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On a certain class of infinite products with an application to arithmetical semigroups

By

KARL-HEINZ INDLEKOFER, EUGENIJUS MANSTAVICIUS[1]) and RICHARD WARLIMONT

1. From J. Knopfmacher's monograph [3], Chapter 1, we quote the following

Definition. An *additive arithmetical semigroup* is a commutative semigroup G with identity element 1 together with a subset $\mathscr{P}$ and a mapping $\delta: G \to \mathbb{N}_0$, such that the following conditions hold:

(i) every element $a \neq 1$ in G admits a factorization into a finite product of elements of $\mathscr{P}$ which is unique, except for the order in which the factors occur,
(ii) $\delta(ab) = \delta(a) + \delta(b)$ for all $a, b \in G$,
(iii) $\delta(p) \in \mathbb{N}$ for all $p \in \mathscr{P}$,
(iv) $\pi(n) := |\{p \in \mathscr{P} : \delta(p) = n\}| < \infty$ for all $n \in \mathbb{N}$.

We put

$$\gamma(n) := |\{a \in G : \delta(a) = n\}|.$$

Then obviously $\gamma(0) = 1$ and $\gamma(n) < \infty$ for all n. More exactly we have

$$\gamma(n) \leqq \sum_{j=1}^{n} \pi(j)\gamma(n-j) \quad (n \in \mathbb{N}).$$

G is countable, and the identity

$$1 + \sum_{n=1}^{\infty} \gamma(n) t^n = \prod_{n=1}^{\infty} (1 - t^n)^{-\pi(n)}$$

(which should be interpreted within $\mathbb{Z}[[t]]$) holds true.

Very fundamental is Axiom $A^{\#}$ (see [3], p. 7) which says that there are real numbers $A > 0$, $q > 1$, $0 \leqq \nu < 1$ such that $\gamma(n) = Aq^n + O(q^{\nu n})$ as $n \to \infty$.

If G satisfies Axiom $A^{\#}$ then the function

$$f(z) := 1 + \sum_{n=1}^{\infty} \gamma(n) z^n \quad (|z| < 1/q)$$

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is holomorphic in the disc $|z| < q^{-v}$ up to a pole of order one in $1/q$. f can be considered as the zeta-function associated with (G, P, δ), and it has a Euler-product representation (cf. [3], Chapter 2)

$$f(z) = \prod_{n=1}^{\infty} (1 - z^n)^{-\pi(n)} \quad (|z| < 1/q).$$

Chapter 8 of [3] deals with a theorem called the *abstract prime number theorem*: *If the additive arithmetical semigroup G satisfies Axiom* $A^{\#}$ *then*

$$\pi(n) = \frac{q^n}{n} + O\left(\frac{q^n}{n^\alpha}\right) \quad as\ n \to \infty \tag{1}$$

is true for any $\alpha > 1$.

The estimation of the remainder term in (1) has quite recently been improved by S. D. Cohen [1] to $O(q^{\theta n})$ where $\max\{1/2, v\} < \theta < 1$.

Knopfmacher's arguments and Cohen's as well are based on Lemma 8.5 in [3] which states that f has no zeros on the circle $|z| = 1/q$. Its proof is modelled after the classic proof by de la Vallée Poussin that Riemann's zeta function $\zeta(s)$ satisfies $\zeta(1 + it) \neq 0$ for all $t \in \mathbb{R}$.

Unfortunately the proof is incomplete since it only yields $f(z) \neq 0$ for $|z| = 1/q$, $z \neq -1/q$, and this gap leaves the lemma and the prime number theorem in suspense.

As a matter of fact there are examples of additive arithmetical semigroups satisfying Axiom $A^{\#}$ such that $f(z)$ has a zero at $z = -1/q$ which affects the leading term in (1) considerably. Therefore the abstract prime number theorem in its current version is false.

The aim of our paper is to remedy the situation. We shall prove a correct version of the prime number theorem (the simplest case is formulated as a corollary to Theorem 1) and a necessary conditon for the existence of a zero at $-1/q$ (Theorem 2). As a corollary (Corollary 2) we obtain that the condition $f(x)f(-x) = o(1/(1 - q^{1/2}x))$ as $x \to q^{-1/2}$, $0 < x < q^{-1/2}$, ensures that $f(-q^{-1}) \neq 0$. Our example $f_3(z)$ shows that this result is sharp.

