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Asymptotic value distribution of additive functions defined on the symmetric group

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Abstract We examine the asymptotic value distribution of additive functions defined via the multiplicities of lengths of the cycles comprising a random permutation taken from the symmetric group with equal probability. We establish necessary and sufficient conditions for the weak law of large numbers and for the relative compactness of the sequence of distributions. Considering particular cases, we demonstrate that long cycles play an exceptional role and that, sometimes, in order to obtain a Poisson limit law, their influence must be negligible. The proofs are based on the ideas going back to the seminal papers of I.Z. Ruzsa on the classical additive arithmetic functions.

Keywords Symmetric group · Random permutation · Weak law of large numbers · Relative compactness · Poisson law

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1 Introduction and results

In recent years there has been an increasing interest in random combinatorial structures. The symmetric group, the set of all mappings of a finite set into itself, general assemblies, multisets, selections, additive arithmetical semigroups and other structures (see [1]) endowed with a probabilistic measure can be studied by applying various analytic and probabilistic approaches. Limit theorems for the number of components or for more general additive mappings on these structures have much in common. Especially striking is the similarity of such theorems with the results of

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probabilistic number theory (see [7, 8] or [20]). Nevertheless, some of the number theoretical ideas still have not been exhibited in combinatorial settings. One of the goals of this paper is to popularize them. We are convinced that these approaches can be developed for general logarithmic combinatorial structures (see [1]).

We deal with asymptotic value distribution problems of mappings defined on the symmetric group $\boldsymbol{S}_n$ with respect to the uniform probability measure

$$\nu_n(\ldots) = (n!)^{-1}\#\{\sigma \in \boldsymbol{S}_n\colon \ldots\}$$

as $n \to \infty$. Let $\sigma \in \boldsymbol{S}_n$ be an arbitrary permutation and

$$\sigma = \varkappa_1 \cdots \varkappa_\omega \tag{1.1}$$

be its representation as the product of independent cycles $\varkappa$ and $\omega = \omega(\sigma)$ be the number of the cycles comprising σ, which is unique up to the order of the cycles. If $k_j(\sigma)$, $1 \le j \le n$, denotes the number of cycles of length j in (1.1), then

$$l(\sigma) := 1k_1(\sigma) + \cdots + nk_n(\sigma) = n. \tag{1.2}$$

This makes the components of the *structure vector* $\bar{k}(\sigma) = (k_1(\sigma), \ldots, k_n(\sigma))$ dependent with respect to the frequency ν_n. Let ξ_j, $j \ge 1$, be independent Poisson random variables (r.v.'s) given on some probability space $\{\Omega, \mathcal{F}, P\}$ with $\mathbf{E}\xi_j = 1/j$. Then it is known [1] that

$$\nu_n\big(\bar{k}(\sigma) = \bar{k}\big) = P\bigg((\xi_1, \ldots, \xi_n) = \bar{k} \,\bigg|\, \sum_{j \le n} j\xi_j = n\bigg), \tag{1.3}$$

where $\bar{k} \in \mathbf{Z}^{+^n}$, and

$$(k_1(\sigma), \ldots, k_n(\sigma), 0, \ldots) \quad \overset{\nu_n}{\Rightarrow} \quad (\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots)$$

in the sense of convergence of the finite dimensional distributions. Here and in what follows we assume that $n \to \infty$. On the other hand, the fundamental lemma (see Theorem 2 in [2]) says that the total variation distance satisfies

$$\frac{1}{2}\sum_{k_1,\ldots,k_r \ge 0} |\nu_n(k_1(\sigma) = k_1, \ldots, k_r(\sigma) = k_r) - P(\xi_1 = k_1, \ldots, \xi_r = k_r)| = o(1) \tag{1.4}$$

if and only if $r = o(n)$. For more precise estimates of the remainder in (1.4) in terms of r/n see [1].

Many statistical properties of a random $\sigma \in \boldsymbol{S}_n$ can be expressed via the structure vector. Here we are concerned with sequences of *additive functions* (statistics) $h^n\colon \boldsymbol{S}_n \to \mathbf{R}$. To define these functions, we start with arbitrary real arrays $\{h_{nj}(k)\}$, $1 \le j \le n$, $k \ge 0$, satisfying the condition $h_{nj}(0) \equiv 0$, and set

$$h^n(\sigma) = \sum_{j=1}^{n} h_{nj}(k_j(\sigma)).$$

If, in addition, $h_{nj}(k) = a_{nj}k$ with some $a_{nj} \in \mathbf{R}$ for all $k \geq 1$ and $1 \leq j \leq n$, then $h^n(\sigma)$ is called a *completely additive* function. Similarly, using products instead of the sums we define multiplicative functions.

To motivate our interest in additive functions, we recall a few results. V.L. Goncharov [10–12] examined the function $w(\sigma) = k_1(\sigma) + \cdots + k_n(\sigma)$ and proved that

$$\nu_n(\omega(\sigma) - \log n < x\sqrt{\log n}) \to \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du.$$

P. Erdős and P. Turán [9] dealt with the additive function equal to the sum of the logarithms of the different cycle lengths. This function serves as a good approximation to the non-additive function $\log Ord(\sigma)$, where $Ord(\sigma)$ denotes the group–theoretic order of the permutation σ. Both of these functions, after appropriate normalization, obey the normal law. The latest development on the Erdős-Turán problem can be found in the papers by our Ph.D. student V. Zacharovas [37, 38].

An interesting example of an additive function has appeared in the recent investigations [13] and [36] of distribution of the number $N(\sigma; x, y)$ of eigenvalues on the unit circumference between the points $e^{2\pi ix}$ and $e^{2\pi iy}$, $0 \leq x < y \leq 1$, of the permutation matrix associated to σ. We have the relation

$$N(\sigma; x, y) - n(y - x) = \sum_{j=1}^{n} \big(\{xj\} - \{yj\}\big) k_j(\sigma) =: W(\sigma),$$

where $\{a\}$ denotes the fractional part of $a \in \mathbf{R}$. Again, the appropriately normalized completely additive function $W(\sigma)$ obeys the normal law.

Finally, many problems on the distribution of the number of permutations having cycles with constraints on the lengths actually concern additive or multiplicative functions. In this direction, we list the papers by A.I. Pavlov [28, 29] and A.L. Jakymiv [16, 17], to name but few.

The general problem on the value distribution of additive functions can be stated as follows:

Let $h^n(\sigma)$ be a sequence of additive functions defined via $\{h_{nj}(k)\}$. Under what conditions do the frequencies

$$F_n(x) := \nu_n(h^n(\sigma) - \alpha(n) < x)$$

for some centralizing sequence $\alpha(n)$ converge weakly to a limit distribution function?

By virtue of (1.2), this problem concerns sums of dependent random variables. A few attempts to solve this problem were taken in [1, 2, 18, 19, 21], and other papers. The invariance principle was examined in a series of papers by the author and G.J. Babu [3–6]. Nevertheless, so far, we do not have necessary and sufficient conditions under which $F_n(x)$ converges to a given law. To describe the obstacles in obtaining such a result, we quote the author's theorem [21].

