Acta Appl Math (2007) 97: 119–127
DOI 10.1007/s10440-007-9133-y

Moments of Additive Functions on Random Permutations

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Published online: 23 March 2007

Abstract We obtain upper and lower bounds for the power moments of additive functions on random permutations. The main ideas of proofs have been originated in probabilistic number theory.

Keywords Symmetric group · Random permutation · Cycle structure · Power moment

Mathematics Subject Classification (2000) 60C05 · 05D40

1 Introduction

The upper and lower bounds for the power moments of order $\beta \geq 2$ for sums of independent random variables (i.r.vs) were established by H.P. Rosenthal [16]. In [19], B. von Bahr and C.-G. Esseen presented two upper estimates in the case $0 < \beta < 2$. Combining them together with truncation and symmetrization of the summands, we [9, 10] obtained sharp upper and lower bounds in this case. Our method based on the concentration function estimates had been originated in the remarkable I.Z. Ruzsa's paper [17] dealing with the second moment for additive number theoretical functions. Such functions can be interpreted as sums of dependent r.vs. An upper bound for this moment was given earlier by the Turán-Kubilius inequality [7]. Extending it, the author [9] and independently A. Hildebrand [5] examined all power moments of additive functions.

The present remark is devoted to sums of dependent r.vs appearing in probabilistic combinatorics which, as the recent results (see, for instance [1]) show, has a lot of parallels with probabilistic number theory. Therefore the aforementioned estimates as well as the others obtained by J. Kubilius [8], D.T.A. Elliott [3], I.Z. Ruzsa [18], A. Hildebrand [5],

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and K.-H. Indlekofer [6] serve us for a good pattern. We now demonstrate the possibility to achieve the similar goals for the power moments of additive functions defined on random permutations.

Let $\mathbf{S}_n$ be the symmetric group of permutations σ acting on $n \geq 1$ letters. Each $\sigma \in \mathbf{S}_n$ has a unique representation (up to the order) by the product of independent cycles κ

$$\sigma = \kappa_1 \cdots \kappa_w, \tag{1}$$

where $w = w(\sigma)$ denotes the number of cycles. Set ν_n for the uniform probability measure on $\mathbf{S}_n$. Denote by $k_j(\sigma)$ the number of cycles of length j in (1), $1 \leq j \leq n$, and let $\bar{k}(\sigma) = (k_1(\sigma), \ldots, k_n(\sigma))$ be the structure vector of $\sigma \in \mathbf{S}_n$. If $\ell(\bar{k}) := 1k_1 + \cdots + nk_n$ for $\bar{k} = (k_1, \ldots, k_n) \in \mathbf{Z}^{+^n}$, then the evident relation $\ell(\bar{k}(\sigma)) = n$ shows that the family of r.vs $\{k_j(\sigma), 1 \leq j \leq n\}$ is dependent with respect to ν_n. Its joint distribution can also be interpreted as the conditional distribution of a set of independent Poisson r.vs $\{\xi_j, 1 \leq j \leq n\}$, $\mathbf{E}\xi_j = 1/j$, defined on some probability space. It is known [1] that, for $\bar{\xi} := (\xi_1, \ldots, \xi_n)$ and $\bar{k} \in \mathbf{Z}^{+^n}$,

$$\nu_n(\bar{k}(\sigma) = \bar{k}) = \mathbf{1}(\ell(\bar{k}) = n) \prod_{j=1}^{n} \frac{1}{j^{k_j} k_j!} = P\big(\bar{\xi} = \bar{k} | \ell(\bar{\xi}) = n\big). \tag{2}$$

