin A\right) + \mathbf{1}\{\theta < 1\}n^{-\theta}. \end{aligned}$$

Corollary 1 is proved.

Corollary 2 *Let* $h\colon \mathcal{A}_n \to \mathbb{R}$ *be an additive function. Uniformly in* $x \in \mathbb{R}$ *and* $y \geq 0$*,*

$$\mu_n\left(|h(\sigma) - x| \geq y\right) \ll P^{\theta}\left(\left|\sum_{j\leq n} h_j(\xi_j) - x\right| \geq y/3\right) + \mathbf{1}\{\theta < 1\}n^{-\theta}.$$

Proof Apply the previous corollary for $\mathbb{G} = \mathbb{R}$ with $A = \{t : \ |t - x| \leq y/3\}$.

Theorem 2, applied for the vector-valued additive function $(h(\sigma, 1), \dots, h(\sigma, n)) \in \mathbb{R}^n$, where

$$h(\sigma, m) := \sum_{j\leq m} h_j\left(k_j(\sigma)\right), \quad 1 \leq m \leq n,$$

leads to analogs of Kolmogorov's and Lévy's inequalities.

Corollary 3 *Uniformly in* $x(m) \in \mathbb{R}$, $1 \leq m \leq n$*, and* $y \geq 0$*,*

$$\mu_n\left(\max_{1\leq m\leq n} |h(\sigma, m) - x(m)| \geq y\right)$$
$$\ll P^{\theta}\left(\max_{1\leq m\leq n}\left|\sum_{j\leq m} h_j(\xi_j) - x(m)\right| \geq y/3\right) + \mathbf{1}\{\theta < 1\}n^{-\theta}.$$

Proof Apply Corollary 1 for $\mathbb{G} = \mathbb{R}^n$ and $A = \{\bar{t} \in \mathbb{R}^n : \ ||\bar{t} - \bar{x}|| \leq y/3\}$, where $\bar{x} \in \mathbb{R}^n$ and $||\cdot||$ is the maximum norm.

5 Proof of Theorem 3

The very idea goes back to [3,9], or even to some earlier author's number-theoretic papers.

Firstly, we observe that the measures $\mu_n \circ \widehat{H}_n^{-1}$ and $\mu_n \circ \widetilde{H}_n^{-1}$ can only converge simultaneously. Indeed, if $\delta > 0$ and $K > 2$ are arbitrary, then, by Corollary 3 of Theorem 3 and by virtue of $\lambda_j \leq \Theta/j$ and $B(n) \to \infty$, we have

$$\begin{aligned}
\mu_n(\delta) := \mu_n&\left(\sup_{0\leq t\leq 1}\left|\widehat{H}_n(\sigma, t) - \widetilde{H}_n(\sigma, t)\right| > \delta\right)\\
&\ll P^{\theta}\left(\exists j \leq K : \ \xi_j \geq K\right) + P^{\theta}\left(\exists j \geq K : \ \xi_j \geq 2\right)\\
&\quad + P^{\theta}\left(\sum_{j\leq K}\left(|h_j(\xi_j)| + |a_j|\xi_j\right) \geq \delta B(n)/3, \ \xi_j \leq K \ \forall\, j \leq K\right) + \mathrm{o}(1)\\
&\ll \left(\sum_{j\leq K}\sum_{k\geq K}\frac{\lambda_j^k}{k!}\right)^{\theta} + \left(\sum_{j\geq K}\sum_{k\geq 2}\frac{\lambda_j}{k!}\right)^{\theta} + \mathrm{o}_K(1)\\
&\ll K^{-\theta} + \mathrm{o}_K(1).
\end{aligned}$$

This yields that $\mu_n(\delta) = o(1)$ for each $\delta > 0$. Thus, from now on we may analyze the processes $\widehat{H}_n$ omitting the "hat".

Secondly, the r. vs $\hat{\xi}_{nj} := a_{nj}(\xi_j - \lambda_j)$, $1 \le j \le n$, are infinitesimal, therefore the equivalence of (I_2) and (I_3) is well known (see, for instance, [7], Section IX.3).

