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DISTRIBUTION OF THE TRADITIONALLY NORMALIZED ADDITIVE FUNCTIONS

E. Manstavičius

1. Results.

Let $h(m)$ be a real–valued additive function,

$$A(x) = \sum_{p^k \le x} \frac{h(p^k)}{p^k}\,, \qquad B(x) = \left(\sum_{p^k \le x} \frac{h^2(p^k)}{p^k} \right)^{1/2},$$

where p denotes a prime number and $k \ge 1$. Put $I(\ldots)$ for the indicator of the set of natural numbers m satisfying the conditions written in the parenthesis. In the present paper we consider the conditions when there exist distribution functions $\nu(v)$ and $\mu(v)$ such that

$$\nu_x(v) := [\,x\,]^{-1} \sum_{m \le x} I\big(h(m) - A(x) < vB(x)\big) \Rightarrow \nu(v) \tag{1}$$

or

$$\mu_x(v) := \left(\sum_{m \le x} \frac{1}{m} \right)^{-1} \sum_{m \le x} \frac{1}{m}\, I\big(h(m) - A(x) < vB(x)\big) \Rightarrow \mu(v)\,. \tag{2}$$

Here and in what follows we suppose that $B(x) \to \infty$, the notation $\Rightarrow$ denotes weak convergence of bounded nondecreasing functions, and the notation of the limiting passage $x \to \infty$ is omitted.

In the first problem only the case of the improper limiting distribution concentrated at $v = a$ (denoted in the sequel by $E_a(v)$) has been investigated completely (see Ruzsa [16]). In the second problem, in addition to the law of large numbers (see Levin and Timofeev [9]), we have solved (Manstavičius [14]) the case of the limiting distribution having the zero mean and the unit variance. Now we shall summarize our results announced in several contributions (Manstavičius [10–13]), the theorems from the papers (Manstavičius [14], Timofeev [20]) will be generalized, and their proofs will be simplified considerably.

It seems likely that the conditions necessary for (1) or (2) with proper distribution functions $\nu(v)$, $\mu(v)$ should contain the existence of a nondecreasing bounded function $K(w)$ such that

$$\frac{1}{B^2(x)} \sum_{\substack{p \le x \\ h(p) < wB(x)}} \frac{h^2(p)}{p} \Rightarrow K(w)\,, \qquad K(-\infty) = 0\,. \tag{3}$$

Moreover, on the contrary to the theory of distribution of independent random variables, these conditions should involve (explicitly or not) the relations

$$\psi_x(u) := \frac{A(x^u) - A(x)}{B(x)} \to \psi(u)\,, \qquad \varphi_x(u) := \frac{B(x^u)}{B(x)} \to \varphi(u)\,, \tag{4}$$

valid uniformly in u belonging to any closed interval of the half–axis $(0,\infty)$, where $\psi(u)$ and $\varphi(u)$ are continuous functions. Then we had (see, Levin and Timofeev [9]) their expressions

$$\psi(u) = c\,(u^\rho - 1)\,, \qquad \varphi(u) = u^\rho\,, \tag{5}$$

where $c \in \mathbb{R}$ and $\rho \geq 0$ ($c = 0$ when $\rho = 0$).

Now the following question arises: *what does the constants c and ρ control*? The role of the limiting distribution should not to be overrated because we can have the standard normal limit distribution for $\nu_x(v)$, when $\rho = 0$ or $\rho = 1/2$ (see Timofeev [21]). As our result in (Manstavičius [14]) shows, the situation with $\mu_x(v)$ is pretty different.

We have observed the quantities determining c and p. Denote

$$\alpha_x = \frac{1}{B(x)\log x} \sum_{p^k \leq x} \frac{h(p^k)\log p^k}{p^k}\,,$$

$$\beta_x = \frac{1}{B^2(x)\log x} \sum_{p^k \leq x} \frac{h^2(p^k)\log p^k}{p^k}\,.$$

Definition. *An additive function $h(m)$ belongs to the class $\mathcal{H}(\alpha)$ if $\alpha_x \to \alpha$; $h(m)$ belongs to the class $\mathcal{H}(\alpha,\beta)$ if $\alpha_x \to \alpha$ and $\beta_x \to \beta$.*