Despite to this dependency, the theory of limit distributions as $n \to \infty$ of the decomposable mappings, called *additive functions*, is under progress (see [1, 2, 11, 12, 14], and the bibliography given in these papers). By definition, such function has the following expression

$$h(\sigma) := \sum_{j=1}^{n} h_j(k_j(\sigma)) \tag{3}$$

with some two-dimensional array $h_j(k) \in \mathbf{R}$, $1 \leq j \leq n$ and $k \geq 0$, satisfying the condition $h_j(0) \equiv 0$. The functions satisfying also the condition $h_j(k) = a_j k$ are called *completely additive*. The above mentioned number of cycles $w(\sigma)$ is the classical example. Many of the statistics for so-called A-permutations (see the recent book by A.L. Yakymiv [22]) fall within this scope. Interesting examples of additive functions appear in the Erdős-Turán problem concerning the group theoretical order of a random permutation (see [1] and [21]) or in the investigations (see [4] and [20]) of distribution of the number of eigenvalues on the unit circumference of the permutation matrix associated to σ.

One completely additive function plays an exceptional role in the present remark. It is just $l(\sigma) := \ell(\bar{k}(\sigma))$. Observe that $l(\sigma) \equiv n$ for all $\sigma \in \mathbf{S}_n$. Thus, we have to bear in mind that any function under investigation can have the "deterministic" component $\lambda l(\sigma) = \lambda n$ with some $\lambda \in \mathbf{R}$. Separation of this component is one of the main technical obstacles.

In the sequel, we denote by $\mathbf{E}_n f(\sigma)$ the mean value of $f : \mathbf{S}_n \to \mathbf{R}$ with respect to ν_n. The purpose of the present paper is to estimate

$$M_n(h, A, \beta) = \big(\mathbf{E}_n |h(\sigma) - A|^{\beta}\big)^{1/\beta}$$

for all $\beta > 0$. Let $h_j(k; \lambda) = h_j(k) - \lambda jk$ and $h^{(\lambda)}(\sigma) = h(\sigma) - \lambda l(\sigma)$ be the additive function defined as in (3) via this sequence. Observe that

$$M_n(h, A, \beta) = M_n(h^{(\lambda)}, A - \lambda n, \beta). \tag{4}$$

The estimates obtained for $M_n(h, A, \beta)$ are expressed in terms of

$$A_n(h) := \sum_{jk\le n} \frac{h_j(k)}{j^k k!}, \qquad B_n(h, \beta) := \Bigl(\sum_{jk\le n} \frac{|h_j(k)|^\beta}{j^k k!}\Bigr)^{1/\beta}$$

but, by virtue of (4), h had to be substituted by $h^{(\lambda)}$ with an appropriate λ.

For brevity, we also use the notation $\ll$ as an analog of $O(\cdot)$ and $a \asymp b$ to denote $a \ll b \ll a$. To show the dependence on parameters, we will add the extra indexes.

2 Upper Estimates

In this section, $n \ge 1$.

Theorem 1 *If $\beta \ge 2$, then*

$$M_n(h, A, \beta) \ll_\beta \bigl|A_n(h^{(\lambda)}) - A + \lambda n\bigr| + B_n(h^{(\lambda)}, 2) + B_n(h^{(\lambda)}, \beta)$$

for every $\lambda \in \mathbf{R}$.

For the remaining case, we need the following additional notation. Set

$$h'_j(k; \lambda, c) = \begin{cases} h_j(k; \lambda) & \text{if } |h_j(k; \lambda)| \le c, \\ 0 & \text{if } |h_j(k; \lambda)| > c \end{cases}$$

and $h''_j(k; \lambda, c) = h_j(k; \lambda) - h_j(k; \lambda, c)$. Using these sequences and (3), we define two additive functions $h'(\sigma) := \hat{h}^{(\lambda,c)}(\sigma)$ and $h''(\sigma) := \check{h}^{(\lambda,c)}(\sigma)$ respectively. In other words, we have split $h^{(\lambda)}(\sigma) = h'(\sigma) + h''(\sigma)$.

