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Mappings on Decomposable Combinatorial Structures: Analytic Approach

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On the class of labelled combinatorial structures called assemblies we define complex-valued multiplicative functions and examine their asymptotic mean values. The problem reduces to the investigation of quotients of the Taylor coefficients of exponential generating series having Euler products. Our approach, originating in probabilistic number theory, requires information on the generating functions only in the convergence disc and rather weak smoothness on the circumference. The results could be applied to studying the asymptotic value distribution of decomposable mappings defined on assemblies.

1. Multiplicative functions on assemblies

We examine mappings defined on labelled combinatorial structures which, in [1], [2], and some other recent papers, are called *assemblies*. Let us recall their definition. Suppose an N-set σ of labelled points is partitioned into subsets so that, among them, there are k_j of size j, $1 \leqslant j \leqslant N$, with $1k_1 + \cdots + Nk_N = N$. In each such subset of size j, independent of the choice of elements, let a structure be defined. Let the number of different structures that can be defined on a subset of size j be m_j, where $1 \leqslant m_j < \infty$. From now on a subset with a given structure is called a *component* of σ. Suppose the number m_j does not depend on the possibility of other subsets forming components. The set σ with a fixed component structure satisfying the aforementioned properties is called an assembly. For various decompositions of the N set into subsets, using the same rule to define their structure, we get the class $\mathscr{S} = \{\sigma\}$ of assemblies of size N. To enumerate them, we observe that there are

$$N!\prod_{j=1}^{N}\left(\frac{1}{j!}\right)^{k_j}\frac{1}{k_j!}$$

ways to partition an N set into subsets of the specified sizes as given above. Thus, for each vector $\bar{k} = (k_1, \ldots, k_N)$ with nonnegative integer coordinates satisfying the equality

$L(\bar{k}) := 1k_1 + \cdots + Nk_N = N$, we get

$$Q_N(\bar{k}) := N! \prod_{j=1}^{N} \left(\frac{m_j}{j!}\right)^{k_j} \frac{1}{k_j!} \tag{1.1}$$

different assemblies. The total number of assemblies in this class equals

$$p(N) := \sum_{L(\bar{k})=N} Q_N(\bar{k}).$$

As can be verified directly, its exponential generating function satisfies the formal relation

$$Z(z) := 1 + \sum_{N\geqslant 1} \frac{p(N)}{N!} z^N = \prod_{j\geqslant 1} \exp\left\{\frac{m_j}{j!} z^j\right\}, \quad z \in \mathbf{C}. \tag{1.2}$$

The function $Z(z)$ characterizes the class of assemblies and, notably, it has the product expression (1.2). This is similar to the Riemann zeta function with its Euler product, very significant in number-theoretical investigations, and leads to the possibility that combinatorial problems analogous to those of analytic and probabilistic number theory might successfully be considered.

Examples of the classes of assemblies include the set $\mathbf{S}_N$ of permutations acting on N letters for which the components are cycles. Thus, $p(N) = N!$, $m_j = (j-1)!$, and $Z(z) = (1-z)^{-1}$ in this case. The next interesting example is the class consisting of all mappings of an N set into itself with the components defined by the connected components of the related functional graphs. Now $p(N) = N^N$,

$$m_j = (j-1)! \sum_{s=0}^{j-1} \frac{j^s}{s!},$$

and $Z(z) = (1-t(z))^{-1}$, where $t(z)$ is defined in $|z| < e^{-1}$ by the relation $t(z) = ze^{t(z)}$. For these and more sophisticated examples see [1, 2].

We are concerned with value distribution of *decomposable* mappings defined on a class of assemblies. Their values are directly related to component structure. If $\sigma \in \mathscr{S}$ and has $k_j = k_j(\sigma)$ components of size j, $1k_1 + \cdots + Nk_N = N$, we also write $\sigma \in \bar{k}$. We call a function $f : \mathscr{S} \to \mathbf{C}$ *multiplicative* if there exists a double sequence of complex numbers $f_j(k)$ with $f_j(0) = 1$ such that

$$f(\sigma) = \prod_{j=1}^{N} f_j(k_j) \tag{1.3}$$

for each $\sigma \in \bar{k}$. If, in addition, $f_j(k) = f_j(1)^k$ for any $k \geqslant 0$ and $j \geqslant 1$, the function f will be called *completely multiplicative*. Apart from the uniform probability measure defined on the set $\mathscr{S}$ of assemblies of size N, there are other important possibilities. The best-known instance is the Ewens sampling formula ascribing the probability

$$\theta^{w(\sigma)}/\theta(\theta+1)\cdots(\theta+N-1), \qquad w(\sigma) = k_1 + \cdots + k_N, \qquad \theta > 0,$$

for each permutation $\sigma \in \mathbf{S}_N$ (see [2] for the further comments). In general we can use other weighted frequencies for probability measures on a class of assemblies. In what follows, let ν_N^q denote the probability measure ascribing probability

$$q(\sigma)\left(\sum_{\sigma\in\mathscr{S}} q(\sigma)\right)^{-1}$$

to each $\sigma \in \mathscr{S}$. Here q is a positive multiplicative function on $\mathscr{S}$. Set ν_N for ν_N^q if $q(\sigma) \equiv 1$.

Being influenced by probabilistic number theory, we are interested in the asymptotic behaviour of the mean values

$$M_N^q(f) := \left(\sum_{\sigma\in\mathscr{S}} q(\sigma)\right)^{-1} \sum_{\sigma\in\mathscr{S}} q(\sigma)f(\sigma),$$

where $f : \mathscr{S} \to \mathbf{C}$ is a multiplicative function. Here and in what follows we mean that $N \to \infty$. Using generating functions we reduce the problem to investigation of the asymptotic behaviour of the complex Nth Taylor coefficient of an analytic function $F(z)$ in $|z| < 1$ having a singularity at $z = 1$. The transfer method cultivated by Flajolet and Odlyzko [4, 5] has proved to be useful in many similar situations (see, for instance, [13]). Nevertheless, it requires an analytic continuation for $F(z)$ into the region

$$\{z \in \mathbf{C} : |z| \leqslant 1 + \eta_1, |\mathrm{Arg}(z-1)| \geqslant \eta_2\}, \quad \eta_1 > 0, 0 < \eta_2 < \pi/2.$$