Before stating this result it is worthwhile to make a few observations. When dealing with $F_n(x)$, we can always assume that $h_{nj}(k) = 0$ for $jk > n$. Observe also that, if $h^n(\sigma) = h(\sigma)/\beta(n)$, where $h(\sigma)$ is an additive function which does not depend on n and $\beta(n) \to \infty$ (or, more generally, if $h_{nj}(k) = o(1)$ for each fixed $j, k \geq 1$), then we can examine the distribution of the completely additive functions defined by $a_{nj} = h_{nj}(1)$ only. Further, the functions $h^n(\sigma)$ under consideration can have a deterministic component $\lambda l(\sigma) \equiv \lambda n$, the influence of which can be eliminated by a suitably chosen centralization. Therefore we introduce the shifted sequences

$$h_{nj}(k, \lambda) = h_{nj}(k) - \lambda jk, \qquad a_{nj}(\lambda) = a_{nj} - \lambda j = h_{nj}(1) - \lambda j,$$

defining the function $h^n(\sigma) - \lambda l(\sigma)$ and the r.v.'s

$$X_{nj}(\lambda) = a_{nj}(\lambda)\xi_j, \quad j \leq n.$$

Henceforth, when using this notation, we will omit the extra index n if no misunderstanding can arise. Put for brevity $u^* = (1 \wedge |u|)\operatorname{sgn} u$, where $a \wedge b := \min\{a, b\}$ and $a, b, u \in \mathbf{R}$. We recall that complete convergence of bounded nondecreasing functions $\Psi_n(u)$ to $\Psi(u)$ means convergence at all continuity points of the limit function $\Psi(u)$ and $\Psi_n(\pm\infty) \to \Psi(\pm\infty)$. In what follows we will use the symbol $\ll$ in the meaning of $O(\cdot)$.

Theorem A ([21]) *Let $h(\sigma) := h^n(\sigma)$ be a sequence of completely additive functions defined on $\mathbf{S}_n$ via a_j as above. Suppose that for some $\lambda := \lambda_n \in \mathbf{R}$,*

$$\sum_{j \leq n} \frac{a_j(\lambda)^{*2}}{j} \ll 1 \tag{1.5}$$

and

$$\sum_{\substack{nu \leq j \leq n \\ |a_j(\lambda)| \geq \varepsilon}} \frac{1}{j} = o(1) \tag{1.6}$$

for each fixed $0 < u < 1$ and $\varepsilon > 0$.

The following assertions are equivalent:

(i) *The distribution functions $F_n(x)$ converge weakly to a limit distribution.*

(ii) *The distribution functions*

$$P(X_{n1}(\lambda) + \cdots + X_{nn}(\lambda) - (\alpha(n) - n\lambda) < x)$$

converge weakly to a limit distribution.

(iii) *There exists a nondecreasing bounded function $\Psi(u)$ defined on $\bar{\mathbf{R}}$ such that*

$$\Psi_n(u) := \sum_{\substack{j \leq n \\ a_j(\lambda) < u}} \frac{a_j(\lambda)^{*2}}{j} \tag{1.7}$$

converges completely to $\Psi(u)$ *and*

$$\alpha(n) = n\lambda + \sum_{j=1}^{n} \frac{a_j(\lambda)^*}{j} + \alpha + o(1) \tag{1.8}$$

for some constant $\alpha \in \mathbf{R}$.

If the condition (iii) *is satisfied, the limit distribution for the sequences in* (i) *or* (ii) *is the same and its characteristic function has the form*

$$\exp\left\{it\alpha + \int_{\mathbf{R}} (e^{itu} - 1 - itu^*)u^{*^{-2}}\, d\Psi(u)\right\}, \quad t \in \mathbf{R}.$$

Using the result of A. Hildebrand [14], it is not difficult to prove that the class of limit distributions agrees with the family of infinitely divisible distributions.

Corollary *If the function* $\Psi_n(u)$ *defined in* (1.7) *with some* λ *satisfies*

$$\Psi_n(u) \to \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u < 0, \end{cases} \tag{1.9}$$

then taking $\alpha(n)$ *as in* (1.8) *with* $\alpha = 0$ *we have*

$$F_n(x) \to \Phi(x). \tag{1.10}$$

The Lindeberg type condition (1.9) implies the truncation condition (1.6). But, in contrast to the central limit theorem for sums of independent r.v.'s, it is not necessary for the convergence (1.10). A counterexample is presented in [3]. It shows that the strongly dependent summands of the additive function corresponding to long cycles can also produce the Gaussian distribution. Note also that, for this example, the limit distribution for the corresponding sum of independent r.v.'s in (ii) is not normal. A similar situation exists for the Poisson limit law (see [26]). For the convergence rate estimates in (1.10) we refer to the author's paper [22].

In this paper we solve the problem posed above in the case of degenerate law. Motivated by the impressive result of I.Z. Ruzsa [32] in probabilistic number theory, we present our theorem in a quantitative form. We recall that the Lévy distance of a random variable X defined on some probability space $\{\Omega, \mathcal{F}, P\}$ from the set of constants is the quantity

$$L(X; P) := \inf\{\varepsilon + P(|X - a| \geq \varepsilon) \colon a \in \mathbf{R}, \varepsilon > 0\}.$$

In particular, let $L(h; \nu_n) =: L_n(h)$ and set $h_j(k, \lambda) = h_j(k) - \lambda jk$,

$$U_n(h, \lambda) := \sum_{jk \leq n} \frac{h_j(k, \lambda)^{*^2}}{j^k k!},$$

$U_n(h) = \min\{U_n(h, \lambda) \colon \lambda \in \mathbf{R}\}$, and $V_n(h) = 1 \wedge U_n(h)$.

Theorem 1 *For every additive function $h(\sigma)$ we have*

$$V_n(h) \ll L_n(h) \ll V_n(h)^{1/3}, \tag{1.11}$$

where the implied constants are absolute.

Observe that $L_n(h) = L_n(h - \lambda l)$ for every $\lambda \in \mathbf{R}$. The upper estimate, actually, will be derived from the inequality

$$L_n(h) \ll L(S_n(\lambda); P), \tag{1.12}$$

where $S_n(\lambda) = h_1(\xi_1, \lambda) + \cdots + h_n(\xi_n, \lambda)$ is the sum of independent r.v.'s.

We now give the answer to the above question in the case of degenerate distribution.

Corollary *Let $h(\sigma) := h^n(\sigma)$ be a sequence of additive functions on $\mathbf{S}_n$ defined via $\{h_j(k)\}$ as above. The frequencies $F_n(x)$ converge to the degenerate distribution at the point* 0 *if and only if*

$$U_n(h, \lambda) = o(1) \tag{1.13}$$

for some $\lambda = \lambda_n \in \mathbf{R}$ and $\alpha(n)$ has the form

$$\alpha(n) = n\lambda + \sum_{j \le n} \frac{a_j(\lambda)^*}{j} + o(1). \tag{1.14}$$

In connection with (1.12), one might guess that the converse estimate

$$\min_{\lambda \in \mathbf{R}} L(S_n(\lambda); P) \ll L_n(h)$$

also holds. Since the case of independent r.v.'s is well understood, this would imply the lower estimate in Theorem 1.

Using similar ideas going back to a paper of J. Šiaulys [33], we can examine relative compactness of the sequence of distribution functions. Recall that a sequence $\{F_n(x)\}$, $n \ge 1$, is relatively compact if every sequence of natural numbers $n' \to \infty$ contains a subsequence $n'' \to \infty$ such that $F_{n''}(x)$ converges weakly to a limit distribution function. The sequence is tight if for every $\varepsilon > 0$ there exists x_ε such that $F_n(x_\varepsilon) - F_n(-x_\varepsilon) > 1 - \varepsilon$ for $n \ge 1$. It is well known that tightness is equivalent to relative compactness. We derive the relevant criteria.

