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Value concentration of additive functions on random permutations

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Abstract. We prove an analog of the Kolmogorov–Rogozin inequality for the value concentration of completely additive functions defined on random permutations.

Key words: symmetric group, random permutation, cycle structure, concentration function

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1. INTRODUCTION AND RESULT

Let $\mathbf{S}_n$ be the symmetric group of permutations σ acting on $n \geq 1$ letters. Each $\sigma \in \mathbf{S}_n$ has a unique representation (up to the order) by the product of independent cycles κ

$$\sigma = \kappa_1 \cdots \kappa_w, \tag{1}$$

where $w = w(\sigma)$ denotes the number of cycles. Set ν_n for the uniform probability measure on $\mathbf{S}_n$. Denote by $k_j(\sigma)$ the number of cycles of length j in (1), $1 \leq j \leq n$, and let $\bar{k}(\sigma) = (k_1(\sigma), \dots, k_n(\sigma))$ be the structure vector of $\sigma \in \mathbf{S}_n$. The evident relation $1k_1(\sigma) + \cdots + nk_n(\sigma) = n$ shows that the family of random variables (r. vs) $\{k_j(\sigma), 1 \leq j \leq n\}$ is dependent with respect to ν_n. Its joint distribution can also be interpreted as the conditional distribution of a set of independent Poisson random variables $\{\xi_j, 1 \leq j \leq n\}$, $\mathbf{E}\xi_j = 1/j$, defined on some probability space. It is known [1] that

$$\begin{aligned} \nu_n(\bar{k}(\sigma) = \bar{k}) &= \mathbf{1}(1k_1 + \cdots + nk_n = n) \prod_{j=1}^{n} \frac{1}{j^{k_j} k_j!} \\ &= P\big((\xi_1, \dots, \xi_n) = \bar{k} \; | 1\xi_1 + \cdots + n\xi_n = n\big), \quad \bar{k} \in \mathbf{Z}^{+^n}. \end{aligned}$$

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Despite to this dependency, the theory of limit distributions as $n \to \infty$of the decomposable mappings, called *additive functions*, is under progress (see [1], [2], [10], [11], and the bibliography given in these papers).

We will examine the value concentration of a completely additive function $h : \mathbf{S}_n \to \mathbf{R}$, maybe, depending on n. By definition, it has the expression

$$h(\sigma) := \sum_{j=1}^{n} a(j) k_j(\sigma) \tag{2}$$

with some $a(j) \in \mathbf{R}$, $1 \le j \le n$. The above mentioned number of cycles $w(\sigma)$ is the classical example. An interesting completely additive function has appeared in the recent investigations [7] and [14] of distribution of the number $N(\sigma; x, y)$ of eigenvalues on the unit circumference between the points $e^{2\pi i x}$ and $e^{2\pi i y}$, $0 \le x < y \le 1$, of the permutation matrix associated to σ. We have the relation

$$N(\sigma; x, y) - n(y - x) = \sum_{j=1}^{n} k_j(\sigma)(\{xj\} - \{yj\}) =: W(\sigma),$$

where $\{a\}$ denotes the fractional part of $a \in \mathbf{R}$. The function $W(\sigma)$ is completely additive. Appropriately normalized, either of $w(\sigma)$ or $W(\sigma)$ obeys the standard normal limit distribution with respect to ν_n as $n \to \infty$. Nevertheless, the limit distribution for

$$n^{-\alpha} h^{\alpha}(\sigma) := n^{-\alpha} \sum_{j=1}^{n} j^{\alpha} k_j(\sigma)$$

with $\alpha > 1$ is proper and concentrated in the interval [0,1] and therefore it is not infinitely divisible (see [10]). Note that $h^1(\sigma) \equiv n$ for all $\sigma \in \mathbf{S}_n$. Thus, for better understanding of our results, it is worth to bear in mind that any function under investigation can have the "deterministic" component as $\lambda h^1(\sigma) = \lambda n$ with some $\lambda \in \mathbf{R}$.

Our purpose is to obtain a Kolmogorov type estimate for the concentration

$$Q_n(l) = \sup_{x \in \mathbf{R}} \nu_n(x \le h(\sigma) < x + l), \quad l \ge 0.$$

General properties of the concentration functions and the estimates for sums of independent r. vs are presented in [8]. We have been mainly influenced by the investigation of I.Z. Ruzsa [13] in probabilistic number theory. We use his ideas and arguments in the sequel.

Let $x \wedge y = \min(x, y)$. For a completely additive function $h(\sigma)$ and $\lambda \in \mathbf{R}$, we set

$$D_n(l; \lambda) = \sum_{j=1}^{n} \frac{l^2 \wedge (a(j) - \lambda j)^2}{j}, \qquad D_n(l) = \min_{\lambda \in \mathbf{R}} D_n(l; \lambda).$$

Throughout the paper $C, C_1, \ldots$ will denote absolute positive constants. The main result of the paper is the following theorem.