This is a serious obstacle if we deal with generating functions of general mappings on $\mathscr{S}$. It is well known (see [13, p. 1127] or [16, p. 451]) that the function

$$\prod_{j\geqslant 1} \left(1 + z^j/j\right),$$

having the natural boundary $|z| = 1$ and representing the exponential generating function of the number $\widehat{p}(N)$ of permutations with distinct cycle lengths, is amenable to Darboux's and transfer methods. Usually, the Hardy–Littlewood–Karamata Tauberian theorem is applied in this case. It yields an asymptotic formula for the summation function of the Taylor coefficients. Later some 'bootstrapping' (see [4, p. 235]) can be done. In this way Greene and Knuth [6] prove that

$$\widehat{p}(N)/N! = e^{-\gamma}(1 + N^{-1}) + O(N^{-2}\log N),$$

where γ denotes the Euler constant. Applied to this particular problem, the approach proposed in our paper gives the main term and a weak estimate of the remainder term. To obtain the formula above, the same bootstrapping is also needed. Nevertheless, our method gains in generality in giving an asymptotic formula for the Nth Taylor coefficient. To show this, at the end of the introduction we will treat the function

$$A(z) := \prod_{j\geqslant 1} \left(1 + a_j z^j/j\right)$$

with a bounded complex sequence a_j. Characteristically, our approach does not require information about functions outside the convergence disc, though some smoothness on the circumference is needed. In this regard, we have to mention the work [3] dealing with more specialized functions. We proceed in developing the ideas of our papers [8], [10],

[11] and [12]. Then we add some refinements originating in probabilistic number theory (see [9]). The main target is an extension of the following result.

Theorem A. **([12])** *Let f be a complex-valued multiplicative function defined by* (1.3) *on the symmetric group* $\mathbf{S}_N$. *Suppose that*

$$|f_j(1)| \leqslant \tau, \quad \tau > 1/2,$$

and, for some $\varepsilon > 0$,

$$\sum_{j\geqslant 1}\sum_{k\geqslant 2}\frac{|f_j(k)|(1+\varepsilon)^{jk}}{k!j^k} < \infty.$$

Then there exist constants $B \in \mathbf{C}$, $t_0 \in (-\pi, \pi]$, *and a function* $L : [1, \infty) \to \mathbf{C}$, $|L(u)| = 1$, $L(uv)/L(u) = 1 + o(1)$ *uniformly in* $u \leqslant v \leqslant 2u$ *as* $u \to \infty$, *such that*

$$M_N(f) = \frac{1}{N!}\sum_{\sigma\in\mathbf{S}_N} f(\sigma) = BL(N)N^{\tau-1}e^{it_0N} + o(N^{\tau-1}). \tag{1.4}$$

Here $B = 0$ is possible as well. The approach used in its proof did not allow us to avoid the condition $\tau > 1/2$. The same difficulties would arise even in the case $\tau = 1$ if, instead of the uniform probability measure ν_N on $\mathbf{S}_N$, we were to use the Ewens sampling formula with parameter $\theta \leqslant 1/2$. The condition of Theorem A with the double sum could be relaxed; it could even be omitted for completely multiplicative functions.

Consider the exponential generating function for the sequence of sums of values of a multiplicative function $g : \mathscr{S} \to \mathbf{C}$. We have, by (1.1) and (1.3),

$$\begin{aligned} G(z;g) &:= 1 + \sum_{N\geqslant 1}\Bigg(\sum_{\sigma\in\mathscr{S}} g(\sigma)\Bigg)\frac{z^N}{N!} = 1 + \sum_{N\geqslant 1}\Bigg(\sum_{L(\bar{k})=N} Q_N(\bar{k})\prod_{j=1}^{N} g_j(k_j)\Bigg)\frac{z^N}{N!} \\ &= \sum_{N\geqslant 0}\Bigg(\sum_{L(\bar{k})=N}\prod_{j=1}^{N}\left(\frac{m_j}{j!}\right)^{k_j}\frac{g_j(k_j)}{k_j!}\Bigg)z^N \\ &= \prod_{j\geqslant 1}\left(1 + \frac{m_jg_j(1)}{j!1!}z^j + \frac{m_j^2g_j(2)}{j!^22!}z^{2j} + \cdots\right) \\ &= \exp\Bigg\{\sum_{j\geqslant 1}\frac{m_jg_j(1)}{j!}z^j\Bigg\}H(z;g), \end{aligned} \tag{1.5}$$

where

$$H(z;g) = \prod_{j\geqslant 1}\left(1 + \frac{m_jg_j(1)}{j!1!}z^j + \frac{m_j^2g_j(2)}{j!^22!}z^{2j}\cdots\right)\exp\Bigg\{-\frac{m_jg_j(1)}{j!}z^j\Bigg\}.$$

We now see that the weighted mean value M_N^q is, in fact, the quotient of the Nth Taylor coefficient of $G(z;qf)$ and that of $G(z;q)$. Each of these generating series has a very particular shape (1.5).

As has been noticed in [1] and [2], for many classes of assemblies there exists a fixed parameter $x > 0$ such that

$$0 < \lambda^-/j \leqslant m_j x^j/j! =: \lambda_j/j \leqslant \lambda^+/j < \infty, \qquad j \geqslant 1. \tag{1.6}$$

Thus, under this condition, the change of arguments $z = sx$ of the functions $Z(z)$ and $G(z;g)$ also unifies the treatment of various classes of assemblies. So, in the instances above, one may take $x = 1$ and $x = e^{-1}$ for permutations and mappings, respectively. This yields a singularity at $s = 1$ for the function $Z(sx)$ in either of the cases.

The asymptotic behaviour of quotients of the Nth Taylor coefficients is examined in the second and third self-contained sections of the paper. The propositions proved there allow us to extend Theorem A. The first of them implies the following result.

Theorem 1.1. *Let the class of assemblies $\mathscr{S}$ satisfy condition* (1.6) *and let $q : \mathscr{S} \to \mathbf{R}^+$ be a completely multiplicative function. Assume that, for a multiplicative function $f : \mathscr{S} \to \mathbf{C}$, the above-defined $H(xz;f)$ is analytic in $|z| < 1$ and continuously differentiable on $|z| = 1$ function. Further, let*

$$0 < q^- \leqslant q_j(1) =: q_j \leqslant q^+ < \infty \tag{1.7}$$

and

$$|f_j(1)| \leqslant q_j \tag{1.8}$$

for each $j \geqslant 1$.