We depart from the original framework of additive arithmetical semigroups since it turned out that the subject matter rather belongs to the elementary theory of holomorphic functions which are represented as infinite products of a certain type.

2. We start with a sequence $\{\pi(n)\}$, $n \in \mathbb{N}$, of real numbers with the following property: There are constants $C > 0$ and $q > 1$ such that

$$|\pi(n)| \leqq Cq^n \quad \text{for all } n. \tag{2}$$

One can easily show that

$$(1 - z^n)^{-\pi(n)} - 1 \ll (q|z|)^n \quad \text{for} \quad |z| \leqq 1/q.$$

Therefore the infinite product

$$\prod_{n=1}^{\infty} (1 - z^n)^{-\pi(n)}$$

converges for $|z| < 1/q$ and yields there a holomorphic function f without zeros. The logarithmic derivative of f is given by

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} d\pi(d)\Big) z^{n-1}.$$

Furthermore, we have the power series expansion

$$f(z) = 1 + \sum_{n=1}^{\infty} \gamma(n) z^n \quad (|z| < 1/q).$$

By a straightforward calculation one can show

$$\pi(n) \in \mathbb{Z} \quad \text{for all } n \Leftrightarrow \gamma(n) \in \mathbb{Z} \quad \text{for all } n$$

and

$$\pi(n) \geqq 0 \quad \text{for all } n \Rightarrow \gamma(n) \geqq \pi(n) \quad \text{for all } n.$$

We now introduce an assumption (V) which corresponds to Axiom $A^{\#}$:

(V): *There is some $R > 1/q$ such that f can be analytically continued on the disc $|z| < R$ to a meromorphic function (which we still denote by f) which has exactly one pole of order* 1 *at $1/q$.*

If (V) holds we get

$$f(z) = \frac{B}{z - 1/q} + g(z)$$

where g is holomorphic for $|z| < R$ and $B \neq 0$. If

$$g(z) = \sum_{n=0}^{\infty} c_n z^n \quad (|z| < R)$$

then $c_n \ll r^{-n}$ for all r, $0 < r < R$, and

$$\gamma^{(n)} = A q^n + c_n, \quad \text{where } A := -Bq.$$

Conversely such an asymptotic formula for $\gamma(n)$ implies (V).

Theorem 1. *Let* (2) *and* (V) *be fulfilled. Put*

$$\sigma(n) := q^{-n} \sum_{d \mid n} d\pi(d).$$

Then the following two assertions hold:

(i) *If f has no zeros of modulus $1/q$ then for every r, $1/q < r < R$, one has*

$$\sigma(n) = 1 - \sum_{j=1}^{l} (qb_j)^{-n} + O((qr)^{-n}),$$

where $b_j (1 \leqq j \leqq l = l(r))$ are the zeros of f with $1/q < |b_j| < r$, counted according to their multiplicities.

(ii) *If* $\pi(n) \geqq 0$ *for all* n, *and if* f *has zeros of modulus* $1/q$ *then* f *has exactly one zero in the disc* $|z| < R$. *It is located at* $-1/q$ *and has order* 1. *Furthermore, in this case one has*

$$\sigma(n) = 1 - (-1)^n + O((qr)^{-n}) \quad \text{for every } r, 1/q < r < R.$$

Proof. Let ε, $0 < \varepsilon < R - 1/q$, be given. Then there is some r, $R - \varepsilon < r < R$, such that $f(z) \neq 0$ for $|z| = r$.

Let $a_j (1 \leqq j \leqq k)$ denote the zeros of f with $|a_j| = 1/q$ and $b_j (1 \leqq j \leqq l = l(r))$ those with $1/q < |b_j| < r$, all listed according to their multiplicities. We observe that the zeros of f lie symmetrically to the real line.

The residue theorem gives

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{f'(z)}{f(z)} \frac{dz}{z^n} = \sum_{d|n} d\pi(d) - q^n + \sum_{j=1}^{k} a_j^{-n} + \sum_{j=1}^{l} b_j^{-n}.$$

Therefore

$$\sigma(n) = 1 - \sum_{j=1}^{k} (qa_j)^{-n} - \sum_{j=1}^{l} (qb_j)^{-n} + O((qr)^{-n}).$$

In the first case ($k = 0$) all is done since there are no a_j's, and r may be chosen arbitrarily near to R.