Theorem 2 *Let $c > 0$ and $\lambda \in \mathbf{R}$ be arbitrary. If $1 \le \beta < 2$, then*

$$M_n(h, A, \beta) \ll_\beta \bigl|A_n(h^{(\lambda)}) - A + \lambda n\bigr| + B_n(\hat{h}^{(\lambda,c)}, 2) + B_n(\check{h}^{(\lambda,c)}, \beta).$$

If $0 < \beta < 1$, then

$$M_n(h, A, \beta) \ll_\beta \bigl|A_n(\hat{h}^{(\lambda,c)}) - A + \lambda n\bigr| + B_n(\hat{h}^{(\lambda,c)}, 2) + B_n(\check{h}^{(\lambda,c)}, \beta).$$

Remark To obtain sharp upper estimates in Theorems 1 and 2, one has to minimize with respect to λ and c the quantities on the right-hand side. In the next section, we will show that there exists a choice of the parameters $\lambda \in \mathbf{R}$ and $c > 0$ such that, apart from a small correction, these estimates can be reverted.

Exploiting (4), we reduce the problem to estimating of the moment when $\lambda = 0$. This is assumed until the end of the section. So, equality $h = h^{(\lambda)}$ will be also kept in mind. The very formulation of our theorems shows that the moments $M_n(h, A, \beta)$ are comparable with that of the sum of i.r.vs

$$Y_n(\lambda) := \sum_{j=1}^{n} h_j(\xi_j; \lambda).$$

Set $Y_n := Y_n(0)$.

Observe that the values of $h_j(k)$ for $jk > n$ make no influence on $h(\sigma)$ therefore we assume that $h_j(k) = 0$ if $jk > n$. We now recall the known estimates of power moments applied for this fairly special sum of i.r.vs.

Lemma 1 *If* $\beta \geq 2$, *then*

$$\left(\mathbf{E}|Y_n - \mathbf{E}Y_n|^\beta\right)^{1/\beta} \asymp_\beta B_n(h, 2) + B_n(h, \beta). \tag{5}$$

If $0 < \beta < 2$, *then*

$$\mathbf{E}|Y_n - \mathbf{E}Y_n|^\beta \leq B_n^\beta(h, 2). \tag{6}$$

Moreover,

$$\mathbf{E}|Y_n - \mathbf{E}Y_n|^\beta \leq 2B_n^\beta(h, \beta) \tag{7}$$

for $1 < \beta \leq 2$ *and*

$$\mathbf{E}|Y_n|^\beta \leq B_n^\beta(h, \beta) \tag{8}$$

for $0 < \beta \leq 1$.

Proof If $\beta \geq 2$, apply the Rosenthal's inequality [16]. Estimate (6) follows from the monotonicity with respect to β of the function $(\mathbf{E}|X|^\beta)^{1/\beta}$. Inequality (7) follows from the result of B. von Bahr and C.-G. Esseen [19] and (8) is trivial.

If $\beta \geq 1$, we also have

$$\begin{aligned} |\mathbf{E}Y_n - A_n(h)| &= \left|\sum_{jk \leq n} \frac{h_j(k)}{j^k k!}\left(e^{-1/j} - 1\right)\right| \\ &\leq B_n(h, \beta)\left(\sum_{jk \leq n} \frac{1}{j^{k+1}k!}\right)^{1-1/\beta} \ll B_n(h, \beta). \end{aligned} \tag{9}$$

So, in what follows we will be able to use $A_n(h)$ instead of $\mathbf{E}Y_n$.

In general, no estimate (6), (7), or (8) is sharp in the case $0 < \beta < 2$. Combining them, we [10] have succeeded in obtaining lower and upper estimates of the same order. To give an impression and to motivate the truncation used in Theorem 2, we present the following corollary of the author's result [10].

