INEQUALITIES FOR THE p-th MOMENT, p, $0 < p < 2$, OF A SUM OF INDEPENDENT RANDOM VARIABLES

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Let $X_1, \ldots, X_n$ be independent random variables (r.v.), $EX_k = 0$, $k = \overline{1, n}$, and $S_n = X_1 + X_2 + \ldots + X_n$. For $p \geqslant 2$ estimates are known for $E|S_n|^p$, giving the exact order of growth with respect to n (cf. [1], [2], and also [3], [4], or [5]). We set

$$D_p(n) = \Big(\sum_{k=1}^{n} EX_k^2\Big)^{p/2} + \sum_{k=1}^{n} E|X_k|^p.$$

Then, not paying attention to constants, the corresponding result can be written in the form

$$c'(p) D_p(n) \leqslant E|S_n|^p \leqslant c''(p) D_p(n),$$

where $c'(p)$ and $c''(p)$ are certain positive quantities depending on p. Here of course it is required that $E|X_k|^p < \infty$, $k = \overline{1, n}$.

When $0 < p \leqslant 2$ one has

$$E|S_n|^p \leqslant \Big(\sum_{k=1}^{n} EX_k^2\Big)^{p/2}, \quad E|S_n|^p \leqslant 2 \sum_{k=1}^{n} E|X_k|^p \tag{1, 2}$$

(cf. [6]) under the assumption of finiteness of the corresponding moments. It is easy to see that neither of the quantities on the right gives the exact order with respect to n. Using both inequalities, we can get a satisfactory upper estimate, however estimation below with the help of the inequalities of [7] (cf. also [5]), as was done in [2], meets with considerable difficulty.

The goal of the present paper is to offer another method for estimating below for $p > 0$. We recount it, getting an optimal estimate of $E|S_n|^p$ in the sense of the dependence on n, when $0 < p < 2$. The original idea of the method arose in probabilistic number theory for getting a lower estimate in the familiar Kubilyus Inequality for additive arithmetic functions [8].

Let $F_k(u)$ be the distribution function of the r.v. X_k, $k = \overline{1, n}$. In what follows, a tilde over a r.v. or a distribution function will denote symmetrization. Let us assume that $E|X_k|^p < \infty$ for $k = \overline{1, n}$. For $t \geqslant 0$ we set

$$\Lambda_n(p, t) = \Big(\sum_{k=1}^{n} \int_{|u|<t} u^2 d\tilde{F}_k(u)\Big)^{p/2} + \sum_{k=1}^{n} \int_{|u|\geqslant t} |u|^p d\tilde{F}_k(u)$$

and

$$\Lambda_n(p) = \inf_{t\geqslant 0} \Lambda_n(p, t).$$

THEOREM 1. Let $X_1, \ldots, X_n$ be independent random variables $EX_k = 0$ and $E|X_k|^p < \infty$ for $1 \leqslant p < 2$ and $k = \overline{1, n}$. Then there exist positive quantities $c_1(p)$ and $c_2(p)$, depending only on p, such that

$$c_1(p) \Lambda_n(p) \leqslant E|S_n|^p \leqslant c_2(p) \Lambda_n(p). \tag{3}$$

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Proof. It is known [6] that for a r.v. X for $1 \leqslant p < 2$, $EX = 0$ and $E|X|^p < \infty$

$$\frac{1}{2} E|\tilde{X}|^p \leqslant E|X|^p \leqslant E|\tilde{X}|^p. \tag{4}$$

For any $t \geqslant 0$ and $k = \overline{1, n}$ we set $X'_k = \tilde{X}_k$ if $|\tilde{X}_k| < t$ and $X'_k = 0$ if $\tilde{X}_k \geqslant t$. Let $X''_k = \tilde{X}_k - X'_k$. Now, using (1) and (2), we have

$$E|S_n|^p \leqslant E|\tilde{S}_n|^p \leqslant 2^{p-1} E\left|\sum_{k=1}^{n} X'_k\right|^p + 2^{p-1} E\left|\sum_{k=1}^{n} X''_k\right|^p \leqslant 2^p \Lambda_n(p, t)$$

for any $t \geqslant 0$. Whence follows the right inequality of (3).