Proof of Sufficiency of (I_3) Let (I_3) be satisfied, $r := \varepsilon n$, $0 < \varepsilon < 1$, and

$$X_n^{(r)}(t) = \sum_{\substack{j \le r \\ \tau_{nj} \le t}} \hat{\xi}_{nj}, \quad H_n^{(r)}(\sigma, t) = \sum_{\substack{j \le r \\ \tau_{nj} \le t}} a_{nj}\left(k_j(\sigma) - \lambda_j\right).$$

Using the conditions of Theorem 2, for every δ, we obtain

$$\begin{aligned}
P_n(\delta) &:= P\left(\sup_{0 \le t \le 1}\left|X_n(t) - X_n^{(r)}(t)\right| \ge \delta\right) = P\left(\sup_{\tau_{nr} \le t \le 1}\left|\sum_{\substack{r < j \le n \\ \tau_{nj} \le t}} \hat{\xi}_{nj}\right| \ge \delta\right) \\
&\le P\left(\max_{r \le k \le n}\left|\sum_{r < j \le k} \hat{\xi}_{nj}\right| \ge \delta\right) \\
&\le P\left(\sum_{r < j \le n} |\hat{\xi}_{nj}| \ge \delta\right) \le \frac{1}{\delta^2}\sum_{r \le j \le n} a_{nj}^2 \lambda_j \\
&\le \frac{C\varepsilon^2}{\delta^2}\sum_{r \le j \le n}\frac{1}{j} + \frac{1}{\delta^2}\sum_{\substack{1 \le j \le n \\ |a_{nj}| \ge \varepsilon}} a_{nj}^2 \lambda_j \\
&\ll \frac{\varepsilon^2}{\delta^2}\log\frac{1}{\varepsilon} + o(1).
\end{aligned}$$

This shows that there exists a sequence $\varepsilon = \varepsilon_n = o(1)$ such that $P_n(\delta) = o(1)$ for every δ. Consequently, it follows from (I_3) that the process $X_n^{(r)}$ for $r = \varepsilon_n n$ also weakly converges to the standard Brownian motion. Moreover, by Corollary 1 of Theorem 2,

$$\mu_n\left(\sup_{0 \le t \le 1}\left|H_n(\sigma, t) - H_n^{(r)}(\sigma, t)\right| \ge \delta\right) \ll P_n^{\theta}(\delta/3) = o(1). \tag{22}$$

By Theorem 1, the distributions of $X_n^{(r)}$ and $H_n^{(r)}$ asymptotically are the same, using (22), we obtain the convergence $\mu_n \circ H_n^{-1} \Rightarrow W$.

Sufficiency of (I_3) is proved.

To prove that (I_1) implies (I_3), we need an extra lemma. For $b_j \in \mathbb{C}$, $1 \le j \le n$, we define a completely multiplicative function $\mathcal{A}_n \to \mathbb{C}$ by setting

$$f(\sigma) = \prod_{j=1}^{n} b_j^{k_j(\sigma)}, \quad 0^0 := 1.$$

Let $\mathbf{E}_n f(\sigma)$ be its mean-value with respect to μ_n, $\mathbf{E}$ stand for the expectation with respect to P, $D(z)$, and D_n be defined previously. Afterwards the conditions of Theorem 2 are assumed.

Lemma 9 *Let $J \subset (n/2, n]$ and $b_j \in \mathbb{C}$, $|b_j| \le 1$, for $1 \le j \le n$. If $b_j = 1$ for all but $j \in J$, then for the multiplicative function defined above, we have*

$$\mathbf{E}_n f(\sigma) = 1 + \sum_{j \in J} (b_j - 1)\lambda_j \frac{D_{n-j}}{D_n}. \tag{23}$$

Moreover, if condition (II) *in the Definition is satisfied, $J \subset ((1-\delta)n, n]$, where $\delta > 0$ is sufficiently small, then*

$$|\mathbf{E}_n f(\sigma)| \ge 1/2$$

provided that n is sufficiently large.