If there exists $t_0 \in (-\pi, \pi]$ such that the series

$$\sum_{j\geqslant 1} \frac{q_j - \Re(f_j(1)e^{-itj})}{j} \tag{1.9}$$

converges for $t = t_0$, then

$$\sum_{\sigma\in\mathscr{S}} f(\sigma) = \left(\exp\left\{ it_0N + \sum_{j\leqslant N} \frac{\lambda_j(f_j(1)e^{-it_0j} - q_j)}{j} \right\} H(xe^{-it_0};f) + o(1) \right) \sum_{\sigma\in\mathscr{S}} q(\sigma).$$

If the series (1.9) *diverges for each $t \in (-\pi, \pi]$, then*

$$\sum_{\sigma\in\mathscr{S}} f(\sigma) = o\left(\sum_{\sigma\in\mathscr{S}} q(\sigma) \right).$$

Theorem 1.1 allows us to reduce the problem of the summatory function of a complex-valued function f to the simpler problem of a positive completely multiplicative function q. In fact, we have to know *a priori* the asymptotic behaviour of the sum of values $q(\sigma)$ over $\sigma \in \mathscr{S}$. The case of $q(\sigma) \equiv 1$ is trivial. For the weight in the Ewens sampling formula, one can use the formula

$$\frac{1}{N!} \sum_{\sigma\in\mathbf{S}_N} \theta^{w(\sigma)} = \frac{N^{\theta-1}}{\Gamma(\theta)} + O(N^{\theta-2}), \tag{1.10}$$

easily obtainable via the transfer method [4]. Here $\Gamma(z)$ denotes the Euler gamma function.

Proposition 2.4 below indicates how to relax the condition on $H(z;f)$.

Theorem 1.2. *Let the class of assemblies $\mathscr{S}$ and the completely multiplicative weight function $q : \mathscr{S} \to \mathbf{R}^+$ satisfy* (1.6), (1.7), *and*

$$\sum_{n<j\leqslant N} \frac{\lambda_j q_j}{j} \geqslant \log \frac{N}{n} + O(1),$$

where $1 \leqslant n \leqslant N$. For a multiplicative function $f : \mathscr{S} \to \mathbf{C}$ satisfying (1.8) *and*

$$\sum_{j\geqslant 1} \sum_{k\geqslant 2} \frac{\lambda^{+^k} |f_j(k)|}{j^k k!} < \infty,$$

both assertions of Theorem 1.1 are true.

The following corollary corresponds to the Delange–Halász result (see [15, Chapter III.4]) on multiplicative number-theoretical functions.

Corollary 1.3. *Let $f : \mathscr{S} \to \mathbf{C}$ be a multiplicative function, $|f(\sigma)| \leqslant 1$ for all $\sigma \in \mathscr{S}$. Assume that the class of assemblies $\mathscr{S}$ and the weight function q satisfy the conditions of Theorem 1.2.*

If the series

$$\sum_{j\geqslant 1} \frac{1 - \Re(f_j(1)e^{-itj})}{j} \tag{1.11}$$

converges for some $t = t_0 \in (-\pi, \pi]$, then

$$M_N^q(f) = \exp\left\{ it_0 N + \sum_{j\leqslant N} \frac{\lambda_j q_j (f_j(1)e^{-it_0 j} - 1)}{j} \right\} H(xe^{-it_0}; fq) + o(1).$$

If the series (1.11) *diverges for each $t \in (-\pi, \pi]$, then*

$$M_N^q(f) = o(1).$$

Remark. If the function f depends on some parameter $u \in I \subset \mathbf{R}$ and the series (1.11) converges for $t = t_0(u)$ uniformly in $u \in I$, then this uniformity is preserved in the remainder term estimate $o(1)$ of $M_N^q(f)$. This is also true for the second assertion of Corollary 1.3.

Corollary 1.4. *Suppose a multiplicative function f defined on the symmetric group $\mathbf{S}_N$ satisfies the conditions of Theorem A with arbitrary $\tau > 0$ and $\varepsilon = 0$; then relation* (1.4) *holds.*

Let us return to the function $A(z)$ defined at the beginning of our discussion. By (1.5), it represents the generating function of $M_N(\hat{f})$, where $\hat{f} : \mathbf{S}_N \to \mathbf{C}$ is multiplicative and defined by (1.3) via $\hat{f}_j(1) = a_j$ and $\hat{f}_j(k) = 0$ for $k \geqslant 2$, $j \geqslant 1$. Thus, from Corollary 1.4 we see that asymptotic formula (1.4) still holds for $M_N(\hat{f})$. Paying more attention to the quantities appearing on the right-hand side of (1.4), for this particular case, we obtain the following.

Corollary 1.5. *Suppose $|a_j| \leqslant \tau$ for all $j \geqslant 1$ and $0 < \tau < \infty$. If the series*

$$B(t) := \sum_{j \geqslant 1} \frac{\tau - \Re(a_j e^{-itj})}{j}$$

converges for some $t = t_0 \in (-\pi, \pi]$, then (1.4) *holds with*

$$B = \frac{e^{-B(t_0)}}{\Gamma(\tau)} \prod_{j \geqslant 1} \left(\left(1 + \frac{a_j e^{it_0 j}}{j} \right) \exp \left\{ - \frac{a_j e^{-it_0 j}}{j} \right\} \right)$$

and

$$L(N) = \exp \left\{ i \sum_{j \leqslant N} \frac{\Im(a_j e^{-it_0 j})}{j} \right\}.$$

If the series $B(t)$ diverges for all $t \in (-\pi, \pi]$, then $B = 0$.

The individual conditions (1.7) and (1.8) are rather restrictive. We guess that our approach still works under some averaged requirements. Maybe we could use

$$\sum_{j \leqslant N} q_j \sim \delta N, \quad \delta > 0, \qquad \sum_{j \leqslant N} |f_j(1)| \leqslant \sum_{j \leqslant N} q_j + O(1).$$

This would lead to generalizations of Pavlov's [14] and a few other results.

2. Quotients of the Taylor coefficients

We now explore the asymptotic behaviour, as $N \to \infty$, of the Taylor coefficients m_N of a function analytic in $|z| < 1$, such that

$$\begin{aligned} F(z) &:= \sum_{N \geqslant 0} m_N z^N = \sum_{k \geqslant 0} b_k z^k \exp \left\{ \sum_{j \geqslant 1} \frac{a_j z^j}{j} \right\} \\ &=: H(z) \exp\{U(z)\} =: H(z) G(z) \end{aligned} \tag{2.1}$$

if $a_j, b_k \in \mathbf{C}$. Different decompositions of $F(z)$ will not affect the asymptotic behaviour of m_N. Since a_j, with $j > N$, does not affect m_N, without loss of generality we may take them equal to zero. After this change, if there is any ambiguity, we let $U(z)$ have the index N. Thus the analytic continuability of $U(z)$ does not involve difficulties. The growth of their modulus values near the circle $|z| = 1$ as $N \to \infty$ is much more significant. Dealing with $H(z)$, which can also be dependent on some parameters, at first we will assume its differentiability on $|z| = 1$ and use the following bound:

$$\sup_{|z|=1} |H'(z)| \leqslant H < \infty. \tag{2.2}$$

In what follows we will assume that

$$|a_j| \leqslant d_j, \qquad 0 < \theta^- \leqslant d_j \leqslant \theta^+ < \infty \tag{2.3}$$

for any $1 \leqslant j \leqslant N$ with some constants θ^- and θ^+. The main purpose of this section is to find the asymptotic behaviour of the quotient $m_N/\tilde{m}_N$, where $\tilde{m}_N$ is defined by

$$D(z) = \sum_{N\geqslant 0} \tilde{m}_N z^N = \exp\left\{\sum_{j\geqslant 1} \frac{d_j z^j}{j}\right\} =: \exp\{V(z)\}.$$