THEOREM. *We have*

$$Q_n(l) \le Cl(D_n(l))^{-1/2}.$$

Of course, if $D_n(l) = o(l^2)$ as $n \to \infty$, the trivial estimate $Q_n(l) \le 1$ is better. Observe that the Kolmogorov–Rogozin theorem (see [12] or Theorem 2.2.4 in [8]) applied for the sum $S_n := a(1)\xi_1 + \cdots + a(n)\xi_n$, where ξ_j are the above mentioned independent Poisson r. vs, yields the estimate

$$\sup_{x \in \mathbf{R}} P(x \le S_n < x + l) \le C_1 l (D_n(l;0))^{-1/2}.$$

Thus, with a successful choice of λ, our concentration estimate for $h(\sigma) - \lambda h^1(\sigma)$ is comparable with that for S_n.

2. PROOF OF THEOREM

We split the proof into several lemmas.

LEMMA 1. *Let $f : \mathbf{S}_n \to \mathbf{C}$ be a completely multiplicative function defined by*

$$f(\sigma) = \prod_{j=1}^{n} b(j)^{k_j(\sigma)}, \qquad 0^0 := 1$$

with $b(j) \in \mathbf{C}$, $|b(j)| \le 1$ for each $1 \le j \le n$. Then

$$\frac{1}{n!}\Big| \sum_{\sigma \in \mathbf{S}_n} f(\sigma) \Big| \le C_2 \exp\Big\{ -\frac{1}{4} \min_{|u| \le \pi} \sum_{j=1}^{n} \frac{1 - \Re(b(j)e^{iuj})}{j} \Big\}.$$

Proof. This is the estimate (17) in the author's paper [10].

Further we will apply it for the Fourier transform of the distribution $\nu_n(h(\sigma) < x)$ (see inequality (11) below). In this case, $b(j) = e^{2\pi i a(j)t}$, $t \in \mathbf{R}$. So, the first task will be investigation of the trigonometrical polynomial

$$m(u,t) := \sum_{j=1}^{n} \frac{1 - \cos 2\pi(a(j)t - uj)}{j}.$$

Consider the values $u(t)$ giving

$$\min_{-1/2 \le u < 1/2} m(u,t) = m(u(t),t), \quad t \in [-1,1].$$

Dealing with the stationary points, we use a criteria for implicit functions and observe that $u(t)$ is well defined continuous function in some nontrivial neighborhood of the point $t = 0$, $u(0) = 0$. Beyond it, if several values of $u(t)$ appear for a fixed t, we can choose the smallest of them and so obtain the function $u(t)$ defined on the whole interval [-1,1] and taking values in [-1/2,1/2). It appears that $u(t)$ is related to a homomorphism of the additive groups $\mathbf{R}$ and $\mathbf{R}/\mathbf{Z} = \mathbf{T}$. For convenience, we identify $\mathbf{T}$ with the interval [0,1) and take addition modulo one. Observe that the group $\mathbf{T}$ is the complete metric space with respect to the metric defined via the distance to the nearest integer $||x|| = \{x\} \wedge (1 - \{x\})$ which is not a norm. Further, if $-1/2 \le u(t) < 0$, we redefine this value to $1 + u(t)$ and so get the function $u : [-1, 1] \to \mathbf{T}$ preserving continuity at the point $t = 0$.

We now examine the approximate Cauchy equations with respect to $||u||$.

Lemma 2. *Let $w : \mathbf{R} \to \mathbf{T}$ be the function satisfying*

$$||w(x + y) - w(x) - w(y)|| < \varepsilon \tag{3}$$

for all $x, y \in \mathbf{R}$ and some $0 < \varepsilon < 1/6$, then there exists a homomorphism $\varphi : \mathbf{R} \to \mathbf{T}$ such that

$$||w(x) - \varphi(x)|| \le \varepsilon$$

for all $x \in \mathbf{R}$.

Proof. See [4] and [5], Thm 1. In the second paper [5], D. Cenzer also shows that the bound 1/6 can not be substituted by 1/4.

Lemma 3. *Let $\varphi : \mathbf{R} \to \mathbf{T}$ be a homomorphism. Assume that there is a nonempty open set $U \subset \mathbf{R}$ and a $0 < \delta < 1/2$ such that*

$$\varphi(U) \subset (-1/2 + \delta, 1/2 - \delta) + \mathbf{Z},$$

then

$$\varphi(x) = \lambda x \bmod 1 \tag{4}$$

for some $\lambda \in \mathbf{R}$.