Lemma 2 *Let* X_k, $1 \leq k \leq n$ *be symmetric i.r.vs such that* $\mathbf{E}|X_k| < \infty$ *and* $0 < \beta \leq 2$. *Then*

$$\begin{aligned} \mathbf{E}\left|\sum_{k \leq n} X_k\right|^\beta \asymp_\beta \inf\Bigg\{&\sum_{k \leq n} \mathbf{E}\left(\mathbf{1}\{|X_k| > c\}|X_k|^\beta\right) \\ &+ \left(\sum_{k \leq n} \mathbf{E}\left(\mathbf{1}\{|X_k| \leq c\}|X_k|^2\right)\right)^{\beta/2} : c > 0\Bigg\}. \end{aligned}$$

Proof See [10].

Let $h'(\sigma) = \hat{h}^{(0,c)}(\sigma)$ and $h''(\sigma) = \check{h}^{(0,c)}(\sigma)$. Set

$$Y' = \sum_{j=1}^n \hat{h}_j(\xi_j; 0, c), \qquad Y'' = \sum_{j=1}^n \check{h}_j(\xi_j; 0, c).$$

The following author's inequality spans the upper estimation of the moments of additive functions with that of sums of i.r.vs.

Lemma 3 *We have*

$$\nu_n(|h(\sigma)-A|\geq u)\leq 32e^2P(|Y_n-A|\geq u/3),$$

where $A\in\mathbf{R}$ *and* $u\geq 0$ *are arbitrary.*

Proof See [12]. Recently we [15] observed that the factor $1/3$ at u can not be substituted by $5/7$.

Corollary 1 *For every additive function* $h(\sigma)$, $A\in\mathbf{R}$, *and* $\beta>0$, *we have*

$$M_n^\beta(h,A,\beta)\leq 32e^2 3^\beta\mathbf{E}|Y_n-A|^\beta. \tag{10}$$

Proof It suffices to use twice the expression

$$\mathbf{E}|X|^\beta=\int_0^\infty P(|X|\geq u)d(u^\beta),$$

valid for each r.v. X having the β-th power moment, and Lemma 3.

Proofs of Theorems 1 and 2 We will repeatedly apply the inequality

$$|x+y|^\beta\leq\min\{1,2^{\beta-1}\}(|x|^\beta+|y|^\beta),\quad \beta>0. \tag{11}$$

Let $\beta\geq 1$. By Corollary 1, we have

$$M_n(h,A,\beta)\ll\big(\mathbf{E}|Y_n-A|^\beta\big)^{1/\beta}\ll\big(\mathbf{E}|Y_n-\mathbf{E}Y_n|^\beta\big)^{1/\beta}+|\mathbf{E}Y_n-A|. \tag{12}$$

Now, if $\beta\geq 2$, the desired upper estimate follows from (5) and (9). If $1\leq\beta<2$, as mentioned above, we split $Y_n=Y'+Y''$. Further we use (11) and apply (6) for $\mathbf{E}|Y'-\mathbf{E}Y'|^\beta$ and (7) for $\mathbf{E}|Y''-\mathbf{E}Y''|^\beta$. Similarly, we change $\mathbf{E}Y_n$ by $A_n(h)$. So, from (12) we obtain the upper estimate in Theorem 2 for this case.

If $\beta<1$, we start with

$$\begin{aligned}M_n(h,A,\beta)\ll{}&(\mathbf{E}|Y'-\mathbf{E}Y'|^\beta)^{1/\beta}+(\mathbf{E}|Y''|^\beta)^{1/\beta}\\&+|\mathbf{E}Y'-A_n(h')|+|A_n(h')-A|.\end{aligned} \tag{13}$$

For the first three terms on the right hand side, we apply (6), (8), and (9) with $\beta=2$ respectively. Observing that the appearing constants do not depend on c, we complete the proof of the upper bound in Theorem 2 for $\lambda=0$. As we have noted, the same holds if an arbitrary λ is involved.

3 A Lower Estimate

We will use the functions $\hat{h}^{(\lambda,c)}$ and $\check{h}^{(\lambda,c)}$ introduced in Sect. 2 but with $c=v:=M_n(h,A,\beta)$.