We have $|\alpha| \leq \min\left\{1/\sqrt{2}\,, \sqrt{\beta}\right\}$ and $0 \leq \beta \leq 1$. The celebrated Kubilius $\mathcal{H}$ class (Kubilius [5]) agrees with $\mathcal{H}(0,0)$. As we shall see, in the case of the standard normalized additive functions the condition $h(m) \in \mathcal{H}(\alpha,\beta)$ substitutes for the following relation

$$\frac{1}{\log x} \sum_{\substack{p \leq x \\ h(p) < wB(x)}} \frac{\log p}{p} \Rightarrow F(w)\,,$$

where $F(u)$ is a distribution function, used in the paper (Levin and Timofeev [8]), or the condition (3) in Theorem 18.1 (Elliott [1]).

In the paper (Levin and Timofeev [9]) the problem of joint convergence of $\nu_x(v)$ and $\mu_x(v)$ has been solved. We modify this problem considering the weak convergence of either of the sequences together with convergence of some sequences of moments.

For a distribution function $F(v)$ put

$$a(F) = \int_{\mathbb{R}} v\, dF(v)\,, \qquad \sigma(F) = \int_{\mathbb{R}} v^2\, dF(v)\,.$$

We have $a(\nu_x) = o(1)$, $\sigma(\nu_x) \leq 3/2 + o(1)$ (see Kubilius [6], [7]), and $a(\mu_x) = -\alpha_x + o(1)$, $\sigma(\mu_x) \leq e^\gamma + o(1)$, where γ denotes the Euler constant (corollary of the estimate (13) in (Manstavičius [14])). Therefore the limiting distributions $\nu(v)$ and $\mu(v)$, when they do exist, will have the first two finite moments satisfying the following relations

$$a(\nu) = 0\,, \qquad \sigma(\nu) \leq \liminf \sigma(\nu_x) \leq 3/2\,,$$

$$a(\mu) = -\alpha\,, \qquad \sigma(\mu) \leq \liminf \sigma(\mu_x) \leq e^\gamma\,, \qquad a(\mu)^2 \leq \sigma(\mu)\,.$$

The first part of the following theorem will consider a special case of Theorem 1 of the paper (Levin and Timofeev [9]).

Theorem 1. *Let the conditions (3) and (4) be satisfied. Then*

$i_1)$ $\nu_x(v) \Rightarrow \nu(v)$ *and* $\mu_x(v) \Rightarrow \mu(v)$ *;*

$i_2)$ $h \in \mathcal{H}(\alpha, \beta)$ *, where*

$$\alpha = -\int_0^1 \psi(u)\,du = -a(\mu)\,, \qquad \beta = 1 - \int_0^1 \varphi^2(u)\,du < 1\,;$$

$i_3)$ $\varphi(u) = u^{\beta/2(1-\beta)}$ *,* $\psi(u) = 0$ *if* $\beta = 0$ *and* $\psi(u) = \alpha(2-\beta)/\beta\big(u^{\beta/2(1-\beta)} - 1\big)$ *if* $\beta \neq 0$ *;*

$i_4)$ $\sigma(\mu) - (2-\beta)\,a(\mu)^2 = (1-\beta)\,\sigma(\nu)$ *, where* $\sigma(\mu) \neq 0$ *if and only if* $K(+\infty) > 0$ *;*

$i_5)$ $\sigma(\nu_x) = o(1)$ *if and only if* $\alpha^2 = 1/2$ *.*

Suppose $h(m) \in \mathcal{H}(\alpha, \beta)$ *and* $\nu_x(v) \Rightarrow \nu(v) \neq E_0(v)$ *. Then the conditions (3) and (4) are satisfied and, in addition to the relations* $i_1) - i_4)$ *, we have* $\alpha^2 = a(\mu)^2 < 1/2$ *and* $\sigma(\mu) \neq (2-\beta)\,a(\mu)^2$ *.*

Thus, considering the necessity of the conditions (3) and (4) there is no need to take $\mu_x(v) \Rightarrow \mu(v) \neq E_a(v)$ in advance. The influence of the moment convergence show following results.