Here also, for convenience, we may assume that $d_j = 0$ for each $j > N$ and set, if necessary, $V_N(z)$ for $V(z)$. Let us stress once more that we allow the dependence on N or on other parameters of the coefficients a_j, d_j for $1 \leqslant j \leqslant N$. Bearing that in mind, we introduce the class $\mathscr{A}$ of sequences $\{a_j\}$, $j \leqslant N$, characterized by the following two conditions:

$$\sum_{j\leqslant N} \frac{d_j - \Re(a_j e^{-it_0 j})}{j} \leqslant D < \infty, \tag{2.4}$$

$$\frac{1}{N}\sum_{j\leqslant N} |d_j - a_j e^{-it_0 j}| \leqslant \mu_N = o(1) \tag{2.5}$$

with some $t_0 \in (-\pi, \pi]$ uniformly for $\{a_j\} \in \mathscr{A}$. For brevity, let $a \wedge b := \min\{a, b\}$ and $\ll$ be the analogue of the symbol $O(\ldots)$.

Proposition 2.1. *Let $F(z)$ be given in* (2.1) *with $H(z)$ satisfying* (2.2). *Assume that $\{a_j\} \in \mathscr{A}$ satisfies condition* (2.3). *Then*

$$m_N/\tilde{m}_N = \exp\left\{it_0 N + U_N(e^{-it_0}) - V_N(1)\right\} H(e^{-it_0}) + O(R_N),$$

where

$$R_N \ll \mu_N^{c_1} + N^{-c_2}$$

with some positive quantities $c_1 = c_1(\theta^-, \theta^+)$ and $c_2 = c_2(\theta^-, \theta^+)$. The constant in the symbol $\ll$ depends only on H, D, θ^-, and θ^+.

It follows from our proof of Proposition 2.1 that

$$R_N \ll T\delta^{-\alpha\theta^+}\left(\mu_N(T + \delta^{-1}) + TN^{-1}\right) + \delta^{\alpha\theta^-} + \delta^{-1}T^{-(1-\alpha)\theta^-/2}$$

provided $N^{-1} \leqslant \delta < 1$, $0 < \alpha < 1$, $1 \leqslant T \leqslant N$, and $\mu_N \leqslant \delta \wedge T^{-1}$. An analysis of this complicated estimate shows that numerical values of c_1 and c_2 in particular cases can also be obtained.

What happens when the sum (2.4) is unbounded for all t_0? The answer is given by the following result.

Proposition 2.2. *Suppose conditions* (2.2) *and* (2.3) *are satisfied. Then there exists a positive constant $c_3 = c_3(\theta^-, \theta^+)$ such that*

$$\frac{m_N}{\tilde{m}_N} \ll \exp\left\{ -c_3 \min_{|t|\leqslant\pi} \sum_{j\leqslant N} \frac{d_j - \Re(a_j e^{-itj})}{j}\right\}.$$

The constant in $\ll$ depends only on H, θ^-, and θ^+.

If $d_j \equiv 1$ and $a_j \geqslant 0$ for all $j \geqslant 1$, this estimate also follows from [7, Theorem 2].

If the sequences $\{a_j\}$ and $\{d_j\}$ do not depend on N, condition (2.4) implies (2.5). In such a case, if $1 \leqslant x < N$ and

$$\sum_{j \geqslant x} \frac{d_j - \Re(a_j e^{-it_0 j})}{j} \leqslant r(x) = o(1)$$

as $x \to \infty$, then the quantity on the left-hand side of (2.5) can be estimated by

$$\begin{aligned} \frac{x}{N} + \sum_{x<j\leqslant N} \frac{|d_j - a_j e^{-it_0 j}|}{j} &\ll \frac{x}{N} + \left(\sum_{x<j\leqslant N} \frac{|d_j - a_j e^{-it_0 j}|^2}{j} \right)^{1/2} \left(\log \frac{N}{x} \right)^{1/2} \\ &\ll \frac{x}{N} + (r(x))^{1/2} \left(\log \frac{N}{x} \right)^{1/2}. \end{aligned}$$

By the choice of $x \asymp Nr(x)$, this enables us to find some $\mu_N = o(1)$ in (2.5). Moreover, if

$$\sum_{j \leqslant N} \frac{d_j - \Re(a_j e^{-itj})}{j} \to \infty \tag{2.6}$$

for any fixed $t \in [-\pi, \pi]$, then this holds uniformly in $t \in [-\pi, \pi]$. Thus we have the following.

Corollary 2.3. *Let the sequences $\{a_j\}$, $\{b_k\}$, and $\{d_j\}$ be independent of N and satisfy conditions* (2.2) *and* (2.3). *If* (2.4) *holds, then the assertion of Proposition 2.1 is true. If condition* (2.6) *is satisfied, then $m_N = o(\tilde{m}_N)$ as $N \to \infty$.* □

In the next result we pay more attention to the function $H(z)$. The bound for its derivative is sometimes too stringent. Instead of (2.2) it suffices to use the bound

$$\sum_{k \leqslant N} |b_k| \leqslant H_1 < \infty \tag{2.7}$$

together with one of the following conditions:

$$r_N(u) := \sum_{uN<k\leqslant N} |b_k| \leqslant N^{(\theta^- -1)\wedge 0} \rho_N(u) \tag{2.8}$$

or

$$r_N(u) \leqslant \rho_N(u), \qquad \sum_{n<k\leqslant N} \frac{d_j}{j} \geqslant \log \frac{N}{n} - C, \quad C > 0, \quad 1 \leqslant n \leqslant N \tag{2.9}$$

with some $\rho_N(u) = o(1)$ for any fixed $u \in (0,1)$. Set $\hat{\rho}(N) = \inf_{0<u<1}\{u + \rho_N(u)\}$.

Proposition 2.4. *Let $F(z)$ be given by* (2.1). *Suppose $\{a_j\} \in \mathscr{A}$ and conditions* (2.3), (2.7) *are satisfied. Then, if one of conditions* (2.8) *or* (2.9) *holds,*

$$m_N/\tilde{m}_N = \exp\left\{ it_0 N + U_N(e^{-it_0}) - V_N(1) \right\} H(e^{-it_0}) + O(\tilde{R}_N),$$

where $\tilde{R}_N = o(1)$. □

Now, the estimate of $\tilde{R}_N$ differs from that of R_N in Proposition 2.1 by the extra summand $\hat{\rho}(N)$. In this way we could relax the conditions on $H'(z)$ in Proposition 2.2.