Theorem 2 *The following assertions are equivalent*:

(I) *The sequence of distribution functions $F_n(x)$ is relatively compact.*

(II) *For some $\lambda = \lambda_n$ and $\tilde{\alpha}(n)$, the sequence*

$$P(S_n(\lambda) - \tilde{\alpha}(n) < x)$$

is relatively compact.

(III) *For some* $\lambda := \lambda_n$,

$$\lim_{u\to\infty}\limsup_{n\to\infty} \sum_{\substack{jk\le n\\ |h_j(k,\lambda)|\ge u}} \frac{1}{j^k k!} + \frac{1}{u^2}\sum_{\substack{jk\le n\\ |h_j(k,\lambda)|<u}} \frac{h_j(k,\lambda)^2}{j^k k!} = 0.$$

Theorems 1 and 2 demonstrate some closeness between limit theorems for additive functions on the symmetric group and those for sums of independent r.v.'s. Nevertheless, obtaining necessary and sufficient conditions for the convergence of $F_n(x)$ to a non-degenerated limit law, will likely be much more difficult. This we demonstrate by proving the following result.

Theorem 3 *Let* $h(\sigma) := h^n(\sigma)$ *be a sequence of completely additive functions defined on* $\mathbf{S}_n$ *via* $a_j = h_j(1) \in \{0,1\}$ *as above and* $\Pi_a(x)$ *be the Poisson distribution function with parameter* $a > 0$. *The distributions* $F_n(x) := \nu_n(h(\sigma) < x)$ *converge weakly to* $\Pi_a(x)$ *if and only if*

$$\sum_{\substack{j\le n\\ a_j=1}} \frac{1}{j} = a + o(1) \tag{1.15}$$

and

$$\sum_{\substack{\varepsilon n<j\le n\\ a_j=1}} \frac{1}{j} = o(1) \tag{1.16}$$

for each fixed $0 < \varepsilon < 1$.

The condition (1.16) means that the influence of long cycles must be negligible. This is related to the additional assumption $a_j \in \{0,1\}$. In this regard, it is worth recalling the following example from author's paper [26].

Let $0 < a < -\log\log(e/2)$ and

$$d_m = (1/2)\exp\left\{e^{-a}\sum_{r=1}^{m}\frac{a^r}{r!}\right\}, \quad m \in \mathbf{N}.$$

The sequence of completely additive functions $h^n(\sigma)$ defined via

$$a_j := \begin{cases} m & \text{if } nd_{m-1} < j \le nd_m,\ 1\le m\le n,\\ 0 & \text{otherwise} \end{cases}$$

obeys the limit law $\Pi_a(x)$. Here only the cycles with lengths in $(n/2, n]$ are used in the definition of $h^n(\sigma)$, so condition (1.16) of Theorem 3 or (1.6) of Theorem A are not satisfied. Excluding cycles with lengths in the interval $(n/2, n]$ we can go even further.

Theorem 4 *Let $h(\sigma) := h^n(\sigma)$ be a sequence of completely additive functions defined on $\boldsymbol{S}_n$ via $a_j = h_j(1) \in \mathbf{Z}$, $|a_j| \le K < \infty$, for some constant $K \in \mathbf{N}$, and let $\Pi_a(x)$ be the Poisson distribution function with parameter $a > 0$. Assume that*

$$\sum_{\substack{n/2<j\le n\\ a_j\ne 0}} \frac{1}{j} = o(1). \tag{1.17}$$

The distributions $F_n(x) := \nu_n(h(\sigma) < x)$ converge weakly to $\Pi_a(x)$ if and only if conditions (1.15, 1.16) of Theorem 3, and

$$\sum_{\substack{j\le n\\ a_j\le -1}} \frac{1}{j} = o(1), \qquad \sum_{\substack{j\le n\\ a_j\ge 2}} \frac{1}{j} = o(1) \tag{1.18}$$

are satisfied.

We see that, under certain conditions, sequences of additive functions defined via $h_{nj}(k)$, where $j \in [\varepsilon n, n]$ and $\varepsilon = \varepsilon_n = o(1)$, or in $j \in (n/2, n]$, can obey the Poisson limit law. Nevertheless, we confess that we have failed to obtain this law for the sequences defined via $h_{nj}(k)$ under the condition $j \in [\varepsilon n, n/2]$.

2 Auxiliary lemmas

We need a few auxiliary results obtained mainly by the author. Here, as above, we allow the function $h(\sigma)$ to depend on n but do not indicate this dependence.

Lemma 1 *As above, let ξ_j, $1 \le j \le n$, be independent Poisson r.v.'s, $\mathbf{E}\xi_j = 1/j$, $h(\sigma)$ be an additive function defined by $\{h_j(k)\}$, $a \in \mathbf{R}$, and $u \ge 0$. Then*

$$\nu_n(|h(\sigma) - a| \ge u) \le 32e^2 P\left(\left|\sum_{j=1}^{n} h_j(\xi_j) - a\right| \ge u/3\right). \tag{2.1}$$

Proof See [23]. □

Integrating over $[0, \infty)$ inequality (2.1) multiplied by u^l, we extend estimates of the power moments known for sums of independent r.v.'s. Set

$$\mathbf{E}_n f(\sigma) = \frac{1}{n!} \sum_{\sigma \in \boldsymbol{S}_n} f(\sigma)$$

for a function $f : \boldsymbol{S}_n \to \mathbf{R}$. Applying Rosenthal's inequality (see [30]) we obtain the following result.

Corollary *For arbitrary $l \geq 2$, we have*

$$\mathbf{E}_n|h(\sigma) - A(n;h)|^l \ll_l \left(\sum_{jk\leq n} \frac{h_j^2(k)}{j^k k!}\right)^{l/2} + \sum_{jk\leq n} \frac{|h_j(k)|^l}{j^k k!},$$

where $A(n;h)$ is either of the sums

$$A_1(n;h) := \sum_{jk\leq n} \frac{h_j(k)}{j^k k!}, \qquad A_2(n;h) := \sum_{j\leq n} \frac{h_j(1)}{j}.$$

Set $x_{(r)} = x(x-1)\cdots(x-r+1)$ if $r \in \mathbf{N}$.

Lemma 2 *For arbitrary natural numbers $j_1 < \cdots < j_m$ and $l_1, \ldots, l_m$, we have*

$$\mathbf{E}_n\big(k_{j_1}(\sigma)_{(l_1)} \cdots k_{j_m}(\sigma)_{(l_m)}\big) = \mathbf{1}\{j_1 l_1 + \cdots + j_m l_m \leq n\}\mathbf{E}\big(\xi_{j_1\,(l_1)} \cdots \xi_{j_m\,(l_m)}\big).$$

Proof See [1]. □

The next result concerns the concentration function. For an additive function $h(\sigma)$ defined via $h_j(k)$, $1 \leq j \leq n$, $k \geq 0$, as above, we set $a_j = h_j(1)$,

$$Q_n(u) = \sup_{x\in\mathbf{R}} \nu_n(|h(\sigma) - x| < u), \quad u \geq 0,$$

and $a_j(\lambda) = a_j - j\lambda$. Denote

$$D_n(u;\lambda) = \sum_{j\leq n} \frac{u^2 \wedge a_j(\lambda)^2}{j}, \qquad D_n(u) = \min_{\lambda\in\mathbf{R}} D_n(u;\lambda).$$

Lemma 3 *We have*

$$Q_n(u) \ll u D_n(u)^{-1/2} \tag{2.2}$$

with an absolute constant in $\ll$.