Theorem 2. *In order that* $\nu_x(v) \Rightarrow \nu(v) \neq E_0(v)$ *or* $\mu_x(v) \Rightarrow \mu(v) \neq E_a(v)$ *jointly with*

$$\sigma(\nu_x) \to \sigma_1 > 0\,, \qquad a(\mu_x) \to a\,, \qquad \sigma(\mu_x) \to \sigma_2 > 0 \tag{6}$$

it is necessary and sufficient that the condition (3), where $K(+\infty) > 0$ *, and the condition (4) be satisfied. Under these conditions the formulae (5) are true,* $h \in \mathcal{H}\Big(\frac{c\rho}{\rho+1}\,,\,\frac{2\rho}{2\rho+1}\Big)$ *,*

$$\sigma_1 = 1 + 2c^2\rho^2 \sum_{r=1}^{\infty} \frac{(-1)^r}{(r+\rho)^2} \prod_{s=1}^{r-1} \left(\frac{s-\rho}{s+\rho}\right)^2, \tag{7}$$

$$a = \frac{-c\rho}{\rho+1}\,, \qquad \sigma_2 = \frac{\sigma_1}{(1+2\rho)} + \frac{2a^2(1+\rho)}{(1+2\rho)}\,. \tag{8}$$

Moreover, the case $a = 0$ *,* $\sigma_1 = \sigma_2 = 1$ *occurs if and only if* $\rho = 0$ *,* $c = 0$ *.*

Remark. Considering $\mu_x(v) \Rightarrow \mu(v) \neq E_a(u)$ the second of the relations (6) is superfluous.

Corollary 1. *In order that* $\nu_x(v) \Rightarrow \nu(v) \neq E_0(u)$ *or* $\mu_x(v) \Rightarrow \mu(v) \neq E_a(u)$ *jointly with*

$$a(\mu_x) \to 0\,, \qquad \sigma(\mu_x) \to \sigma > 0 \tag{9}$$

it is necessary and sufficient that the conditions (3) with $K(+\infty) > 0$ *and (4) with* $\psi(u) = 0$ *,* $\varphi(u) = u^{(1-\sigma)/2\sigma}$ *,* $0 < \sigma \leq 1$ *, be satisfied. Under these conditions* $\sigma(\nu_x) \to 1$ *too.*

Corollary 2. (Manstavičius [14]). *In order that* $\mu_x(v) \Rightarrow \mu(v)$ *with* $a(\mu) = 0$ *and* $\sigma(\mu) = 1$ *it is necessary and sufficient that* $h \in \mathcal{H}(0,0)$ *and the condition (3) with* $K(+\infty) = 1$ *be satisfied.*

Sometimes the behavior of moments is given indirectly. In part it is shown by the following theorem.

Theorem 3. *Let* $\varepsilon > 0$ *be arbitrary, the star* $\star$ *denote the condition* $h(p) < -\varepsilon B(x)$, *and*

$$T_x := \frac{1}{\log x} \sum_{p \le x}^{\star} \frac{\log p}{p} \to 0\,.$$

In order that $\nu_x(v) \Rightarrow \nu(v)$ *with* $\sigma(\nu) = 1$ *it is necessary and sufficient that* $h \in \mathcal{H}(0,0)$ *and the condition (3) with* $K(+\infty) = 1$ *be satisfied.*

In order that $\nu_x(v) \Rightarrow \nu(v) \ne E_a(v)$ *jointly with* $\sigma(\nu_x) \to 1$ *it is necessary and sufficient that* $h \in \mathcal{H}(0,0)$ *and the condition (3) with* $K(+\infty) > 0$ *be satisfied. Under this condition* $\sigma(\nu) = K(+\infty)$.

This result in the case of the standard normal law has been announced by N. M. Timofeev ([20]). His rather complicated proof is presented in the thesis ([23]). Later a new attempt has been done (Timofeev [22]). We follow the idea of P. D. T. A. Elliott given in Supplement of the book [2]. Our criticism concerning the original calculations is expressed in Concluding remark at the end of this paper.

2. Proofs.

Our approach is based upon relations of the sequences of distribution functions $\nu_x(v)$ and $\mu_x(v)$. On the contrary to the paper (Levin and Timofeev [9]), we do not use their joint convergence in advance. We observe that either of them is relatively compact. Further, adding other conditions we prove that the convergence of one of the sequences implies that for the other one. Thus, then the result of (Levin and Timofeev [9]) can be applied.