3. Proofs of propositions

Proof of Proposition 2.1. We start the proof of Proposition 2.1 with an explanation of the idea due to Levin and Timofeev (see [8]). For simplicity, considering $a_j e^{-it_0}$ instead of a_j, we reduce conditions (2.4) and (2.5) to the case when $t_0 = 0$. Therefore, in what follows we may set $t_0 = 0$. Let $\Delta_1 = \{z = e^{it} : T/N \leqslant |t| \leqslant \pi\}$, $\Delta_0 = [-\pi, \pi] \setminus \Delta_1$, $K = \delta N \geqslant 1$, and let $0 < \delta, \alpha < 1$, $1 \leqslant T \leqslant N$ be some parameters to be chosen later. Denote

$$G_1(z) = \exp\left\{\alpha \sum_{j\leqslant K} \frac{a_j}{j} z^j\right\}, \quad G_2(z) = \exp\left\{-\alpha \sum_{K<j\leqslant N} \frac{a_j}{j} z^j\right\}, \quad G_3(z) = G^{\alpha}(z) - G_1(z).$$

Let $D_l(z)$ be the functions defined by $G_l(z)$, $l = 1, 2, 3$, but substituting a_j with d_j.

Recalling our agreement that $a_j = d_j = 0$ for $j > N$ and setting, for brevity, $U_N = U$, $V_N = V$, we use Cauchy's formula. We start from the following identity:

$$\begin{aligned} m_N &= \frac{1}{2\pi i N} \int_{|z|=1} \frac{F'(z)}{z^N}\, dz = \frac{1}{2\pi i N} \int_{\Delta_0} \frac{F'(z)(1 - G_2(z))}{z^N}\, dz \\ &\quad + \frac{1}{2\pi i N} \int_{\Delta_1} \frac{F'(z)(1 - G_2(z))}{z^N}\, dz + \frac{1}{2\pi i N} \int_{|z|=1} \frac{F'(z) G_2(z)}{z^N}\, dz \\ &=: J_0 + J_1 + J_2. \end{aligned} \tag{3.1}$$

We at first obtain the estimate $J_1 + J_2 \ll \tilde{m}_N R_N$ uniformly for $\{a_j\} \in \mathscr{A}$ and, further, change the integrand of J_0 to get

$$J_0 = \frac{H(1) \exp\{U(1) - V(1)\}}{2\pi i N} \int_{\Delta_0} D'(z)\big(1 - D_2(z)\big) \frac{dz}{z^N} + O(\tilde{m}_N R_N).$$

Since the sequence $\{d_j\}$ belongs to $\mathscr{A}$ itself and the integral in the last formula divided by $2\pi i N$ also gives the main asymptotic term of $\tilde{m}_N$, we obtain the desired approximation of the quotient $m_N/\tilde{m}_N$. Technical details are more complicated; some of them can be found in the author's paper [11].

The integrals in (3.1) should be estimated in terms of $\tilde{m}_N$; therefore we need to know its order expressed in more convenient quantities. Observe that the expression of $\tilde{m}_N$ is available by multiplication of the expansions of exponential functions in $D(z)$. In this way, we have

$$\tilde{m}_N = \sum_{L(\bar{k})=N} \prod_{j=1}^{N} \left(\frac{d_j}{j}\right)^{k_j} \frac{1}{k_j!}, \tag{3.2}$$

where, as above, the summation is extended over vectors $\bar{k} = (k_1, \ldots, k_N)$ with nonnegative integer coordinates and satisfying the relation $L(\bar{k}) = N$. Recall that

$$D(1) = \exp\left\{\sum_{j\leqslant N} \frac{d_j}{j}\right\}.$$

Lemma 3.1. *Let $\{d_j\}$ be the sequence satisfying* (2.3). *Then*

$$\theta^- c(\theta^+) D(1) N^{-1} \leqslant \tilde{m}_N \leqslant \theta^+ D(1) N^{-1} \tag{3.3}$$

for $N \geqslant 1$ with some positive constant $c(\theta^+)$.

Proof. Differentiating $D(z)$ and comparing the coefficients in the equality obtained, we derive the recurrence relation

$$\tilde{m}_N = \frac{1}{N} \sum_{n=0}^{N-1} d_{N-n} \tilde{m}_n, \quad M_0 = 1. \tag{3.4}$$

Further, we have from the expression of $D(z)$

$$\sum_{n=0}^{N} \tilde{m}_n \leqslant D(1).$$

By virtue of (2.3), this together with (3.4) implies the second inequality in (3.3).

To derive the left inequality in (3.3), we first obtain a lower estimate of the sum of $\tilde{m}_N$. We now use formula (3.2). Let $0 < \varepsilon < 1$ be arbitrary, $N_1 = [\varepsilon N] \geqslant 1$. We bound the region of summation in (3.2) to the vectors $\bar{k}$ with $k_j = 0$ for each $N_1 < j \leqslant N$. Thus

$$\begin{aligned}
\sum_{n=0}^{N-1} \tilde{m}_n &\geqslant \sum_{k_1,\dots,k_{N_1} \geqslant 0} \prod_{j=1}^{N_1} \left(\frac{d_j}{j}\right)^{k_j} \frac{1}{k_j!} - \sum_{\substack{L(\bar{k}) \geqslant N \\ k_{N_1+1},\dots,k_N = 0}} \prod_{j=1}^{N_1} \left(\frac{d_j}{j}\right)^{k_j} \frac{1}{k_j!} \\
&\geqslant \exp\left\{\sum_{j=1}^{N_1} \frac{d_j}{j}\right\} - e^{-\varepsilon^{-1}} \sum_{k_1,\dots,k_{N_1} \geqslant 0} \prod_{j=1}^{N_1} \left(\frac{d_j \exp\{N_1^{-1} j\}}{j}\right)^{k_j} \frac{1}{k_j!} \\
&= \exp\left\{\sum_{j=1}^{N_1} \frac{d_j}{j}\right\} - \exp\left\{\sum_{j=1}^{N_1} \frac{d_j \exp\{N_1^{-1} j\}}{j} - \varepsilon^{-1}\right\} \\
&\geqslant \exp\left\{\sum_{j=1}^{N_1} \frac{d_j}{j}\right\} \bigl(1 - \exp\{e\theta^+ - \varepsilon^{-1}\}\bigr).
\end{aligned}$$

The choice $\varepsilon^{-1} = e\theta^+ + \log 3$ now yields

$$\sum_{n=0}^{N-1} \tilde{m}_n \geqslant \frac{2}{3} \exp\left\{\sum_{j=1}^{N_1} \frac{d_j}{j}\right\} \geqslant c(\theta^+) D(1)$$

with some positive constant $c(\theta^+)$ when $N \geqslant N_0(\theta^+)$. Since the case of bounded N is trivial, this and (3.4) imply the desired estimate.