Proof. This result belongs to J.G. Van der Corput [6], p. 64. The proof was restated by K. Baron and P. Volkmann [3]. For an extensive bibliography concerning the approximate Cauchy equations, see the survey [9].

In what follows we will use the following corollary of Lemmas 2 and 3.

LEMMA 4. *Let $v : [-1, 1] \to \mathbf{T}$ be continuous at the point $t = 0$ and $v(0) = 0$. Suppose that, for some $0 < \eta < 1/18$,*

$$||v(t_1 + t_2) - v(t_1) - v(t_2)|| \le \eta \tag{5}$$

whenever $t_1, t_2, t_1 + t_2 \in [-1, 1]$. Then

$$||v(t) - \lambda t|| \le 3\eta \tag{6}$$

for some $\lambda \in \mathbf{R}$ and all $t \in [-1, 1]$.

Proof. Define $w : \mathbf{R} \to \mathbf{T}$ by

$$w(2k + t) = 2ku(1) + v(t) \bmod 1, \qquad t \in [-1, 1).$$

This function extends $v(t)$ to the real line, except at $t = 1$, where $w(1) = 2v(1) + v(-1) \bmod 1$. The function $z(t) := w(t) - v(1)t \bmod 1$ is 2-periodical. Hence estimating, as in (3) for all $x, y \in \mathbf{R}$,

$$d := ||w(x + y) - w(x) - w(y)|| = ||z(x + y) - z(x) - z(y)||$$

we may confine ourselves to the values $x, y \in [-1, 1)$. If also $x + y \in [-1, 1)$, then by (5) $d \le \eta$. If $x + y \ge 1$, using (5) again, we obtain

$$\begin{aligned} d &= ||2v(1) + v(x + y - 2) - v(x) - v(y)|| \\ &\le ||v(x + y - 2) - v(x - 1) - v(y - 1)|| + ||v(x - 1) + v(1) - v(x)|| \\ &+ ||v(y - 1) + v(1) - v(y)|| \le 3\eta. \end{aligned}$$

Similarly, $d \le 3\eta$ in the case $x + y < -1$. Thus, w satisfies (3) with $\varepsilon = 3\eta$. By Lemma 2 there exists a homomorphism $\varphi : \mathbf{R} \to \mathbf{T}$ such that $||w(x) - \varphi(x)|| \le 3\eta$ for all $x \in \mathbf{R}$. By the continuity condition of the lemma, $||w(x)|| = ||v(x)|| \le \eta$ for all x belonging to some neighborhood U of the point $x = 0$. Hence $||\varphi(x)|| \le \eta + ||w(x) - \varphi(x)|| \le 4\eta < 1/2$ for $x \in U$. Lemma 3 now gives the expression (4) of $\varphi(x)$. Lemma 4 is proved.

We now return to the trigonometrical polynomials $m(u, t)$ with $u \in \mathbf{T}$ and $t \in [-1, 1]$.

LEMMA 5. *For $0 \le x \le 1/2$ and $0 \le \Theta \le 10$, we have*

$$m(\Theta x, 0) \le m(x, 0) + C_3.$$

Proof. Let $r := e^{-1/n}$ and

$$\begin{aligned}\Psi(y) \quad : &= \sum_{j=1}^{\infty} \frac{1-\cos 2\pi jy}{j} r^j = \log \frac{|1-re^{2\pi iy}|}{1-r} \\ &= \frac{1}{2}\log\left(1+\frac{4r}{(1-r)^2}\sin^2 \pi y\right).\end{aligned}$$

By virtue of

$$\sum_{j=1}^{n} \frac{1-r^j}{j} + \sum_{j>n} \frac{r^j}{j} \le 1 + \int_n^\infty \frac{r^x}{x} dx < 1 + e^{-1}, \quad n \ge 1,$$

we have $|m(y,0) - \Psi(y)| < 3$. Thus it remains to check if $\Psi(\theta x) \le \Psi(x) + C_4$ in the region assumed in the lemma. We have

$$C(\Theta) := \max_{0 \le x \le 1/2} \frac{\sin^2 \pi\Theta x}{\sin^2 \pi x} \le C_5$$

for $0 \le \Theta \le 10$. Hence

$$\Psi(\Theta x) \le \frac{1}{2}\log\left(1+\frac{4rC(\Theta)}{(1-r)^2}\sin^2 \pi x\right) \le \Psi(x) + \frac{1}{2}\log(\max(1, C_5)),$$

as desired. Lemma 5 is proved.

Lemma 6. *Let M be a constant such that, for a continuous at the point $t = 0$ function $u : [-1,1] \to \mathbf{T}$, $u(0) = 0$, we have*

$$m(u(t), t) \le M \tag{7}$$

for all $t \in [-1,1]$. Then, for some $\lambda \in \mathbf{R}$,

$$m(\lambda t, t) \le 20M + 2C_3, \quad t \in [-1,1] \tag{8}$$

and

$$D_n(1, \lambda) \le (10M + C_3)C_6. \tag{9}$$

Remark. Since for $u(t)$ one could imagine the above mentioned function, realizing the minimum of $m(u,t)$, we have not used another notation.