Theorem 3 *There exist* $\lambda \in \mathbf{R}$ *and* $n_0 = n_0(\beta) \in \mathbf{N}$ *such that, for* $n \geq n_0$,

$$\left|A - \lambda n - A_n(\hat{h}^{(\lambda,c)})\right| + B_n(\hat{h}^{(\lambda,c)}, 2) + \left(\sum_{jk \leq n-n_0} \frac{|h_j''(k; \lambda, v)|^\beta}{j^k k!} \right)^{1/\beta} \ll_\beta v.$$

For the proof, as in [18] and [10], we will exploit concentration function and sieve estimates. Denote $h_j(1) = a_j$, $a_j(\lambda) = a_j - j\lambda$, and

$$Q_n(u) = \sup_{x \in \mathbf{R}} \nu_n(|h(\sigma) - x| < u), \quad u \geq 0.$$

Set

$$D_n(u; \lambda) = \sum_{j=1}^n \frac{u^2 \wedge a_j(\lambda)^2}{j}, \qquad D_n(u) = \min_{\lambda \in \mathbf{R}} D_n(u; \lambda).$$

Lemma 4 *We have*

$$Q_n(u) \ll u D_n(u)^{-1/2}.$$

The constant in the symbol $\ll$ *is absolute.*

Proof See [13].

Let J be an arbitrary subset of $\{1, \ldots, n\}$, maybe, depending on n and $\bar{J} = \{1, \ldots, n\} \setminus J$. Define

$$S_n^J = \{\sigma \in \mathbf{S}_n \colon k_j(\sigma) = 0 \; \forall j \in J \;\&\; k_i(\sigma) \leq 1 \; \forall i \in \bar{J}\}$$

and

$$S_n^{j,k} := \{\sigma \in \mathbf{S}_n \colon \; k_j(\sigma) = k, \; k_i(\sigma) = 0 \; \forall i \in J \setminus \{j\} \\ \& \, k_l(\sigma) \leq 1 \; \forall l \in \bar{J} \setminus \{j\}\}.$$

Given a constant $K \geq 0$, set $\mu_n(K) = \min_J \nu_n(S_n^J)$, where minimum is taken over all subsets J satisfying the condition

$$\sum_{j \in J} \frac{1}{j} \leq K. \tag{14}$$

Lemma 5 *Let* $J = J_n \subset \{j \colon \; j \leq n\}$ *be an arbitrary nonempty set satisfying* (14). *There exists a number* $n_1 = n(K) \in \mathbf{N}$ *and a constant* $c(K) > 0$ *such that, for* $n > n_1$, *we have* $\mu_n(K) > c(K)$ *and*

$$\nu_n\left(S_n^{j,k}\right) \geq c(K) \frac{1}{j^k k!}$$

for all

$$(j,k) \in I := I(J, n - n_1) := ((J \times \{1\}) \cup \{(j,k) \in \mathbf{N}^2 \colon k \geq 2\}) \\ \cap \{(j,k) \in \mathbf{N}^2 : jk \leq n - n_1\}.$$

Proof See [14].

Proof of Theorem 3 An application of Lemma 4 gives the first needed relation. If $u_0 := 2^{1/\beta} M_n(h, A, \beta) = 2^{1/\beta} v > 0$, then by Markov's inequality

$$Q_n(u_0) \geq 1 - \nu_n(|h(\sigma) - A| \geq u_0) \geq 1 - u_0^{-\beta} M_n^{\beta}(h, A, \beta) = 1/2.$$