Lemma 3.1 is proved. □

We now start to estimate the integrals in (3.1). In what follows we indicate only the dependence on the newly introduced parameters, leaving aside those given by the conditions of Theorem 1.2.

Lemma 3.2. *We have*

$$J_2 \ll \tilde{m}_N \delta^{\alpha\theta^-}$$

if $N \geqslant 1/\delta$.

Proof. The estimation is based on the following equality:

$$F'(z)G_2(z) = H(z)G^{1-\alpha}(z)G_1(z)U'(z) + H'(z)G^{1-\alpha}(z)G_1(z).$$

Let

$$G^{\alpha}(z) =: \sum_{s\geqslant 0} g_s(\alpha)z^s, \qquad g_s(1) = g_s.$$

Applying the relations between $g_s(\alpha)$ and a_j and (2.3), we obtain

$$\sum_{s\leqslant N} |g_s(\alpha)| \leqslant \exp\left\{\alpha \sum_{j\leqslant N} \frac{|a_j|}{j}\right\} \leqslant \exp\left\{\alpha \sum_{j\leqslant N} \frac{d_j}{j}\right\}.$$

Similarly, if $\tilde{g}_s(\alpha)$ denotes the Taylor coefficient of $G_1(z)$, then

$$\sum_{s\leqslant N} |\tilde{g}_s(\alpha)| \leqslant \exp\left\{\alpha \sum_{j\leqslant K} \frac{d_j}{j}\right\}.$$

By condition (2.2) the function $H'(z)$ belongs to the Hardy class $\mathbf{H}_1$; thus the series of the coefficients $|b_k|$ is bounded by a constant depending on H only. Cauchy's formula, previous estimates, and Lemma 3.1 yield

$$\begin{aligned}
J_{21} &:= \frac{1}{2\pi i N}\int_{|z|=1} \frac{H(z)G^{1-\alpha}(z)G_1(z)U'(z)}{z^N}\,dz \\
&= \frac{1}{N} \sum_{\substack{k,n,s\geqslant 0\\ k+n+s\leqslant N-1}} b_k g_n(1-\alpha)\tilde{g}_s(\alpha)a_{N-s-k-n} \\
&\ll \frac{1}{N}\sum_{k\leqslant N}|b_k| \sum_{n\leqslant N}|g_n(1-\alpha)| \sum_{s\leqslant N}|\tilde{g}_s(\alpha)| \\
&\ll \frac{1}{N} D(1)\exp\left\{-\alpha \sum_{K<j\leqslant N} \frac{d_j}{j}\right\} \ll \tilde{m}_N\, \delta^{\alpha\theta^-} \qquad (3.5)
\end{aligned}$$

provided $N \geqslant 1/\delta$.

Further, using conditions (2.2), (2.3), and Lemma 3.1, we obtain

$$\begin{aligned}
J_{22} &:= \frac{1}{2\pi i N}\int_{|z|=1} \frac{H'(z)G^{1-\alpha}(z)G_1(z)}{z^N}\,dz \\
&\ll \frac{1}{N}\exp\left\{(1-\alpha)\sum_{j\leqslant N}\frac{|a_j|}{j} + \alpha\sum_{j\leqslant K}\frac{|a_j|}{j}\right\} \ll \tilde{m}_N\, \delta^{\alpha\theta^-}, \qquad (3.6)
\end{aligned}$$

if $N \geqslant 1/\delta$.

Since $J_2 = J_{21} + J_{22}$, Lemma 3.2 follows from (3.5) and (3.6). □

Lemma 3.3. *We have*

$$J_1 \ll \tilde{m}_N \delta^{-1} T^{-(1-\alpha)\theta^-/2}$$

if $1 \leqslant T \leqslant N$.

Proof. Recalling the previous notation and condition (2.2), we have

$$\begin{aligned} J_1 &= \frac{1}{2\pi i N} \int_{\Delta_1} \frac{F'(z)}{G(z)} G^{1-\alpha}(z) G_3(z) \frac{dz}{z^N} \\ &\ll \frac{1}{N} \max_{z\in\Delta_1} |G^{1-\alpha}(z)| \int_{|z|=1} (1 + |U'(z)|)|G_3(z)|\,|dz|. \end{aligned} \tag{3.7}$$

To estimate the quantity

$$\max_{z\in\Delta_1} \frac{|G(z)|}{D(1)} = \max_{T/N\leqslant|t|\leqslant\pi} \exp\left\{ \sum_{j\leqslant N} \frac{\Re(a_j e^{itj}) - d_j}{j} \right\},$$

we use (2.3), (2.4), and the following trigonometrical estimates:

$$\begin{aligned} |a| - \Re(ae^{iu}) - d(1-\cos u) &= -(d-|a|)(1-\cos u) + (|a| - \Re a)\cos u + \Im a \sin u \\ &\ll (d - \Re a) + |\Im a|\,|\sin u| \end{aligned}$$

for $|a| \leqslant d$ and $|u| \leqslant \pi$. Since the sum in (2.4) remains bounded if $\Re a_j$ are changed by $|a_j|$, using (2.3) we obtain

$$\begin{aligned} \max_{z\in\Delta_1} \frac{|G(z)|}{D(1)} \ll \max_{T/N\leqslant|t|\leqslant\pi} \exp\Biggl\{ \theta^- \sum_{j\leqslant N} \frac{\cos tj - 1}{j} \\ + O\left(\left(\sum_{j\leqslant N} \frac{d_j - \Re a_j}{j} \right)^{1/2} \left(\sum_{j\leqslant N} \frac{1-\cos tj}{j} \right)^{1/2} \right) \Biggr\} \ll T^{-\theta^-/2}. \end{aligned} \tag{3.8}$$

Here the calculations of the maximum were based on the relations

$$\begin{aligned} \sum_{j\leqslant N} \frac{1-\cos tj}{j} &= \sum_{j\geqslant 1} \frac{1-\cos tj}{j} e^{-j/N} + O(1) \\ &= \log \frac{|1 - e^{-1/N+it}|}{1 - e^{-1/N}} + O(1) = \frac{1}{2} \log\left(1 + \frac{2e^{-1/N}(1-\cos t)}{(1-e^{-1/N})^2} \right) + O(1). \end{aligned}$$

The next steps are based on Parseval's equality. We have

$$\int_{|z|=1} |U'(z)|^2\,|dz| \ll \int_{|z|=1} |V'(z)|^2\,|dz| \ll \sum_{j\leqslant N} d_j^2 \ll N. \tag{3.9}$$