To consider the relative compactness as well as features of the normalizing sequences $A(x)$ and $B(x)$, we shall apply Lemmas from our paper (Manstavičius [14]). While the proof of Lemma 1 [14] has been then omitted, having this opportunity we present it in more abstract setting influenced by the Ruzsa's paper [18]. That is motivated by our future plans too.

Let $\mathcal{G}$ be a commutative metric group with respect to additively written algebraic operation. Put $d\,(\,.\,,\,.)$ for the metric and $\|g\| = d\,(g, 0)$. We consider the asymptotic distribution of additive functions $f : \mathbb{N} \to \mathcal{G}$. In the set $\mathcal{M}(\mathcal{G})$ of probabilistic measures defined on the Borel σ-algebra $\mathcal{B}(\mathcal{G})$ of the sets of $\mathcal{G}$ we introduce the Lévy–Prokhorov metric

$$L(\eta_1, \eta_2) = \inf\Big\{\varepsilon \mid \eta_1(A) \le \eta_2(A^{\varepsilon}) + \varepsilon,\ \eta_2(A) \le \eta_1(A^{\varepsilon}) + \varepsilon;\ A \in \mathcal{B}(\mathcal{G})\Big\}\,,$$

where A^{ε} denotes the ε-neighborhood of the set A , and $\eta_1, \eta_2 \in \mathcal{M}(\mathcal{G})$. Let further, U_{ε} denote the ε-neighborhood of the zero element,

$$\Lambda(\eta) = \inf\big\{\varepsilon + \eta\,(\bar{U}_{\varepsilon}) \mid \varepsilon > 0\big\}\,, \qquad \eta \in \mathcal{M}(\mathcal{G})\,, \qquad \bar{U}_{\varepsilon} = \mathcal{G} \setminus U_{\varepsilon}\,.$$

The total variation of the signed measure Δ will be denoted by $|\Delta|$. Denote by $\delta(v)$ the unit mass at a point v and $\delta = \delta(0)$. Put

$$\Theta_y = [\,y\,]^{-1} \sum_{m\le y} \delta(f(m) - g)\,,$$

where the function $f = f_x$, $g = g_x \in \mathcal{G}$, and $y = y_x \to \infty$ as $x \to \infty$.

Lemma 1. *Let* $z = y^{1+u}$, $|u| \le 1/5$, $y_1 = y^{\sqrt{|u|}}$, $y \ge 3$,

$$R = \sum_{y_1<p\le y} \frac{1}{p}\,, \quad \tau_y = R^{-1} \sum_{y_1<p\le y} \frac{\delta\big(h(p)\big)}{p}\,.$$

Then

$$L(\Theta_y, \Theta_z) \ll \Lambda(\tau_y) + R^{-1/2}\,,$$

where the constant implied is absolute.

Let $t^\star = \min\big\{|\,t\,|\,, 1\big\}\,\mathrm{sgn}\,t$, where $t \in \mathbb{R}$. In the notations of Lemma 1 we obtain

Corollary. *We have*

$$L(\Theta_y, \Theta_z) \ll R^{-1/3} \Bigg(\sum_{p\le y} \frac{\|f(p)\|^{\star 2}}{p} \Bigg)^{1/3} + R^{-1/2}\,.$$

In the case $\mathcal{G} = \mathbb{R}$ having in advance a limiting distribution for the values $f_x(m) - \gamma_x$, $\gamma_x \in \mathbb{R}$, we can derive the boundiness of the sum in the last estimate, when $f(p)$ is replaced by $f(p) - \lambda_x \log p$ with some $\lambda_x \in \mathbb{R}$. Then applied for $f_x(m) - \gamma_x - \lambda_x \log(m/x)$ Corollary is quite useful. Lemma 2 below shows only one of its applications.

Proof of Lemma 1 is based on A. Hildebrand's arguments (see Lemma 4 in (Hildebrand [3])).