Proof For completely additive functions this result has been obtained in [27]. To generalize it, we recall the idea going back to [31]. The main steps were to prove the estimate

$$\varphi_n(t) := \frac{1}{n!}\sum_{\sigma\in\boldsymbol{S}_n} e^{ith(\sigma)} \ll \exp\left\{-c \min_{|u|\leq\pi} \sum_{j\leq n} \frac{1-\cos(a_j t - uj)}{j}\right\}, \tag{2.3}$$

where $c > 0$, and using it to obtain

$$Q_n(1) \ll \int_{-1}^{1} |\varphi_n(t)|dt \ll (D_n(1))^{-1/2}.$$

We will show that (2.3) remains true for arbitrary additive functions $h(\sigma)$. Hence the above inequality for $Q_n(1)$ also holds in the general case, and by rescaling (2.2) follows.

To establish (2.3), we apply a corollary of our estimate [25] of the Taylor coefficients of an analytic function in $|z|<1$ having a particular form. We present a few of the details. Let $[z^n]G(z)$ denote the nth Taylor coefficient of $G(z)$ and take

$$G(z)=\sum_{k\geq 0} b_k z^k \exp\left\{\sum_{j\geq 1}\frac{g_j z^j}{j}\right\}, \tag{2.4}$$

where $g_j, b_j \in \mathbf{C}$. As it has been proved in [25], if $|g_j|\leq 1$ for $1\leq j\leq n$ and

$$\sum_{j\leq n}|b_j|\leq B, \quad j|b_j|\leq b,\ 1\leq j\leq n, \tag{2.5}$$

then

$$[z^n]G(z)\ll \exp\left\{-c\min_{|u|\leq\pi}\sum_{j\leq n}\frac{1-\Re(g_j e^{-iuj})}{j}\right\} \tag{2.6}$$

for some absolute constant $c>0$ and the constant in $\ll$ depending on b and B only.

In our case, using the definition of an additive function, we have

$$1+\sum_{n\geq 1}\varphi_n(t)z^n=\prod_{j\geq 1}\left(1+\frac{g_j(1)}{j1!}z^j+\frac{g_j(2)}{j^2 2!}z^{2j}+\cdots\right)=:\prod_{j\geq 1}\chi_j(z),$$

where $g_j(k)=\exp\{ith_j(k)\}$. Thus, representing this function in the form (2.4), we take $g_j=g_j(1)$ and

$$\sum_{k\geq 0} b_k z^k:=\prod_{j\geq 1}\chi_j(z)\exp\left\{-\frac{g_j}{j}z^j\right\}. \tag{2.7}$$

So, it remains to verify (2.5).

It is easy to prove that

$$[z^n]\prod_{j\leq j_0}\chi_j(z)\ll n^{-2}$$

for arbitrary fixed $j_0\geq 1$. Since

$$|\chi_j(z)-1|\leq e^{1/j}-1\leq e^{1/j}j^{-1}\leq\sqrt{e}/2<1$$

for $j\geq j_0\geq 2$, we can take the main branch of logarithm and examine

$$H_1(z):=\sum_{j\geq j_0}\left(\log\chi_j(z)-\frac{g_j}{j}z^j\right)=\sum_{j\geq j_0}\sum_{k\geq 2}\frac{g_j(k)}{j^k k!}z^{jk}$$
$$+\sum_{j\geq j_0}\sum_{r\geq 2}\frac{(-1)^{r-1}}{r}\left(\sum_{k\geq 1}\frac{g_j(k)}{j^k k!}z^{jk}\right)^r=:H_2(z)+H_3(z).$$

Now $[z^n]H_2(z) \le 2n^{-2}$ for $n \ge 1$ and $[z^n]H_3(z) \le [z^n]H_4(z)$, where

$$H_4(z) = \sum_{j\ge j_0}\sum_{r\ge 2}\frac{1}{r}\left(\sum_{k\ge 1}\frac{1}{j^k k!}z^{jk}\right)^r = \sum_{j\ge j_0}\sum_{r\ge 2}\frac{1}{r}\sum_{k_1,\dots,k_r\ge 1}\prod_{i=1}^{r}\left(\frac{z^j}{j}\right)^{k_i}\frac{1}{k_i!}.$$

Hence

$$[z^n]H_4(z) = \sum_{\substack{j_0\le j\le n/2\\ j|n}} j^{-n/j}\sum_{2\le r\le n/j}\frac{1}{r}\sum_{\substack{k_1,\dots,k_r\ge 1\\ k_1+\cdots+k_r=n/j}}\prod_{i=1}^{r}\frac{1}{k_i!}.$$

The inner most sum is the number of surjections of an $m = n/j$ set onto an r set. Therefore it does not exceed $r^m/m!$. This and Stirling's formula imply

$$\begin{aligned}
[z^n]H_4(z) &\ll \sum_{\substack{j_0\le j\le n/2\\ j|n}}\left(\frac{e}{n}\right)^{n/j}\sum_{2\le r\le n/j} r^{n/j-1}\\
&\ll \sum_{\substack{j_0\le j\le n/2\\ j|n}}\left(\frac{e}{j}\right)^{n/j} \ll n^{-2} + \int_{j_0}^{n/3}\left(\frac{e}{x}\right)^{n/x}dx \ll n^{-2}
\end{aligned}$$

provided that $j_0 > e^2$. K.-H. Hwang [15] has noticed that $[z^n]G(z) \ll n^{-2}$ implies $[z^n]\exp\{G(z)\} \ll n^{-2}$ for an analytic in $|z| < 1$ function $G(z)$. In our case the above estimate leads to the inequality

$$[z^n]\prod_{j\ge j_0}\chi_j(z)\exp\left\{-\frac{g_j}{j}z^j\right\} \ll n^{-2}, \quad n \ge 1.$$

Collecting all of these estimates we see that the function (2.7) satisfies both of the conditions (2.5).

The estimate (2.6) implies (2.3) for the characteristic function. This, as we have mentioned above, yields the inequality for the concentration function for an arbitrary additive function. Lemma 3 is proved. □

The following local version is also of interest.

Lemma 4 *For an arbitrary additive function $h\colon \boldsymbol{S}_n \to \mathbf{R}$ we have*

$$\nu_n(h(\sigma) = b) \ll \left(\sum_{\substack{j\le n\\ a_j\ne\lambda j}}\frac{1}{j}\right)^{-1/2}$$

for some $\lambda \in \mathbf{R}$ with an absolute constant in $\ll$.

Proof We can take $0 < \nu := \nu_n(h(\sigma) = b) \le Q_n(u)$ with an arbitrary $u > 0$. Now (2.2) can be rewritten in the form

$$D_n(u) \ll u^2/Q_n(u)^2 \ll u^2/\nu^2$$

implying

$$\sum_{\substack{j \le n \\ |a_j(\lambda)| \ge u}} \frac{1}{j} \ll \frac{1}{\nu^2}.$$

Taking $u < \min\{|a_j(\lambda)|\colon a_j(\lambda) \ne 0, 1 \le j \le n\}$ we obtain the desired estimate.

Estimates of the number of permutations with different cycle lengths and missing some cycle lengths play also an important role. Let J be an arbitrary subset of $\{1, \dots, n\}$, possibly depending on n, $\bar{J} = \{1, \dots, n\} \setminus J$, and

$$\nu_n(J) := \nu_n(k_j(\sigma) = 0 \text{ for all } j \in J \text{ and } k_i(\sigma) \le 1 \text{ for all } i \in \bar{J}).$$

Given a constant $K \ge 0$, set $\mu_n(K) = \min_J \nu_n(J)$, where the minimum is taken over all subsets J satisfying the condition

$$\sum_{j \in J} \frac{1}{j} \le K. \tag{2.8}$$

□

Lemma 5 *We have*

$$\liminf_{n \to \infty} \mu_n(K) \ge \exp\{-e^{cK}\}$$

with some absolute constant $c > 0$.