Observe that, by virtue of the definition of $G_3(z)$,

$$G_3(z) = \sum_{n>K} \left(\sum_{L(\bar{k})=n}{}' \prod_{j=1}^{n} \left(\frac{\alpha a_j}{j} \right)^{k_j} \frac{1}{k_j!} \right) z^n.$$

where $'$ denotes that the summation is extended over $\bar{k} = (k_1, \ldots, k_n)$ such that there exists $j > K$ with $k_j \geqslant 1$. The modulo of the nth Taylor coefficient of $G_3(z)$ in brackets does not exceed the nth Taylor coefficient $q_n(\alpha)$ of the function $D^\alpha(z)$. Hence Parseval's equality and (3.9) imply

$$\int_{|z|=1} |G_3(z)|^2\,|dz| \ll \frac{1}{K^2}\sum_{n\geqslant 1} q_n^2(\alpha)n^2 \ll \frac{1}{K^2}\int_{|z|=1} \left|\big(D^\alpha(z)\big)'\right|^2 |dz|$$
$$\ll \frac{D^{2\alpha}(1)}{K^2}\int_{|z|=1} |V'(z)|^2\,|dz| \ll D^{2\alpha}(1)\delta^{-2}N^{-1}. \tag{3.10}$$

From (3.7), (3.8), (3.9), the last estimate, and Lemma 3.1, we obtain

$$J_1 \ll \frac{1}{N}\cdot\left(\frac{D(1)}{T^{\theta^-/2}}\right)^{1-\alpha}\cdot\sqrt{N}\cdot\frac{D^\alpha(1)}{\delta\sqrt{N}} \ll \tilde{m}_N\delta^{-1}T^{-(1-\alpha)\theta^-/2}.$$

Lemma 3.3 is proved. □

Lemma 3.4. *We have*

$$J_0 = \frac{H(1)\exp\big\{U(1)-V(1)\big\}}{2\pi iN}\int_{\Delta_0} D'(z)\big(1-D_2(z)\big)\frac{dz}{z^N}$$
$$+O\big(\tilde{m}_N T\delta^{-\alpha\theta^+}\big(\mu_N(T+\delta^{-1})+TN^{-1}\big)\big)$$

provided N is sufficiently large, $\mu_N \leqslant \delta\wedge T^{-1}$, $\delta N\geqslant 1$, and $1\leqslant T\leqslant N$.

Proof. Under the conditions of Proposition 2.1 we have

$$D_2(z)\ll\delta^{-\alpha\theta^+},\qquad G_2(z)=D_2(z)\big(1+O(\delta^{-1}\mu_N)\big)=D_2(z)+O\big(\mu_N\delta^{-1-\alpha\theta^+}\big) \tag{3.11}$$

if $\mu_N\leqslant\delta$ and

$$G(z)=D(z)\exp\{U(1)-V(1)\}\big(1+O(T\mu_N)\big)$$
$$=D(z)\exp\{U(1)-V(1)\}+O\big(TD(1)\mu_N\big)$$

if $T\mu_N\leqslant 1$. Observing, in addition, that $U'(z)=V'(z)+O(N\mu_N)$, we obtain

$$G'(z)(1-G_2(z))=\exp\{U(1)-V(1)\}D'(z)(1-D_2(z))$$
$$+O\big(ND(1)\mu_N\delta^{-\alpha\theta^+}(T+\delta^{-1})\big) \tag{3.12}$$

provided $\mu_N\leqslant\delta\wedge T^{-1}$. Hence, by virtue of Lemma 3.1,

$$J_{00}:=\frac{1}{2\pi iN}\int_{\Delta_0} G'(z)(1-G_2(z))\frac{dz}{z^N}$$
$$=\frac{\exp\{U(1)-V(1)\}}{2\pi iN}\int_{\Delta_0} D'(z)(1-D_2(z))\frac{dz}{z^N}$$
$$+O\big(\tilde{m}_N T\mu_N\delta^{-\alpha\theta^+}(T+\delta^{-1})\big).$$

The remaining parts of the integral J_0 give new error terms. Let

$$J_{01}:=\frac{1}{2\pi iN}\int_{\Delta_0} H'(z)G(z)(1-G_2(z))\frac{dz}{z^N}.$$

Conditions (2.2) and (2.3), (3.11), and Lemma 3.1 imply

$$J_{01}\ll D(1)TN^{-2}\delta^{-\alpha\theta^+}\ll\tilde{m}_N TN^{-1}\delta^{-\alpha\theta^+}.$$

Similarly,

$$J_{02} := \frac{1}{2\pi iN}\int_{\Delta_0}(H(z)-H(1))G(z)U'(z)(1-G_2(z))\,\frac{dz}{z^N}$$
$$\ll \tilde{m}_N T^2 N^{-1}\delta^{-\alpha\theta^+}.$$

Since

$$J_0 = H(1)J_{00} + J_{01} + J_{02},$$

the proposition of Lemma 3.4 follows from the above estimates. □

Lemma 3.5. *We have*

$$\frac{1}{2\pi iN\tilde{m}_N}\int_{\Delta_0} D'(z)\big(1-D_2(z)\big)\,\frac{dz}{z^N}$$
$$= 1 + O\big(T\delta^{-\alpha\theta^+}\mu_N\big(T+\delta^{-1}\big) + \delta^{\alpha\theta^-} + \delta^{-1}T^{-(1-\alpha)\theta^-/2}\big)$$

provided N *is sufficiently large,* $\mu_N \leqslant \delta \wedge T^{-1}$, $\delta N \geqslant 1$, *and* $1 \leqslant T \leqslant N$.

Proof. Since $\{d_j\} \in \mathscr{A}$, the estimates obtained in the previous lemmas hold for this sequence. In addition, in this case we may consider that $H(z) \equiv 1$ and $U(z) \equiv V(z)$. Lemma 3.5 is proved. □

Inserting the estimates obtained in Lemmas 3.2, 3.3, 3.4, and 3.5 into (3.1), we end the proof of Proposition 2.1. □

Proof of Proposition 2.2. Instead of (3.1), we now start with

$$m_N = \frac{1}{2\pi iN}\int_{|z|=1}\frac{F'(z)}{G(z)}G^{1-\alpha}(z)G_3(z)\,\frac{dz}{z^N} + \frac{1}{2\pi iN}\int_{|z|=1}\frac{F'(z)G_2(z)}{z^N}\,dz =: J + J_2.$$