Observe that it is sufficient to deal with the case $0 < u \le 1/4$ only. Later, in the case $-1/5 \le u < 0$, one can use the inversions $z \to y$ and $y \to z$. For $0 \le u \le \log^{-3/2} y$ our estimate follows from the simple inequality

$$|\Theta_z - \Theta_y| \le \frac{(z - y + 1)}{[\,y\,]}\,. \tag{10}$$

In that follows we suppose $\log^{-3/2} y < u \le 1/4$ and y sufficiently large. Then

$$R - \frac{|\log u|}{2} \ll \big(\sqrt{u}\log y\big)^{-1} \ll \min\big\{u^{1/6}, \log^{-1/4} y\big\}\,. \tag{11}$$

We start with an averaging of Θ_y. For an arbitrary $b_m \in \mathbb{C},\ 1 \le m \le x$, we have the inequality (see Chapter 4 in Elliott [1])

$$\sum_{p\le y} p \Bigg| \sum_{\substack{m\le y\\ m\equiv 0 \bmod p}} b_m - \frac{1}{p}\sum_{m\le y} b_m \Bigg|^2 \ll y \sum_{m\le y} |b_m|^2 .$$

Applying it for $b_m = I\{f(m)\in A+g\}$ with arbitrary set $A\subset \mathcal{G}$ we obtain

$$\sum_{y_1<p\le y} p^{-1}\Big(\Theta_{y/p}\big(A-f(p)\big)-\Theta_y(A)\Big)^2 \ll 1 .$$

Now the Cauchy inequality yields

$$\Theta_y(A) = R^{-1}\sum_{y_1<p\le y} p^{-1}\Theta_{y/p}\big(A-f(p)\big) + BR^{-1/2} . \tag{12}$$

Here and in what follows B denotes some quantity bounded by an absolute constant $C>0$ (in different places we shall not use indeces). Hence for the probabilistic measure

$$\Xi_y := R^{-1}\sum_{y_1<p\le y} p^{-1}\Theta_{y/p}$$

we obtain

$$L(\Theta_y\,, \Xi_y) \ll \Lambda(\tau_y) + R^{-1/2} . \tag{13}$$

Let us repeat our considerations with Θ_z taking $z_1 = y^{\sqrt{u}+u}$ instead of the previous y_1. If τ_z is defined from τ_y, substituting $y\to z$ and $y_1 \to z_1$, via the estimate (10) and

$$R_1 := \sum_{z_1<p\le z}\frac{1}{p} = \log\frac{1+u}{u+\sqrt{u}} + \frac{B}{\sqrt{u}\log y} = R + Bu^{1/6} \tag{14}$$

we have $|\tau_y - \tau_z| \ll R^{-1}$. If Ξ_z is obtained from Ξ_y after these substitutions, then the same calculations as in the proof of (13) yield

$$L(\Theta_z\,, \Xi_z) \ll \Lambda(\tau_y) + R^{-1/2} . \tag{15}$$

In the next step we shall obtain the integral representations of the measures Ξ_z and Ξ_y. All the estimates will mean the estimates of the total variations of the signed measures. Since the primes $p\in(\sqrt{y}, y\,]$ contribute to Ξ_y the term BR^{-1} only, we divide the interval $(y_1, \sqrt{y}\,]$ by the points $w = w_k := (1+u)^k$, $k = 0, 1, \ldots$. For the typical interval (the last one may be incomplete) in virtue of the prime number theorem with logarithmical error term and (10) we have

$$\sum_{w<p\le w(1+u)} p^{-1}\Theta_{y/p} = \big(\Theta_{y/w}+Bu\big)w^{-1}(1+Bu)\Big(\pi\big(w(1+u)\big)-\pi(w)\Big) =$$

$$= \frac{(\Theta_{y/w}+Bu)u}{\log w} = \int_w^{w(1+u)} \frac{\Theta_{y/t}+Bu}{t\log t}\,dt .$$

Therefore summing these estimates with respect to k and adding one with respect to the interval $(\sqrt{y}\,, y)$ we obtain

$$\Xi_y = R^{-1} \int_{y_1}^{y} \frac{\Theta_{y/t} + Bu}{t \log t}\, dt + BR^{-1} = R^{-1} \int_{1}^{y/y_1} \frac{\Theta_t}{t\, \log y/t}\, dt + BR^{-1}\,.$$

Similarly, using the points $z_1(1+u)^k$, $k \geq 0$, we deduce

$$\Xi_y = R_1^{-1} \int_{1}^{z/z_1} \frac{\Theta_t}{t\, \log z/t}\, dt + BR_1^{-1}\,.$$

Hence and from the estimate (11), (14) in virtue of $y/y_1 = z/z_1$ we obtain

$$\big|\Xi_y - \Xi_z\big| \ll R^{-1} \int_{1}^{y/y_1} \frac{\log z/y}{\log^2(y/t)}\, \frac{dt}{t} + R^{-1} \ll R^{-1}\,.$$

Thus, in the case $\log^{-3/2} \leq u \leq 1/4$ the assertion of Lemma 1 follows from (13), (15), and the last estimate.