Consequently, by Lemma 4 we have $D_n(u_0, \lambda) \ll u_0^2$ with some $\lambda \in \mathbf{R}$ depending on h, n, and β. This value of λ will be used up to the end of the proofs. By virtue of (4), as in the upper estimation, we may substitute $h^{(\lambda)}$ for h. For simplicity, we just assume that $\lambda = 0$. We also preserve the previous notation of functions $h'(\sigma) = \hat{h}^{(0,v)}(\sigma)$, and $h''(\sigma) = \check{h}^{(0,v)}(\sigma)$. Observe that we also have

$$D_n(v, 0) \leq D_n(u_0, 0) \ll u_0^2 = 2^{2/\beta} v^2. \tag{15}$$

This leads to

$$B_n^2(h', 2) \leq D_n(v, 0) + \sum_{\substack{jk \leq n, k \geq 2 \\ |h_j(k)| \leq v}} \frac{h_j^2(k)}{j^k k!} \ll (2^{2/\beta} + 1) v^2. \tag{16}$$

The last estimate trivially holds if $v = 0$ because of $h'(\sigma) \equiv 0$ in this case.

Set

$$J := \{j \leq n : |a_j| > v\} = \{j \leq n : h_j''(1; 0, v) \neq 0\}.$$

Again from (15), we obtain

$$\sum_{\substack{j \leq n \\ |a_j| > v}} \frac{1}{j} \ll 2^{2/\beta}. \tag{17}$$

So, condition (14) is satisfied with K depending on β. We may apply Lemma 5 keeping the notation of the sets S_n^J, $S_n^{j,k}$, and I. Now n_1 depends on β. If $\sigma \in S_n^J$, then $h''(\sigma) = 0$. Hence

$$M_n^{\beta}(h'', A - A_n(h'), \beta) \geq |A - A_n(h')|^{\beta} \nu_n\big(S_n^J\big) \gg_{\beta} |A - A_n(h')|^{\beta} \tag{18}$$

provided that $n \geq n_1 = n_1(\beta)$. If $\sigma \in S_n^{j,k}$, then $h''(\sigma) = h_j''(k; 0, v)$. Lemma 5 now implies

$$M_n^{\beta}(h'', A - A_n(h'), \beta) \gg_{\beta} \sum_{(j,k) \in I} \frac{|h_j''(k; 0, v)) - (A - A_n(h'))|^{\beta}}{j^k k!}.$$

Combining this with (18), by (11) and (17), we obtain

$$\begin{aligned}\sum_{\substack{jk \leq n - n_1 \\ |h_j(k)| > v}} \frac{|h_j(k)|^{\beta}}{j^k k!} &\ll_{\beta} M_n^{\beta}(h'', A - A_n(h'), \beta) \\ &\quad + |A - A_n(h')|^{\beta} \sum_{\substack{jk \leq n \\ |h_j(k)| > v}} \frac{1}{j^k k!} \ll_{\beta} M_n^{\beta}(h'', A - A_n(h'), \beta) \\ &\ll_{\beta} v^{\beta} + M_n^{\beta}(h', A_n(h'), \beta).\end{aligned} \tag{19}$$

In either of the cases $\beta < 2$ or $\beta \geq 2$, using Theorem 1 and (16), we have

$$M_n(h', A_n(h'), \beta) \ll_{\beta} v.$$

This inequality together with (18) and (19) imply the desired estimate in Theorem 3 with $\lambda = 0$. As we have observed, by virtue of (4), this does not cause generality.

Theorem 3 is proved.

Concluding remark Under the extra condition $h_j(k; \lambda) = 0$ for $n - n_2 \le jk \le n$, where $n_2 \in \mathbf{N}$ is sufficiently large, the obtained upper and lower estimates of moments differ just by constants depending on β. We stress that, using the quantities $A_n(h)$ and $B_n(h, \beta)$, one can not obtain such result without any restriction. For instance, by the definition, $M_n(h, A, \beta)$ depends on the values $h_j(k_j)$, where $\ell(\bar{k}) = n$. Since there is no permutation satisfying $k_1(\sigma) = n - 1$, the value $h_1(n-1)$ never appears in the moment. Consequently, estimating the moment more exactly, one must exclude such value at least. A detailed analysis leads to some complicated formulations, however.

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