Proof Without the condition $k_i(\sigma) \le 1$ for all $i \in \bar{J}$ this estimate has been obtained in [24]. The proof in our case requires only minor changes, so we omit it. □

Lemma 6 *Let $J = J_n \subset \{j\colon j \le n\}$ be an arbitrary nonempty set satisfying* (2.8). *There exists a number $n(K) \in \mathbf{N}$ such that, for $n > n(K)$, an arbitrary subset*

$$I \subset \big((J \times \{1\}) \cup \{(j,k) \in \mathbf{N}^2\colon k \ge 2\}\big) \cap \big\{(j,k) \in \mathbf{N}^2, jk \le n - n(K)\big\}$$

and

$$\widetilde{S_n} := \bigcup_{(j,k) \in I} S_n^{j,k},$$

where

$$S_n^{j,k} := \big\{\sigma \in \mathbf{S}_n\colon k_j(\sigma) = k, k_i(\sigma) = 0\ \forall i \in J \setminus \{j\} \text{ and } k_l(\sigma) \le 1\ \forall l \in \bar{J} \setminus \{j\}\big\},$$

we have $\mu_n(K) > c(K)$ and

$$\nu_n(\widetilde{S_n}) \ge c(K) \sum_{(j,k) \in I} \frac{1}{j^k k!},$$

where $c(K) > 0$ is a constant depending at most on K.

Proof From Lemma 5 we get $n(K) \in \mathbf{N}$ and $c_1(K) > 0$ such that $\mu_n(K) \geq c_1(K)$ if $n > n(K)$. We may assume that $c(K)$ is monotonically decreasing in K.

Set $J_m = J \cap [1, m]$ and $\bar{J}_m = \{j \in \mathbf{N}\colon j \leq m\} \setminus J_m$ for $0 \leq m \leq n$. We use (1.3) to obtain

$$\nu_n(S_n^{j,k}) = P\biggl(\xi_j = k, \xi_i = 0\ \forall i \in J \setminus \{j\}, \xi_l \leq 1\ \forall l \in \bar{J} \setminus \{j\} \biggm| \sum_{i \leq n} i\xi_i = n\biggr).$$

The probability of the condition equals

$$P_n := P\biggl(\sum_{i \leq n} i\xi_i = n\biggr) = \sum_{\substack{k_1,\dots,k_n \geq 0\\ 1k_1+\cdots+nk_n=n}} \prod_{i \leq n} \frac{e^{-1/i}}{i^k k!} = \exp\biggl\{-\sum_{i \leq n} \frac{1}{i}\biggr\} \tag{2.9}$$

by Cauchy's identity. If $j \leq m := n - jk$, then observing that $1\xi_1 + \cdots + n\xi_n = m$ implies $\xi_i = 0$ for each $m < i \leq n$ and using (2.9), we obtain

$$\begin{aligned}
\nu_n(S_n^{j,k}) &= \frac{e^{-1/j}}{j^k k!} \exp\biggl\{-\sum_{m<i\leq n} \frac{1}{i}\biggr\} P_n^{-1}\\
&\quad \times P\biggl(\xi_i = 0\ \forall i \in J_m \setminus \{j\}, \xi_l \leq 1\ \forall l \in \bar{J}_m \setminus \{j\}, \sum_{\substack{i \leq m\\ i \neq j}} i\xi_i = m\biggr)\\
&= \frac{1}{j^k k!} P\biggl(\xi_i = 0\ \forall\, i \in J_m, \xi_j = 0,\ \xi_l \leq 1\ \forall l \in \bar{J}_m \biggm| \sum_{i \leq m} i\xi_i = m\biggr)\\
&= \frac{1}{j^k k!} \nu_m(J_m \cup \{j\}) \geq c_1(K+1) \frac{1}{j^k k!}
\end{aligned} \tag{2.10}$$

in either of the cases $j \in J_m$ or $j \in \bar{J}_m$ if $m \geq n(K)$.

By the same argument one obtains this estimate for $m < j \leq n$. Since the sets $S_n^{j,k}$ for $(j, k) \in I$ are pairwise disjoint, summing up (2.10) we complete the proof of the lemma. □

3 Proof of Theorem 1

Inequality (1.12) follows from Lemma 1. Set $h'_j(k, \lambda) = h_j(k, \lambda)$ if $|h_j(k, \lambda)| < 1$ and $h'_j(k, \lambda) = 0$ otherwise. We have

$$P\biggl(S_n(\lambda) \neq \sum_{j \leq n} h'_j(\xi_j, \lambda)\biggr) \ll \sum_{\substack{jk \leq n\\ |h_j(k,\lambda)| \geq 1}} \frac{1}{j^k k!} \leq U_n(h)$$

and, by Cauchy's inequality,

$$\alpha(n) - n\lambda - \sum_{j \leq n} \mathbf{E}h'_j(\xi_j, \lambda) \ll U_n(h)^{1/2}. \tag{3.1}$$

Further applying Chebyshev's inequality to the sum of independent r.v.'s $h'_j(\xi_j,\lambda)$, $1\le j\le n$, we easily obtain the upper estimate (1.11).

The main difficulties lie in the lower estimate. If $L_n(h)\ge c>0$ for some constant c, the task is trivial. In what follows, let $\delta:=2L_n(h)<c$ for an absolute constant $0<c<1/2$ to be specified later. Thus, $\nu_n(|h(\sigma)-a|\ge\delta)\le\delta$ for some $a\in\mathbf{R}$ and

$$Q_n(\delta)\ge\nu_n(|h(\sigma)-a|<\delta)\ge 1-\delta.$$

Lemma 3 now yields

$$\sum_{j\le n}\frac{\delta^2\wedge a_j(\lambda)^2}{j}\ll\frac{\delta^2}{(1-\delta)^2}\ll\delta^2 \tag{3.2}$$

for some $\lambda\in\mathbf{R}$. Since $L_n(h)=L_n(h-\lambda l)=\delta/2$, we can further examine the function $h(\sigma)-\lambda l(\sigma)=h(\sigma)-\lambda n$ where $\sigma\in\boldsymbol{S}_n$. For simplicity, we just assume $\lambda=0$ and use inequalities (3.2) with $a_j(0)=a_j=h_j(1)$. Keeping this in mind, we split $h(\sigma)$ into a sum of two additive functions $h'(\sigma)$ and $h''(\sigma)$ now defined by

$$h'_j(k)=\begin{cases}h_j(k) & \text{if } |h_j(k)|<\delta,\\ 0 & \text{if } |h_j(k)|\ge\delta,\end{cases}$$

and $h''_j(k)=h_j(k)-h'_j(k)$, respectively. Adding some sums bounded by an absolute constant, we can rewrite (3.2) in the form

$$B^2(n;h'):=\sum_{jk\le n}\frac{h'_j(k)^2}{j^k k!}\le C_1\delta^2,\qquad \sum_{\substack{jk\le n\\ h''_j(k)\ne 0}}\frac{1}{j^k k!}\le C_1. \tag{3.3}$$

We now apply Lemma 6 with

$$J=\{j\le n:\ h''_j(1)\ne 0\}$$

and

$$I=\{(j,k):\ 1\le jk\le n-n_1,\ |h''_j(k)|\ge\sqrt{\delta}\},$$

where $n>n_1:=n(C_1)$. Let $S_n^{j,k}$ and $\widetilde{S}_n$ be defined as in Lemma 6. Then by (3.3) we obtain

$$\nu_n(\widetilde{\mathbf{S}}_n)\ge c_1\sum_{(j,k)\in I}\frac{1}{j^k k!}=:c_1\alpha$$

with a constant $c_1:=c(C_1)>0$ if $n>n_1$. Observe that

$$\sum_{\substack{jk\le n\\ |h_j(k)|\ge\sqrt{\delta}}}\frac{1}{j^k k!}-\alpha\le\sum_{n-n_1\le jk\le n}\frac{1}{j^k k!}\le\log\frac{n}{n-n_1}+o(1)=o(1).$$

If $\alpha \ll \delta$, then from the last estimate and (3.3) we obtain the desired inequality $U_n(h,0) \ll \delta$. Therefore in what follows we assume that $\alpha \geq M\delta$, where M is a fixed constant to be chosen later.