Now we return to $\mathcal{G} = \mathbb{R}$. Proofs of Theorems are based on few elementary observations.

Lemma 2. *Suppose the proposition (1) or (2) is valid with a proper limiting distribution. Then for each sequence $y_1 \to \infty$ there exist a subsequence $y \to \infty$, continuous functions $\psi(u)$ and $\varphi(u)$ given on $(0, +\infty)$ and such that the condition (4) is satisfied for the subsequence $x = y$.*

Proof. In the case (2) the proposition has been prooved in (Levin and Timofeev [9]). We shall consider the first case. The existence of a subsequence $y \to \infty$ such that $\varphi_y(u) \to \varphi(u)$ for $u > 0$ has been proved in Theorem 4 (Šiaulys [19]). The arguments given on page 379 of this paper (in virtue of $\mu(n) = 0$ there) yield, in fact, $\psi_y(u) \to \psi(u)$ too. Observe that a possibility to choose $y \to \infty$ such that $\varphi_y(u) \to \varphi(u)$ and $\psi_y(u) \to \psi(u)$ is contained in Lemma 3 of the paper (Manstavičius [14]). Our idea can be seen in the next step of the proof.

To prove continuity of $\psi(u)$ and $\varphi(u)$, introduce

$$\nu_x(u, v) = [x^u]^{-1} \sum_{m \leq x^u} I\big(h(m) - A(x) < vB(x)\big)\,.$$

In what foolows $L(.\,,\,.)$ denotes the Lévy metric in the distribution function space. We have

$$L\Big(\nu_y(u,\,.), \nu\big(\,.\,\varphi(u) + \psi(u)\big)\Big) \to 0$$

as $y \to \infty$. But then Corollary of Lemma 1 yields

$$L\Big(\nu\big(\,.\,\varphi(u+\delta)+\psi(u+\delta)\big)\,,\,\nu\big(\,.\,\varphi(u)+\psi(u)\big)\Big) \to 0$$

as $\delta \to 0$. Now by the well–known probabilistic lemma $\varphi(u+\delta) \to \varphi(u)$ and $\psi(u+\delta) \to \psi(u)$ as $\delta \to 0$. Lemma is proved.

Lemma 3. *Let for a subsequence $x = y \to \infty$ the relations (4) with some continuous functions $\psi(u)$ and $\varphi(u)$, $u > 0$, hold. If $h(m) \in \mathcal{H}(\alpha)$, then almost averywhere in the Lebesgue sense*

$$\psi'(u) = \alpha\left(\varphi'(u) + \frac{\varphi(u)}{u}\right). \tag{16}$$

If $h(m) \in \mathcal{H}(\alpha,\beta)$, then the functions $\psi(u)$ and $\varphi(u)$ do not depend on the subsequence $y \to \infty$, the formulae (5) are valid with $\rho = \beta/2(1-\beta)$, $\beta < 1$; $c = 0$ if $\beta = 0$ and $c = \alpha(2-\beta)/\beta$ if $\beta \neq 0$.

Proof. While $\varphi(u)$ is continuous and nondecreasing, it is differentiable almost everywhere. Further, summing by parts we have

$$A\big(y^{u+\Delta}\big) - A(y^u) = \frac{1}{(u+\Delta)\log y}\sum_{y^u<p\le y^{u+\Delta}}\frac{h(p)\log p}{p} + \int\limits_u^{u+\Delta}\sum_{y^u<p\le y^t}\frac{h(p)\log p}{p}\,\frac{dt}{t^2\log y}\ ,$$

where $0 < u-|\Delta| \le u+|\Delta| \le 1$. Dividing by $B(y)$ and letting $y \to \infty$ we derive

$$\begin{aligned}\psi(u+\Delta)-\psi(u) =&\alpha\big(\varphi(u+\Delta)-\varphi(u)\big)\frac{u}{u+\Delta} + \varphi(u+\Delta)\frac{\Delta}{u+\Delta} + \\ &+\alpha\int\limits_u^{u+\Delta}\frac{\varphi(t)\,dt}{t} - \alpha\varphi(u)\frac{\Delta}{u+\Delta}\ .\end{aligned}$$

Now dividing by Δ and letting $\Delta \to 0$ we obtain (16) almost everywhere in u .