The estimate for J_2 is given in Lemma 3.2. As in the proof of Lemma 3.3, we have by (3.9) and (3.10)

$$J \ll \frac{1}{N}\max_{|z|=1}|G^{1-\alpha}(z)|\int_{|z|=1}(1+|U'(z)|)|G_3(z)|\,|dz|$$
$$\ll \frac{D^{1-\alpha}(1)}{N}\exp\left\{(1-\alpha)\min_{|t|\leqslant\pi}\sum_{j\leqslant N}\frac{\Re(a_je^{itj})-d_j}{j}\right\}\int_{|z|=1}(1+|U'(z)|)|G_3(z)|\,|dz|$$
$$\ll \tilde{m}_N\delta^{-1}\exp\left\{(1-\alpha)\min_{|t|\leqslant\pi}\sum_{j\leqslant N}\frac{\Re(a_je^{itj})-d_j}{j}\right\} =: \tilde{m}_N\delta^{-1}E^{1-\alpha}.$$

Thus it follows from Lemma 3.2 and the last estimate that $m_N/\tilde{m}_N \ll \delta^{\alpha\theta^-} + \delta^{-1}E^{1-\alpha}$ provided $\delta \geqslant 1/N$. Now we choose $\delta = \max\{E^{(1-\delta)/(1+\alpha\theta^-)}, N^{-1}\}$ and get

$$m_N/\tilde{m}_N \ll E^{\alpha(1-\alpha)\theta^-/(1+\alpha\theta^-)} + N^{-\alpha\theta^-}.$$

Further, by virtue of condition (2.3), we see that $E \gg N^{-2\theta^+}$. Thus, if $\alpha = 2\theta^+/(\theta^- + 2\theta^+)$, the first term dominates the second one. So we can find an expression of $c_3 = c_3(\theta^-, \theta^+)$ in Proposition 2.2. The proof is complete. □

Proof of Proposition 2.4. We now start with the formula

$$m_N = \frac{1}{2\pi i N}\int_{|z|=1}\frac{H(z)G'(z)}{z^N}\,dz + \frac{1}{2\pi i N}\int_{|z|=r}\frac{H'(z)G(z)}{z^N}\,dz =: I_1 + I_2$$

with a fixed $0 < r < 1$. The integral I_1 can be calculated repeating all steps in the proof of Proposition 2.1. Actually we have the same integrals estimated in Lemmas 3.2 and 3.3 but without the summands containing $H'(z)$. Thus we obtain in this way

$$I_1 = \frac{1}{2\pi i N}\int_{\Delta_0}\frac{H(z)G'(z)(1-G_2(z))}{z^N}\,dz + O(\tilde m_N\delta^{\alpha\theta^-}) + O(\tilde m_N\delta^{-1}T^{-(1-\alpha)\theta^-/2})$$

provided $N \geqslant 1/\delta$, $1 \leqslant T \leqslant N$. If $z \in \Delta_0$, the functions $G'(z)$ and $G_2(z)$ have been considered in Lemma 3.4. For $H(z)$, we use the inequality

$$\max_{|t|\leqslant T/N}|H(e^{it}) - H(1)| \leqslant \frac{T}{N}\sum_{k\leqslant uN}k|b_k| + 2\sum_{uN<k\leqslant N}|b_k| \ll uT + \rho_N(u)$$

for arbitrary $0 < u < 1$. This after a relevant choice of u, by (2.8) or (2.9), becomes $O(T\hat\rho(N))$. So in the previous notation we obtain

$$\begin{aligned} I_1 &= H(1)J_{00} + O\left(\frac{T\hat\rho(N)}{N}\int_{\Delta_0}|G'(z)||1-G_2(z)||dz|\right) \\ &\quad + O(\tilde m_N\delta^{\alpha\theta^-}) + O(\tilde m_N\delta^{-1}T^{-(1-\alpha)\theta^-/2}) \end{aligned} \tag{3.13}$$

provided $N \geqslant 1/\delta$, $1 \leqslant T \leqslant N$, and $\mu_N \leqslant \delta \wedge T^{-1}$. The integral J_{00} has been considered in the proof of Lemma 3.4. Applying also Lemma 3.5, as in the proof of Proposition 2.1, we can express it with the required accuracy in terms of $\tilde m_N$.

By (3.11), (3.12), and the inequalities $|D'(z)| \ll ND(1) \ll N^2\tilde m_N$, the integral in the second term on the right-hand side of (3.13) does not exceed

$$\int_{\Delta_0}|D'(z)|(1+|D_2(z)|)|dz| + O(\tilde m_N TN\mu_n\delta^{-\alpha\theta^+}(T+\delta^{-1}) \ll \tilde m_N TN\delta^{-\alpha\theta^+}$$

if $\mu_N \leqslant \delta \wedge T^{-1}$. Gathering these estimates we obtain

$$\begin{aligned} I_1/\tilde m_N &= H(1)\exp\{U(1) - V(1)\} + O\big(T^2\delta^{-\alpha\theta^+}(\mu_N + \hat\rho(N))\big) + \\ &\quad + O(T\delta^{-1-\alpha\theta^+}\mu_N) + O(\delta^{\alpha\theta^-} + \delta^{-1}T^{-(1-\alpha)\theta^-/2}) \end{aligned}$$

provided $N \geqslant 1/\delta$, $1 \leqslant T \leqslant N$, and $\mu_N \leqslant \delta \wedge T^{-1}$.

By Cauchy's formula, we have

$$I_2 = \frac{1}{N}\sum_{n<N}(N-n)b_{N-n}g_n.$$

Lemma 3.1 yields $|g_n| \leqslant \tilde m_n \ll \tilde m_N(N/n)^{1-\theta^-}$. Hence, for $0 < u < 1$,

$$I_2/\tilde m_N \ll N^{1-\theta^-}\sum_{uN\leqslant k\leqslant N}|b_k| + u\sum_{k\leqslant uN}|b_k|.$$

This, in the case of (2.7) and (2.8), implies $I_2 \ll \tilde m_N\hat\rho(N)$. Similarly, under condition (2.9), we use the inequality $\tilde m_n \ll \tilde m_N$.

The estimates of I_1 and I_2 yield the assertion of Proposition 2.4. □

4. Proofs of theorems

Proof of Theorem 1.1. In Corollary 2.3 it suffices to take $a_j = \lambda_j f_j(1)$ and $d_j = \lambda_j q_j$. □

Proof of Theorem 1.2. Apply Proposition 2.4 also with $a_j = \lambda_j f_j(1)$ and $d_j = \lambda_j q_j$. The conditions of Theorem 1.2 imply (2.7) and (2.9). □

Proof of Corollary 1.3. This is straightforward. Applying Theorem 1.2, we just have to change $f(\sigma)$ to $f(\sigma)q(\sigma)$. □

Proof of Corollary 1.4. In addition to the previous argument, it suffices to use (1.10). □

Proof of Corollary 1.5. To get the quantities appearing on the right-hand side of (1.4), one can exploit the expression given in Proposition 2.1. □

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