If $\sigma \in \mathbf{S}_n^{j,k}$, then $h''(\sigma) = h_j(k)$, where $|h_j(k)| \geq \sqrt{\delta}$. So, $|h''(\sigma)| \geq \sqrt{\delta}$ for each $\sigma \in \widetilde{\mathbf{S}}_n$. Hence, if $\sigma \in \widetilde{\mathbf{S}}_n$ and $|h(\sigma) - a| < \delta$, then $|h'(\sigma) - a| \geq \sqrt{\delta} - \delta$ and

$$\begin{aligned} \nu_n\big(|h'(\sigma) - a| \geq \sqrt{\delta} - \delta\big) &\geq \nu_n\big(\sigma \in \widetilde{\mathbf{S}}_n\big) - \nu_n\big(|h(\sigma) - a| \geq \delta\big) \\ &\geq c_1\alpha - \delta \geq (c_1 M - 1)\delta. \end{aligned} \tag{3.4}$$

Denote

$$\widehat{S_n} = \{\sigma \in \mathbf{S}\colon\ k_j(\sigma) = 0\ \forall j \in J \text{ and } k_j(\sigma) \leq 1\ \forall j \in \bar{J}\}.$$

By (3.3) and Lemma 6 we also have $\nu_n(\widehat{S_n}) \geq c_1 > 0$ if $n > n_1$. Hence

$$\begin{aligned} \nu_n(|h'(\sigma) - a| \leq \delta) &\geq \nu_n(\sigma \in \widehat{S_n}\colon\ |h(\sigma) - a| < \delta) \\ &\geq c_1 - \nu_n(|h(\sigma) - a| \geq \delta) \geq c_1 - \delta \geq c_1/2 \end{aligned} \tag{3.5}$$

if $\delta < \delta_0 \leq c_1/2$, where the choice of δ_0 is at our disposition. We now can apply the following estimate of the variance $\operatorname{Var} X$ of a random variable X. If $P(X \in A) \geq p_1$, $P(X \in B) \geq p_2$, and $d = \inf\{|x - y|\colon\ x \in A, y \in B\}$, then

$$\operatorname{Var} X = \frac{1}{2}\operatorname{Var}(X - X') = \frac{1}{2}\int_{\mathbf{R}}\int_{\mathbf{R}} (x - y)^2 dF(x)dF(y) \geq \frac{1}{2} p_1 p_2 d^2.$$

Here X' denotes an independent copy of X. Hence, for the variance of $h'(\sigma)$ with respect to ν_n, we obtain from (3.4) and (3.5)

$$\operatorname{Var} h'(\sigma) \geq \frac{1}{4}(c_1 M - 1)c_1\delta \tag{3.6}$$

if $n \geq n_1$. On the other hand, by the Corollary to Lemma 1 and (3.3), we have $\operatorname{Var} h'(\sigma) \ll B^2(n; h') \leq C_1\delta^2$. This contradicts (3.6) if M is sufficiently large. So, the inequality $U_n(h,0) \ll \delta$ is proved for $n > n_1$ and sufficiently small δ. In the remaining cases, it is trivial.

Theorem 1 is proved.

To prove Corollary, it suffices to exploit (3.1) and the assertion of Theorem 1.

4 Proof of Theorem 2

The equivalence of (II) and (III) is just a corollary of well known general results on relative compactness for sums of i.r.v.'s. The claim that (III) implies (I) follows from Lemma 1 and (II). Thus it remains to prove the implication (I) $\Rightarrow$ (II).

(I) $\Rightarrow$ (II). Since $F_n(x)$ is also tight, there exists a number $L \geq 1$ such that $Q_n(L) \geq 1/2$ for all $n \geq 1$. Thus, by Lemma 3, we obtain

$$\sum_{\substack{j \leq n \\ |h_j(1,\lambda)| < L}} \frac{h_j(1,\lambda)^2}{j} \ll L^2, \qquad \sum_{\substack{j \leq n \\ |h_j(1,\lambda)| \geq L}} \frac{1}{j} \ll 1 \tag{4.1}$$

for some $\lambda = \lambda_n \in \mathbf{R}$. Taking $\hat{h}(\sigma) = h(\sigma) - \lambda l(\sigma)$ and $\hat{\alpha}(n) = \alpha(n) - \lambda n$, by virtue of $F_n(x) = \nu_n(\hat{h}(\sigma) - \hat{\alpha}(n) < x)$, we reduce the problem to the case with $\lambda = 0$. Adding the summands corresponding to $k \geq 2$ we see that inequalities (4.1) with $\lambda = 0$ are equivalent to

$$\sum_{jk \leq n} \frac{h_j(k)^2 \wedge L^2}{j^k k!} \leq C_2 L^2. \tag{4.2}$$

Let δ, $0 < \delta < 1$, be arbitrary and $u_\delta > L/\sqrt{\delta}$ be such that

$$\nu_n\big(|h(\sigma) - \alpha(n)| \geq u_\delta\big) < \delta, \quad n \geq 1. \tag{4.3}$$

Define the additive functions $h'(\sigma)$ and $h''(\sigma)$ by

$$h'_j(k) = \begin{cases} h_j(k) & \text{if } |h_j(k)| < L, \\ 0 & \text{otherwise} \end{cases}$$

and by $h''_j(k) = h_j(k) - h'_j(k)$, respectively.

First we estimate $\alpha(n)$. Applying the Corollary of Lemma 1 to $h'(\sigma)$ and (4.2) we obtain

$$\begin{aligned} \frac{1}{n!} & \sum_{\substack{\sigma \in \boldsymbol{S}_n \\ |h(\sigma)-\alpha(n)|<u_\delta}} (h''(\sigma) - \alpha(n) + A(n;h'))^2 \\ & \leq \frac{2}{n!} \sum_{\substack{\sigma \in \boldsymbol{S}_n \\ |h(\sigma)-\alpha(n)|<u_\delta}} (h(\sigma) - \alpha(n))^2 + 2\mathbf{E}_n\big(h'(\sigma) - A(n;h')\big)^2 \\ & \ll u_\delta^2 + B^2(n;h') \ll u_\delta^2 + L^2 \leq C_3 u_\delta^2. \end{aligned} \tag{4.4}$$

Now we bound the summation region on the left-hand side of these inequalities. As in the proof of Theorem 1, we take

$$J = \{j \leq n \colon\ h''_j(1) \neq 0\}$$

and

$$\widehat{\boldsymbol{S}_n} = \big\{\sigma \in \mathbf{S} \colon\ k_j(\sigma) = 0\ \forall j \in J \text{ and } k_j(\sigma) \leq 1\ \forall j \in \bar{J}\}.$$