Summation by parts (here and in what follows we do not indicate trivial estimates to be done at first in order to avoid the influence of the zero neighborhood) applied in the definition of β_y yields

$$\beta = 1 - \int\limits_0^1 \varphi^2(u)\,du\,, \tag{17}$$

which in virtue of $\varphi(1) = 1$ implies $\beta < 1$. The same idea as earlier leads to the relation

$$(1-\beta)\big(\varphi^2(u+\Delta)-\varphi^2(u)\big) = \beta\int\limits_u^{u+\Delta}\frac{\varphi^2(t)}{t}\,dt\,,$$

which gives the desired expression for $\varphi(u)$. Further, solving (16) we use the condition $\psi(1)=0$. Lemma 3 is proved.

Lemma 4. *Let for a subsequence* $x=y\to\infty$ *the relations (4) with some continuous functions* $\psi(u)$ *and* $\varphi(u)$, $u>0$, *hold. If* $a(\mu_x)\to a$, $\sigma(\nu_x)\to\sigma_1$, $\sigma(\mu)\to\sigma_2>0$, *then the functions* $\psi(u)$ *and* $\varphi(u)$ *do not depend on the subsequence* $y\to\infty$ *and the formulae (5) are true. We have two possibilities:*

$i_1)$ $\rho=0$, $c=0$ *if and only if* $a=0$, $\sigma_1=\sigma_2=1$;

$i_2)$ $\rho=\frac{2a^2+\sigma_1-\sigma_2}{2\left(\sigma_2-a^2\right)}>0,\qquad c=-a\frac{\sigma_1+\sigma_2}{2a^2+\sigma_1-\sigma_2},\qquad 2a^2+\sigma_1-\sigma_2>0$.

Proof. At first we observe that the condition $a(\mu_x)\to a$ implies $h\in\mathcal{H}(-a)$ and by Lemma 3

$$\psi'(u)=-a\biggl(\varphi'(u)+\frac{\varphi(u)}{u}\biggr).$$

almost everywhere.

Let ε, $0<\varepsilon<1$, be fixed and $\varepsilon\le v\le\varepsilon^{-1}$. In virtue of $a(\nu_x)=o(1)$ and $\psi_x(u)\ll|\log u|^{1/2}$ summing by parts we obtain

$$\frac{1}{B^2(x)v\log x}\sum_{m\le x^v}m^{-1}\bigl(h(m)-A(x^v)\bigr)^2=O(\varepsilon)+v^{-1}\int_\varepsilon^v\sigma(\nu_{x^u})\varphi_x^2(u)\,du+$$

$$+v^{-1}\int_\varepsilon^v\psi_x^2(u)\,du-2\psi_x(v)v^{-1}\int_\varepsilon^v\psi_x(u)\,du+\psi_x^2(v)+o_\varepsilon\bigl(\varphi_x^2(\varepsilon^{-1})\bigr).$$

Applying it for the subsequence $y\to\infty$ we deduce the equality

$$\sigma_2v\varphi^2(v)=\sigma_1\int_0^v\varphi^2(u)\,du+\int_0^v\psi^2(u)\,du-2\psi(v)\int_0^v\psi(u)\,du+v\psi^2(v),\tag{18}$$

which together with the differential equation above leads to the following system of equations for $\psi=\psi(v)$ and $\varphi=\varphi(v)$:

$$\begin{cases}v\psi'=-av\varphi'-a\varphi\\ 2\sigma_2v\varphi\varphi'=(\sigma_1-\sigma_2)\varphi^2+2v\psi\psi'-2\psi'\int_0^v\psi(u)\,du,\end{cases}$$

when $\varphi(1)=1-\psi(1)=1$. The system easily reduces to

$$\begin{cases}v\psi'=-av\varphi'-a\varphi\\ 2(\sigma_2-a^2)v\varphi'=(\sigma_1-\sigma_2+2a^2)\varphi.\end{cases}$$

Observe that the factors on the left side of the second equation are nonnegative. If $\sigma_2=a^2$, then $\sigma_1=a^2=0$, which contradicts to the condition $\sigma_2>0$. The equality $\sigma_1-\sigma_2+2a^2=0$ yields the solution $\varphi(v)=1$. Further, having $a=0$ we obtain $\psi(v)=0$. In this case

$h \in \mathcal{H} = \mathcal{H}(0,0)$, hence $\sigma_1 = 1 = \sigma_2$. When $\sigma_1 - \sigma_2 + 2a^2 > 0$, the only solution of the system is given in Lemma. That ends the proof.