Let us turn to the second case ($k \geqq 1$). We put $\varrho := \min_{1 \leqq j \leqq l} |b_j|$ if there are zeros b_j, and $\varrho := r$ otherwise. Furthermore, put $qa_j =: e(-\alpha_j)$ $(= \exp(-2\pi i \alpha_j)$ where $0 < \alpha_j < 1$ and observe that $f(q^{-1} e(-\alpha_j)) = 0$ implies $f(q^{-1} e(\alpha_j)) = 0$. Then we get

$$\sigma(n) = 1 - k + \sum_{j=1}^{k} (1 - e(n\alpha_j)) + O((q\varrho)^{-n}).$$

Dirichlet's approximation theorem shows that the sum on the right hand side may become arbitrarily small in absolute value for arbitrarily large n. Since $\sigma(n) \geqq 0$ for all n, k must be 1. Thus f has just one zero of modulus $1/q$. This zero must be real and equals $-1/q$. Thus

$$\sigma(n) = 1 - (-1)^n + O((q\varrho)^{-n}).$$

Assume now there exist zeros b_j. Let $b_j (1 \leqq j \leqq m \leqq l)$ denote those zeros with modulus ϱ. Put

$$\varrho' = \begin{cases} \min_{m < j \leqq l} |b_j|, & \text{if } m < l \\ r, & \text{otherwise}. \end{cases}$$

Further, put $b_j = \varrho e(-\beta_j)$ for $1 \leqq j \leqq m$. Then we have

$$\sigma(n) = 1 - (-1)^n - (q\varrho)^{-n} \sum_{j=1}^{m} e(n\beta_j) + O((q\varrho')^{-n}),$$

and therefore

$$\sigma(2n) = -(q\varrho)^{-2n} \left(m + \sum_{j=1}^{m} (e(n\beta_j) - 1)\right) + O((q\varrho')^{-2n}).$$

Dirichlet's approximation theorem will again produce a contradiction since $\sigma(2n)$ is always $\geqq 0$. Thus we get

$$\sigma(n) = 1 - (-1)^n + O((qr)^{-n}).$$

But r may be chosen arbitrarily close to R.

This ends the proof of Theorem 1.

Möbius's inversion formula gives

$$\pi(n) = \frac{1}{n} \sum_{d \mid n} \mu(n/d)\, q^d \sigma(d). \tag{3}$$

We use

$$\pi(n) = \frac{q^n}{n} \sigma(n) + O(q^{n/2})$$

and obtain a simple form of the abstract prime number theorem.

Corollary 1. *Let (G, P, δ) be an additive arithmetical semigroup satisfying Axiom* $\mathrm{A}^{\#}$. *Then there exist some θ,* $\max\{1/2, v\} < \theta < 1$, *such that either*

$$\pi(n) = \frac{q^n}{n} + O(q^{\theta n})$$

or

$$\pi(n) = \frac{q^n}{n}(1 - (-1)^n) + O(q^{\theta n})$$

holds. The first alternative takes place if and only if the associated zeta-function has no zero on the circle $|z| = 1/q$.

Remarks. 1) Theorem 1 and Formula (3) give of course a more exact prime number theorem than Corollary 1 with a remainder term of order $O(q^{\theta n})$ where $\theta = v + \varepsilon$ for every $0 < \varepsilon < 1 - v$, and the constant in $O(.)$ may depend on ε.

2) It is clear how a prime number theorem with remainder term $O(q^{\theta n})$ implies Axiom $\mathrm{A}^{\#}$ with $v > \theta$. In [2] a new Axiom $\bar{\mathrm{A}}^{\#}$ has been introduced which is equivalent to an exact formula for $\pi(n)$.

3. Here we consider some examples. To begin with we observe that

$$\begin{aligned}\log f(z) &= \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} d\pi(d)\Big) \frac{z^n}{n} \quad (|z| < 1/q)\\ &= \sum_{n=1}^{\infty} \pi(n) z^n + h(z),\end{aligned}$$

where h is holomorphic for $|z| < R := q^{-1/2}$.

Now, put

$$\pi_1(n) := [q^n/n]$$

$$\pi_2(n) := [q^n/n - b^n/n] \quad \text{where } q^{1/2} < b < q,$$

$$\pi_3(n) := \begin{cases} [2q^n/n] & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

(Additive arithmetical semigroups with prescribed $\pi(n)$ can easily be constructed.)