We can check that $\sigma \in \widehat{\boldsymbol{S}_n}$ implies $h''(\sigma) = 0$. By Lemma 6 and (4.2) we again have $\nu_n(\widehat{\boldsymbol{S}_n}) \geq c_2 := c(C_2) > 0$ for $n \geq n_2 := n(C_2)$. From this and (4.3) it follows that

$$\nu_n(\sigma \in \widehat{S}_n,\ |h(\sigma) - \alpha(n)| < u_\delta) \geq c_2 - \delta \geq c_2/2$$

if $\delta \leq c_2/2$ and $n > n_2$. This and (4.4), with $\widehat{\boldsymbol{S}_n}$ instead of $\boldsymbol{S}_n$, imply

$$|\alpha(n) - A(n;h')| \leq (2C_3/c_2)^{1/2} u_\delta =: C_4 u_\delta. \tag{4.5}$$

Let now $u > 2u_\delta$ and

$$I = \{(j,k)\colon 1 \le jk \le n - n_2,\ |h_j(k)| \ge u\}.$$

As in Lemma 6, let

$$S_n^{j,k} = \{\sigma \in \boldsymbol{S}_n\colon k_j(\sigma) = k, k_i(\sigma) = 0\ \forall i \in J \setminus \{j\} \text{ and } k_l(\sigma) \le 1\ \forall l \in \bar{J} \setminus \{j\}\}$$

and

$$\widetilde{\boldsymbol{S}_n} = \bigcup_{(j,k)\in I} S_n^{j,k}.$$

If $\sigma \in S_n^{j,k}$, then $h(\sigma) = h_j(k) + h'(\sigma)$. From this and (4.3) we obtain

$$\begin{aligned}
&\delta > \nu_n(|h(\sigma) - \alpha(n)| \ge u/2) \ge \nu_n(\sigma \in \widetilde{\boldsymbol{S}_n}\colon |h(\sigma) - \alpha(n)| \ge u/2) \\
&\quad = \sum_{(j,k)\in I} \nu_n(\sigma \in S_n^{j,k}\colon |h_j(k) + h'(\sigma) - \alpha(n)| \ge u/2) \\
&\quad \ge \sum_{(j,k)\in I} \nu_n(\sigma \in S_n^{j,k}\colon |h'(\sigma) - \alpha(n)| < u/2) \\
&\quad \ge c_2 \sum_{\substack{jk \le n-n_2 \\ |h_j(k)|\ge u}} \frac{1}{j^k k!} - \nu_n\big(|h'(\sigma) - \alpha(n)| \ge u/2\big)
\end{aligned}$$

for $n > n_2$. By virtue of (4.2), we could apply Lemma 6 in the last step. Using (4.5), the Corollary of Lemma 1, and completing the sum by the summands over $n - n_2 < jk \le n$, we further obtain

$$\begin{aligned}
2\delta &\ge c_2 \sum_{\substack{jk\le n \\ |h_j(k)|\ge u}} \frac{1}{j^k k!} - \nu_n(|h'(\sigma) - A(n;h')| \ge u/4) \\
&\ge c_2 \sum_{jk\le n} \sum_{|h_j(k)|\ge u} \frac{1}{j^k k!} - 16u^{-2}\mathbf{E}_n(h'(\sigma) - A(n;h'))^2 \\
&\ge c_2 \sum_{\substack{jk\le n \\ |h_j(k)|\ge u}} \frac{1}{j^k k!} - Cu^{-2}B^2(n;h')
\end{aligned}$$

for arbitrary $0 < \delta < (c_2/2)^*$, $u > \max\{4C_4, 2\}u_\delta =: v_\delta$ if $n > n_2' > n_2$, where $n_2' = n_2(\delta)$. This and (4.2) imply

$$\limsup_{n\to\infty} \sum_{\substack{jk\le n \\ |h_j(k)|\ge u}} \frac{1}{j^k k!} \ll \delta + L^2/u^2 \le C_5\delta \tag{4.6}$$

if $u \geq v_\delta$. Further, let $u > v_\delta/\sqrt{\delta}$. Estimating the sums when $|h_j(k)| \in [0, L] \cup (L, v_\delta] \cup (v_\delta, u]$, we obtain from (4.2) and (4.6)

$$\limsup_{n\to\infty} \frac{1}{u^2} \sum_{\substack{jk\leq n\\ |h_j(k)|<u}} \frac{h_j(k)^2}{j^k k!} \leq \frac{C_2 L^2}{u^2} + \frac{C_2 v_\delta^2}{u^2} + C_5\delta \ll \delta.$$

This and (4.6) yield the condition (III).

Theorem 2 is proved.

5 Proofs of Theorems 3 and 4

We now exploit some ideas proposed by J. Šiaulys [34, 35] in the context of number theoretic functions.

In either of the theorems the sufficiency parts follow from Theorem A.

The necessity part of Theorem 4. If

$$F_n(x) \Rightarrow \Pi_a(x), \tag{5.1}$$

then $\nu_n(h(\sigma)=0) = e^{-a} + o(1)$. Thus, by Lemma 4,

$$\sum_{\substack{j\leq n\\ a_j\neq \lambda j}} \frac{1}{j} \leq C(a) < \infty. \tag{5.2}$$

Since $a_j \in \mathbf{Z}$ and $|a_j| \leq K$, there must be a fixed integer b, $|b| \leq K$, such that

$$\sum_{j\leq n} \mathbf{1}\{a_j = b = \lambda j\}\frac{1}{j} \to \infty.$$

This is only possible in the case $b = \lambda = 0$. So (5.2) holds with $\lambda = 0$.

Let $J^-{:=}\{j\colon\ j\leq n, a_j \leq -1\}$, $J^+{:=}\{j\colon\ j\leq n, a_j\geq 1\}$, and $J = J^-\cup J^+$. Set

$$S_n^j = \{\sigma\in \mathbf{S}_n\colon\ k_j(\sigma) = 1, k_i(\sigma) = 0\ \forall i\in J\setminus\{j\}\}$$

and observe that $h(\sigma) = h_j(k_j(\sigma)) = h_j(1) = a_j \leq -1$ for each $\sigma\in S_n^j$ if $j\in J^-$. By Lemma 5, we obtain for $n\geq n_1{:=}n_0(C(a))$

$$o(1) = \nu_n(h(\sigma)\leq -1) \geq \nu_n\Bigg(\bigcup_{\substack{j\leq n-n_1\\ j\in J^-}} S_n^j\Bigg) \geq c(C(a)) \sum_{\substack{j\leq n-n_1\\ j\in J^-}} \frac{1}{j}.$$

This implies the first estimate of (1.18). Lemma 1 combined with this estimate yields

$$
\begin{aligned}
\nu_n\left(\left|h(\sigma)-\sum_{j\le n}a_j^+k_j(\sigma)\right|\ge\varepsilon\right)
&\ll P\left(\sum_{j\le n}|a_j-a_j^+|\xi_j\ge\varepsilon/3\right)\\
&\ll\sum_{a_j\le -1}P(\xi_j\ge 1)=\sum_{a_j\le -1}(1-e^{-1/j})=o(1)
\end{aligned}
\tag{5.3}
$$

for each $\varepsilon>0$. Thus, without loss of generality we may henceforth restrict ourselves to nonnegative functions $h(\sigma)$ defined via $a_j\ge 0$.