Proof. Set

$$\alpha = \sup\{||u(t_1+t_2) - u(t_1) - u(t_2)|| : \ t_1, t_2, t_1+t_2 \in [-1,1]\}.$$

If $\alpha = 0$, then by Lemma 4, $||u(t) - \lambda t|| = 0$ and inequality (8) follows from (7). If $\alpha > 0$, we chose $t_1, t_2, t_1 + t_2 \in [-1, 1]$ so that

$$\beta := ||u(t_1 + t_2) - u(t_1) - u(t_2)|| \geq \frac{9}{10}\alpha.$$

For arbitrary $t \in [-1, 1]$, by Lemma 4 with $\eta = \alpha$, we have $\beta_1 := ||u(t) - \lambda t|| \leq 9\alpha \leq 10\beta$. Since the first inequality is trivial for $\alpha \geq 1/18$, applying Lemma 4 we have avoided the condition on α. Now, by virtue of Lemma 5, $m(\beta_1, 0) \leq m(\beta, 0) + C_3$. The inequality

$$1 - \cos(x_1 + \cdots + x_k) \leq k\big((1 - \cos x_1) + \cdots + (1 - \cos x_k)\big) \qquad (10)$$

and (7) yield

$$m(\beta, 0) \leq 3\bigg(m(u(t_1 + t_2), t_1 + t_2) + m(u(t_1), t_1) + m(u(t_2), t_2)\bigg) \leq 9M.$$

Again by (10), we obtain bound (8):

$$m(\lambda t, t) \leq 2m(u(t), t) + 2m(\beta_1, 0) \leq 2M + 2m(\beta, 0) + 2C_3 \leq 20M + 2C_3.$$

Integrating the trigonometrical polynomial $m(\lambda t, t)$ over the interval $[0, 1]$ and using the inequality $1 - (\sin x)/x \geq c_1 \min\{1, x^2\}$, where $x \in \mathbf{R}$ and $c_1 > 0$ is an absolute constant, we obtain assertion (9). Lemma 6 is proved.

Lemma 7. *If $X \subset [-1, 1]$ is a set of positive Lebesgue measure, symmetric to the origin and containing it, then we have*

$$X^r := \{x_1 + \cdots + x_r :\ x_1, \ldots, x_r \in X\} \supset [-1, 1]$$

provided that $r = [12/\mathrm{meas}(X)]$.

Proof. See [13].

Proof of Theorem. For $l > 0$, it suffices to deal with $Q_n(1)$ only and then apply the result for $h(\sigma)/l$. By Lemma 2.2.1 of [8] and our Lemma 1, we have

$$Q_n(1) \leq \frac{C_7}{n!} \int_{-1}^{1} \bigg| \sum_{\sigma \in \mathbf{S}_n} e^{2\pi i t h(\sigma)} \bigg| dt \leq C_8 \int_{-1}^{1} \exp\bigg\{ -\frac{1}{4} \min_{u \in \mathbf{T}} m(u, t) \bigg\} dt. \quad (11)$$

Set

$$X_k = \{t \in [-1, 1] :\ \min_{u \in \mathbf{T}} m(u, t) \leq k\}, \quad k = 1, 2, \ldots.$$

These sets are nonempty measurable, symmetric with respect to the origin, and having the Lebesgue measure $\mu_k := \mathrm{meas}(X_k) > 0$. Applying Lemma 7 for $X = X_k$ we have that the set sum X^r covers the interval [-1,1] if $r = [12/\mu_k]$. In other words, this means that each $t \in [-1,1]$ has an expression $t = t_1 + \cdots + t_r$ such that $m(u_m, t_m) \le k$ with some $u_m \in \mathbf{T}$, $1 \le m \le r$. Hence using (10) we obtain

$$m(u,t) \le kr^2$$

for any $t \in [-1,1]$ and $u = u_1 + \cdots + u_r \bmod 1$, $u \in \mathbf{T}$. The same holds for the function $u(t)$ discussed above. Thus, by Lemma 6,

$$D_n(1,\lambda) \le C_6(10kr^2 + C_3) \le C_9 k \mu_k^{-2}$$

for some $\lambda \in \mathbf{R}$, or equivalently,

$$\mu_k \le C_{10}(k/D_n(1))^{1/2}.$$

This and (11) imply

$$Q_n(1) \le C_8 \sum_{k\ge1} e^{-k/4}\mu_{k+1} \le C_{11}(D_n(1))^{-1/2}.$$

Theorem is proved.

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