For the corresponding zeta functions f_1, f_2 and f_3, respectively, we find

$$f_1(z) = \frac{1}{1 - qz} H_1(z),$$

$$f_2(z) = \frac{1 - bz}{1 - qz} H_2(z),$$

$$f_3(z) = \frac{1 + qz}{1 - qz} H_3(z),$$

where $H_j(z)$ is holomorphic and $\neq 0$ for $|z| < R$ $(j = 1, 2, 3)$.

f_1 and f_2 are examples for the first case in Theorem 1, whereas f_3 is an example for the second case.

Remark. We observe that $H_3(z)$ has the form

$$H_3(z) = \left(\frac{1 + qz^2}{1 - qz^2}\right)^{1/2} \tilde{H}_3(z),$$

where $\tilde{H}_3(z)$ is holomorphic and different from zero in the closed disc $|z| \leqq q^{-1/2}$.

4. A necessary condition for the zeta function $f(z)$ having a zero at $-q^{-1}$ is given in

Theorem 2. *Assume* (2) *and* (V) *with* $R \geqq q^{-1/2}$, *and let* $\pi(n) \in \mathbb{N}_0$ *for all* n. *Then, if* $f(-q^{-1}) = 0$, *we have*

$$f(x) f(-x) \gg 1/(1 - q^{1/2} x) \quad \textit{for } x \to q^{-1/2}, \quad 0 < x < q^{-1/2}.$$

Proof. For $|z| < 1/q$ we have

$$f(z) f(-z) = g(z) f(z^2),$$

where

$$g(z) := \prod_{\substack{n=1 \\ n \text{ even}}}^{\infty} \left(\frac{1 + z^n}{1 - z^n}\right)^{\pi(n)}.$$

Since g is holomorphic for $|z| < 1/q$ and $g(0) = 1$ we have the power series expansion

$$g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (|z| < 1/q).$$

Because of

$$\frac{1+z^n}{1-z^n} = (1+z^n) \sum_{k=0}^{\infty} z^{nk}$$

and $\pi(n) \in \mathbb{N}_0$ all factors in the infinite product for g have coefficients from $\mathbb{N}_0$. Therefore the c_n's are also elements of $\mathbb{N}_0$. In particular they are all nonnegative.

The function

$$F(z) := \frac{1-qz}{1+qz} f(z)$$

is holomorphic for $|z| < R$.

Further the function $f(z^2)$ is holomorphic and $\neq 0$ for $|z| < q^{-1/2}$. Since $R \geqq q^{-1/2}$ the function g is holomorphic for $|z| < q^{-1/2}$ because of

$$g(z) = \frac{f(z) f(-z)}{f(z^2)} = \frac{F(z) F(-z)}{f(z^2)}.$$

Therefore the power series expansion of g introduced above holds still true for $|z| < q^{-1/2}$. Thus we obtain

$$g(x) \geqq 1 \quad \text{for} \quad 0 \leqq x < q^{-1/2},$$

which implies

$$f(x) f(-x) \geqq f(x^2) \quad \text{for} \quad 0 \leqq x < q^{-1/2}.$$

Since $f(z^2)$ has a pole of order one at $q^{-1/2}$ we see

$$f(x^2) \gg 1/(1 - q^{1/2} x) \quad \text{for} \quad 0 \leqq x < q^{-1/2}.$$

This ends the proof of Theorem 2.

Corollary 2. *Under the assumptions of Theorem 2 the following holds: If*

$$f(x) f(-x) = o(1/(1 - q^{1/2} x)) \quad \text{as } x \to q^{-1/2}, \quad 0 < x < q^{-1/2},$$

then f has no zeros on the circle $|z| = 1/q$.

Remark. The above example $f_3(z)$ shows that Corollary 2 is sharp, for

$$f_3(z) f_3(-z) = \frac{1+qz^2}{1-qz^2} \tilde{H}_3(z) \tilde{H}_3(-z)$$

where $\tilde{H}_3(z)$ is holomorphic and $\neq 0$ for $|z| \leqq q^{-1/2}$.

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Anschrift der Autoren:

K.-H. Indlekofer
Universität-GH Paderborn
D-4790 Paderborn

E. Manstavicius
Universität Vilnius
232006 Vilnius, Litauen

R. Warlimont
Universität Regensburg
D-8400 Regensburg

The relevance of a general approach to additive and multiplicative functions lies in the fact that many interesting results on the mentioned familiar arithmetic functions depend essentially only on the additivity or multiplicativity of the function in question and not on its particular definition.

Adolf Hildebrand

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