Next, we examine the power moments. Since a_j are bounded, by the Corollary of Lemma 1 and (5.2) with $\lambda=0$,

$$
\mathbf{E}_n|h(\sigma)|^l\ll_l\mathbf{E}_n\left|h(\sigma)-A_2(n;h)\right|^l+|A_2(n;h)|^l\ll_{l,K}1
$$

for $l=2,3,\ldots$. Trivially, this is also true for $l=1$. Since the Poisson law is uniquely determined by the sequence of its moments, the Fréchet and Shohat theorem and (5.1) imply

$$
\mathbf{E}_n(h(\sigma))^l\to\mathbf{E}X^l=\sum_{r=1}^{l}S(l,r)a^r,
\tag{5.4}
$$

where X is the random variable with distribution $\Pi_a(x)$ and $S(l,r)$ denotes the Stirling number of the second kind. Using Lemma 2 one can verify that, for $1\le i<j\le n$,

$$
\begin{aligned}
&\mathbf{E}_nk_j(\sigma)=\frac{1}{j},\qquad \mathbf{E}_nk_i(\sigma)k_j(\sigma)=\mathbf{1}\{i+j\le n\}\frac{1}{ij}\\
&\mathbf{E}_nk_j^2(\sigma)=\frac{1}{j}+\mathbf{1}\{j\le n/2\}\frac{1}{j^2}.
\end{aligned}
$$

Thus, direct calculations and (5.4) yield

$$
\mathbf{E}_nh(\sigma)=\sum_{j\le n}\frac{a_j}{j}=a+o(1)
\tag{5.5}
$$

and

$$
\mathbf{E}_nh(\sigma)^2=\sum_{j\le n}\frac{a_j^2}{j}+\sum_{i+j\le n}\frac{a_ia_j}{ij}=a^2+a+o(1).
\tag{5.6}
$$

Observe that (1.17) and (5.2) with $\lambda=0$ imply

$$
\sum_{\substack{i,j\le n\\ i+j>n}}\frac{a_ia_j}{ij}\le o_K(1)+2K^2\sideset{}{^*}\sum_{j\le n/2}\frac{1}{j}\sideset{}{^*}\sum_{n-j<i\le n}\frac{1}{i}=o_{K,a}(1),
$$

where the asterisk on the sums indicates the extra condition $a_i \geq 1$ or $a_j \geq 1$. Now (5.5) and (5.6) yield

$$\sum_{j\leq n} \frac{a_j^2}{j} = a + o(1). \tag{5.7}$$

Since by (5.2) the sums

$$s_n(k) := \sum_{\substack{j\leq n\\ a_j=k}} \frac{1}{j}, \quad k = 1, \dots, K,$$

are bounded, from an arbitrary increasing sequence $\{n'\}$ of natural numbers we can extract an infinite subsequence $\{n''\}$ such that $s_{n''}(k) = s_k + o(1)$ for each $k = 1, \dots, K$ as $n'' \to \infty$. By (5.5) and (5.7) the limits satisfy the relations

$$s_1 + 2s_2 + \cdots + Ks_K = a, \qquad s_1 + 4s_2 + \cdots + K^2 s_K = a.$$

Hence we see that the only possibility is $s_1 = a$, $s_2 = \cdots = s_K = 0$. This shows also the existence of the limits for $s_n(k)$ as $n \to \infty$ and the necessity of (1.15) and of the second estimate in (1.18). The argument shown in (5.3) allows us to reduce the problem for the sequence of additive functions defined via $a_j \in \{0, 1\}$. So at this place we continue with the proof of the necessity of the conditions of Theorem 3.

The necessity part of Theorem 3. As in the proof of Theorem 4, via (5.4) we arrive at (5.5) giving the necessity of (1.15) without the use of (1.17).

We now compute the power moments. In what follows let $j, j_1, \dots$ run through the set $\{j \in \mathbf{N}\colon a(j) = 1\}$, $l, r_1, \dots, r_m \in \mathbf{N}$, and let $S(l, m)$ denote the Stirling numbers of the second kind. Applying Lemma 2 and (5.4), we obtain

$$\begin{aligned}
&\sum_{r=1}^{l} S(l,r)a^l + o(1) = \mathbf{E}_n h(\sigma)^l \\
&\quad = \mathbf{E}_n\Big(\sum_j k_j(\sigma)\Big)^l \\
&\quad = \sum_{m=1}^{l} \frac{l!}{m!} \sum_{r_1+\cdots+r_m=l} \frac{1}{r_1!\cdots r_m!} \sum_{j_1}\sum_{j_2\neq j_1}\cdots\sum_{j_m\neq j_1,\dots,j_{m-1}} \mathbf{E}_n\big(k_{j_1}(\sigma)^{r_1}\cdots k_{j_m}(\sigma)^{r_m}\big) \\
&\quad = \sum_{m=1}^{l} \frac{l!}{m!} \sum_{r_1+\cdots+r_m=l} \frac{1}{r_1!\cdots r_m!} \sum_{j_1}\sum_{j_2\neq j_1}\cdots\sum_{j_m\neq j_1,\dots,j_{m-1}} \sum_{l_1=1}^{r_1} S(r_1,l_1) \\
&\qquad \cdots \sum_{l_m=1}^{r_m} S(r_m,l_m) \times \mathbf{1}\{j_1l_1+\cdots+j_ml_m\leq n\}\frac{1}{j_1^{l_1}\cdots j_m^{l_m}}.
\end{aligned} \tag{5.8}$$

The sum $\sum_j \xi_j$ is a Poisson r.v. with parameter $\sum_j 1/j = a + o(1)$. The identities in (5.8) yield the same formula for

$$\mathbf{E}\Big(\sum_j \xi_j\Big)^l = \sum_{r=1}^{l} S(l,r)\Big(\sum_j \frac{1}{j}\Big)^r = \sum_{r=1}^{l} S(l,r)a^r + o(1),$$

except for the lack of an indicator function on the last line of (5.8). Subtracting we arrive at

$$\sum_{m=1}^{l} \frac{l!}{m!} \sum_{r_1+\cdots+r_m=l} \frac{1}{r_1!\cdots r_m!} \sum_{j_1} \sum_{j_2\neq j_1} \cdots \sum_{j_m\neq j_1,\ldots,j_{m-1}} \sum_{l_1=1}^{r_1} S(r_1,l_1)\cdots \sum_{l_m=1}^{r_m} S(r_m,l_m)$$
$$\times \mathbf{1}\{j_1l_1+\cdots+j_ml_m>n\}\frac{1}{j_1^{l_1}\cdots j_m^{l_m}} = o(1).$$

We now check that, if in addition $j_1,\ldots,j_l > n/l$, then $j_1l_1+\cdots+j_ml_m > n$ and the indicator function equals one. This means that the iterated sum above contains the moment

$$\mathbf{E}\Big(\sum_{\substack{a_j=1\\ n/l<j\le n}} \xi_j\Big)^l$$

and it tends to zero as $n\to\infty$ for all $l\ge 1$. The last assertion implies that the factorial moments of all orders of the Poisson r.v. in the braces vanish also. Consequently,

$$\mathbf{E}\Big(\sum_{\substack{a_j=1\\ n/l<j\le n}} \xi_j\Big)_{(l)} = \Big(\sum_{\substack{a_j=1\\ n/l<j\le n}} \frac{1}{j}\Big)^l = o(1)$$

for arbitrary $l\ge 1$. The necessity of the condition (1.16) is proved.

Theorems 4 and 3 are proved.

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