We now see that there exists a quantity $t_n = t_n(p) \geqslant 0$ such that

$$E|S_n|^p \geqslant c_1(p) \Lambda_n(p, t_n).$$

We shall repeatedly refer to the inequality

$$E|X+Y|^p \geqslant E|X|^p, \quad p \geqslant 1, \tag{5}$$

which holds for independent r.v. X and Y, for which $E|X|^p < \infty$, $E|Y|^p < \infty$ and $EY = 0$ (cf. [9]). If $E|S_n|^p = 0$, then (5) implies $E|X_k|^p = 0$, so $\Lambda_n(p) = \Lambda_n(p, 0) = 0$ and the theorem is proved. If $E|S_n|^p \neq 0$, we set $\Psi_n^p = 2^{p+1} E|S_n|^p$, $Y_k = \Psi_n^{-1} X_k$, $k = \overline{1, n}$, $S'_n = Y_1 + \ldots Y_n$. Let $V_k(u) = F_k(u\Psi_n)$ be the distribution function of the r.v. Y_k, $k = \overline{1, n}$. We have that $E|S'_n|^p = 2^{-p-1}$, so Markov's inequality gives a lower estimate of the concentration function

$$\sup_y P\{S'_n \in [y, y+1]\} \geqslant 1 - P\{|S'_n| \geqslant 1/2\} \geqslant 1/2.$$

The latter in conjunction with Theorem 3 of [5] implies the inequality

$$\sum_{k=1}^{n} \int_{|u|<1} u^2 d\tilde{V}_k(u) + \sum_{k=1}^{n} \int_{|u|\geqslant 1} d\tilde{V}_k(u) \leqslant 4A^2, \tag{6}$$

where A is an absolute positive constant.

We set $\xi_k = \tilde{Y}_k$, if $|\tilde{Y}_k| < 1$, and $\xi_k = 0$, if $|\tilde{Y}_k| \geqslant 1$. Let $\eta_k = \tilde{Y}_k - \xi_k$, $k = \overline{1, n}$. Noting that $E\xi_k = 0$, $k = \overline{1, n}$, we get from (1) and (6),

$$E\left|\sum_{k \in K} \xi_k\right|^p \leqslant (2A)^p \tag{7}$$

for any $K \subset \{k, 1 \leqslant k \leqslant n\}$. Let $L = \{k, 1 \leqslant k \leqslant n, P\{\eta_k \neq 0\} \geqslant 1/2\}$ and $\bar{L} = \{k, 1 \leqslant k \leqslant n\} \backslash L$. From (6) follows the following inequality for the number of elements of the set L:

$$|L| \leqslant 8A^2. \tag{8}$$

We note further that from (4) we have the inequality

$$E\left|\sum_{k=1}^{n} \tilde{Y}_k\right|^p \leqslant 2E|S'_n|^p = 2^{-p},$$

which with (5) implies

$$E\left|\sum_{k \in L} \tilde{Y}_k\right|^p \leqslant 2^{-p} \text{ and } E\left|\sum_{k \in \bar{L}} \tilde{Y}_k\right|^p \leqslant 2^{-p}. \tag{9}$$

This allows us to estimate $E\left|\sum_{k \in \bar{L}} \eta_k\right|^p$. In fact, in view of (7), we have

$$E\left|\sum_{k \in \bar{L}} \eta_k\right|^p \leqslant 2^{p-1} E\left|\sum_{k \in \bar{L}} \tilde{Y}_k\right|^p + 2^{p-1} E\left|\sum_{k \in \bar{L}} \xi_k\right|^p \leqslant 2^{-1} + 2^{2p-1} A^p = c_3(p). \tag{10}$$

Here and below $c_j(p)$, $j = 3, 4, \ldots$ denote positive quantities depending only on p. Any of them can be calculated explicitly. For the quantity on the left in (10) we apply the idea of the sieve, arising from the above-mentioned problem of probabilistic number theory [8]. First we define the events. For $j \in L$ we set

$$A_j=\{\eta_j\neq 0\}\cap\{\eta_i=0,\ \forall i\in\bar{L},\ i\neq j\}=\{\eta_j\neq 0\}\cap A_j'.$$

Let $\chi(\ )$ be the indicator of the event given in parentheses. Noting that the events A_j for distinct $j \in \bar{L}$ are disjoint, we deduce from (10) that

$$c_3(p)\geqslant\sum_{j\in\bar{L}}E\left(\Big|\sum_{k\in\bar{L}}\eta_k\Big|^p\chi(A_j)\right)=\sum_{j\in\bar{L}}E\left(|\eta_j|^p\chi(A_j)\right)=$$
$$=\sum_{j\in\bar{L}}E\left(|\eta_j|^p\chi(A_j')\right)=\sum_{j\in\bar{L}}E|\eta_j|^pP(A_j')\geqslant\prod_{i\in\bar{L}}P\{\eta_i=0\}\sum_{j\in\bar{L}}E|\eta_j|^p.$$