Proof We can use the Conditioning Relation and (13) to obtain

$$\begin{aligned}
\mathbf{E}_n f(\sigma) &:= \mathbf{E}\left(\prod_{j=1}^{n} b_j^{\xi_j} \mid \ell(\bar{\xi}) = n \right) \\
&= \frac{1}{P\left(\ell(\bar{\xi}) = n\right)} \sum_{\ell(\bar{s})=n} \prod_{j=1}^{n} b_j^{s_j} P(\bar{\xi} = \bar{s}) \\
&= \frac{1}{D_n} \sum_{\ell(\bar{s})=n} \prod_{j=1}^{n} (b_j\lambda_j)^{s_j} \frac{1}{s_j!} \\
&= \frac{1}{D_n} [z^n] \exp\left\{ \sum_{j=1}^{\infty} b_j \lambda_j z^j \right\} \\
&= \frac{1}{D_n} [z^n] \left(D(z) \exp\left\{ \sum_{j \in J} (b_j - 1)\lambda_j z^j \right\} \right).
\end{aligned}$$

Expanding the exponential function we easily find the relation (23). From it, recalling (18), we obtain

$$|\mathbf{E}_n f(\sigma) - 1| \le \frac{2\Theta}{D_n} \sum_{j \in J} \frac{D_{n-j}}{j} \le \frac{4\Theta}{nD_n} \sum_{k \le \delta n} D_k \le \frac{4\Theta}{\theta_1(\delta)} \le 1/2$$

provided that δ is sufficiently small and n is sufficiently large.

The lemma is proved.

Proof that (I_1) *implies* (I_3) Let $\mu_n \circ H_n^{-1} \Rightarrow W$. Then for each $0 \le t < 1$, the difference $H_n(\sigma, 1) - H_n(\sigma, t)$ converges weakly to the normal distribution with zero mean and variance $1-t$. Let $\psi_n(u,t)$, $u \in \mathbb{R}$, denote the characteristic function of $H_n(\sigma,1)-H_n(\sigma,t)$. Then

$$\psi_n(u,t) = \mathbf{E}_n \exp\left\{iu \sum_{t<\tau_{nj}\le 1} a_{nj}\left(k_j(\sigma)-\lambda_j\right)\right\}$$
$$= \exp\left\{-\frac{u^2}{2(1-t)}\right\} + \mathrm{o}(1) \tag{24}$$

uniformly in $|u| \le T$ for each $T>0$.

Define $b(j)=\exp\{iua_{nj}\}$, $u\in\mathbb{R}$, if $t<\tau_{nj}\le 1$ and $b(j)=1$ elsewhere. For the completely multiplicative function f defined via such $b(j)$, we will apply Lemma 9. Let $0<\delta<1/2$ be as in the lemma, $m=[(1-\delta)n]$, and $\tau_n=\tau_{nm}$. Observe that $\tau_n\to 1$. Indeed, if $\tau_n\to t_0<t_1<1$ for some subsequence $n:=n'\to\infty$, then by the lemma $|\psi_n(u,t_1)|>1/2$ uniformly in $u\in\mathbb{R}$. This contradicts to (24).

Now it follows that

$$1+\mathrm{o}(1)\le \tau_n \le \frac{B^2\,(m+1)}{B^2(n)} \le \frac{B^2\,((1-\delta))n+1)}{B^2(n)} \le 1.$$

Hence $B(un)\sim B(n)$ for each $u\in[(1-(\delta/2))n,n]$ and some $\delta>0$. Substituting $(1-(\delta/2))n$ for n repeatedly, we deduce the existence of $r=r(n)\to\infty$ such that $r=o(n)$ and $B(r)\sim B(n)$. Now repeating the arguments of the proof of the sufficiency part we obtain (22) and

$$\mu_n\left(H_n^r(\sigma,1)<x\right)=\Phi(x)+\mathrm{o}(1),$$

where Φ denotes the standard normal distribution function. This, by virtue of Theorem 1 and (22), leads to the central limit theorem

$$P\left(X_n(1)<x\right)=\Phi(x)+\mathrm{o}(1).$$

Since the r. vs $\{\hat{\xi}_{nj},\, 1\le j\le n\}$ form an infinitesimal array, we can apply the necessity part of the Lindeberg-Feller theorem. By it, the last relation implies (I_3).

Theorem 3 is proved.

In the forthcoming paper, we intend to explore the weak convergence of processes defined via sequences of additive functions on $\mathcal{A}_n$. Moreover, we will present generalizations of Theorem 3 to the more general case when the limiting process is an arbitrary stochastically continuous and has independent increments. Such attempts for permutations have been made in [4,10].

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