But in view of (6) and the inequality $\ln(1-x)\geqslant -2x$, $0\leqslant x\leqslant 1/2$,

$$\prod_{i\in\bar{L}}P\{\eta_i=0\}\geqslant\exp\left\{-2\sum_{i\in\bar{L}}P\{\eta_i\neq 0\}\right\}\geqslant e^{-8A^2},$$

so

$$\sum_{j\in\bar{L}}E|\eta_j|^p\leqslant c_3(p)e^{8A^2}=c_4(p). \tag{11}$$

For $k \in L$ one gets an analogous estimate even more easily. From (9) and (7) we have

$$E\Big|\sum_{k\in L}\eta_k\Big|^p\leqslant 2^{p-1}E\Big|\sum_{k\in L}\tilde{Y}_k\Big|^p+2^{p-1}E\Big|\sum_{k\in L}\xi_k\Big|^p\leqslant 2^{-1}+2^{2p-1}A^p=c_3(p).$$

But $E\eta_k = 0$, $k=\overline{1,n}$, so from (5) follows the inequality $E|\eta_k|^p \leqslant c_3(p)$ for any $k \in L$. Now (8) and (11) give the estimate

$$\sum_{k=1}^{n}E|\eta_k|^p\leqslant 8A^2c_3(p)+c_4(p)=c_5(p). \tag{12}$$

We turn to the variables $\tilde{Y}_k$. From (6) and (12) we conclude

$$B_n(p)=\Big(\sum_{k=1}^{n}\int_{|u|<1}u^2d\tilde{V}_k(u)\Big)^{p/2}+\sum_{k=1}^{n}\int_{|u|\geqslant 1}|u|^pd\tilde{V}_k(u)\leqslant(2A)^p+c_5(p)=c_6(p).$$

But since

$$B_n(p)=\Psi_n^{-p}\Lambda_n(p,\Psi_n)=(2^{p+1}E|S_n|^p)^{-1}\Lambda_n(p,\Psi_n),$$

one has from the last inequality that the assertion of Theorem 1 follows with $c_1(p)=2^{-p-1}c_6^{-1}(p)$.

We study the case $0 < p < 1$. It is now natural to set that the medians of the summable r.v. are equal to zero. Let μX be the median of the r.v. X. One has the following

THEOREM 2. Let $X_1, \ldots, X_n$ be independent random variables, $\mu X_k=0$, $E|X_k|^p<\infty$ for $k=\overline{1,n}$ and $0 < p < 1$. Then there exists a positive quantity $c(p)$, depending only on p, such that

$$c(p)\Lambda_n(p)\leqslant E|S_n|^p\leqslant 2\Lambda_n(p).$$

The proof of Theorem 2 coincides word for word with the proof of Theorem 1, only instead of (4) and (5) one uses special cases of the following simple lemma.

LEMMA. Let X and Y be independent random variables, $E|X|^p < \infty$, $E|Y|^p < \infty$, where $0 < p < 1$, and $\mu Y = 0$. If A is the event defined by the random variable X, then

$$E\left(|X|^p\chi(A)\right)\leqslant 2E\left(|X+Y|^p\chi(A)\right).$$

Proof of the Lemma. For a r.v. Z for which $E|Z|^p < \infty$, Markov's inequality gives

$$P\{|Z|\geqslant(2E|Z|^p)^{1/p}\}\leqslant 1/2.$$

Consequently, $|\mu Z| \leqslant (2E|Z|^p)^{1/p}$, and for any real x we have

$$|x|^p = |\mu(x+Y)|^p \leqslant 2E|x+Y|^p.$$

Hence

$$E\left(|X+Y|^p \chi(A)\right) = \int\limits_A dF_X(x) \left(\int\limits_{-\infty}^{\infty} |x+y|^p dF_Y(y) \right) \geqslant \frac{1}{2} \int\limits_A |x|^p dF_X(x) = \frac{1}{2} E\left(|X|^p \chi(A)\right).$$

The lemma is proved.

We omit the rest of the details of the proof of Theorem 2. We note only that the estimation below by this method is carried out by word for word repetition of arguments. In addition one should enlarge the integrals in the definition of $\Lambda_n(p, t)$ to the corresponding moments and pass to nonsymmetrized random variables.

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