In the sequel we shall use the following corollary of the Hildebrand's result [4].

Lemma 5. *If* $h \in \mathcal{H}(0)$, *then* $\sigma(\nu_x) = 1 + o(1)$. *If the conditions (4) and (5) are satisfied, then* $\sigma(\nu_x) = \sigma_1 + o(1)$, *where* σ_1 *is given by the formula (7).*

Proof. According to Theorem (Hildebrand [4]) we have

$$\sigma(\nu_x) = \Big(1 + \frac{\Theta}{\sqrt{k}} + o_k(1)\Big) V_k(x) . \qquad |\Theta| \le 1 ,$$

where

$$V_k(x) = 1 + \sum_{r=1}^{k} \frac{\lambda_x^2(r)(-1)^r}{r} + o_k(1) + o_k\Bigg(\sum_{r=1}^{k} \lambda_x^2(r)\Bigg) ,$$

$$\lambda_x(r) = \sum_{p \le x} \frac{h(p) l_{rx}(p)}{pB(x)} ,$$

$$l_{rx}(p) = q_r\left(\frac{\log p}{\log x}\right), \quad q_r(t) = t g_{r-1}(t) , \ \ r \in \mathbb{N} .$$

Here $g_{r-1}(t)$, $r \in \mathbb{N}$, denote the sequence of orthogonal polynomials in the interval $[0,1]$ with the weight t , that is,

$$g_{r-1}(t) = (2r)^{1/2} P_{r-1}^{(1,0)}(1-2t) = (2r)^{1/2} \sum_{s=0}^{r-1} \binom{r-1}{s}\binom{r}{s} (-1)^{r-1-s} t^{r-1-s} (1-t)^s, \quad r \ge 1 .$$

While $\alpha_x \to 0$ implies $\lambda_x(r) \to 0$, the first assertion evidently foolows from the Hildebrand's formulae above.

Further, since $|\lambda_x(r)| \le 1$ (see, [4]), the last term in the formula for $V_k(x)$ tends to zero. We shall prove convergence of the main term as $x \to \infty$ and then convergence as $k \to \infty$. All we need for this purpose is to prove that $\lambda_x(r) \to \lambda(r)$, to find these limits, and to verify that $\lambda^2(r) \le \lambda^2(r-1)$, $r \ge r_0$. As in the proof of Lemma 3 summing by parts we have $\lambda_x(1)/\sqrt{2} = \alpha_x \to \alpha = c\rho/(\rho+1)$ and, further for $r \ge 2$,

$$\lambda_x(r) = \alpha \int_0^1 g_{r-1}(t)\big(\varphi(t)t\big)' \, dt + o(1) = \tag{19}$$

$$= c\rho \, (2r)^{1/2} \sum_{s=0}^{r-1} \binom{r-1}{s}\binom{r}{s} (-1)^{r-1-s} \int_0^1 t^{r-s-1+\rho} \, (1-t)^s \, dt + o(1) =$$

$$= c\rho \, (2r)^{1/2} \sum_{s=0}^{r-1} \binom{r-1}{s}\binom{r}{s} (-1)^{r-1-s} \, \frac{\Gamma(r+\rho-s)\,\Gamma(s+1)}{(r+\rho)\,\Gamma(r+\rho)} + o(1) =$$

$$= c\rho \, (2r)^{1/2} (r+\rho)^{-1} \sum_{s=0}^{r-1} (-1)^{r-1-s} \binom{r-1}{s}\binom{r}{s}\binom{r-1+\rho}{s}^{-1} + o(1) .$$

Applying the summation formula 35.3 on page 633 of the book (Prudnikov *et all* [15]) we have

$$\lambda_x(r) = c\rho\,(2r)^{1/2}(r+\rho)^{-1}(-1)^{r-1}\binom{\rho-1}{r-1}\binom{r-1+\rho}{r-1}^{-1} + o(1)\,.$$

If $c\rho = 0$, the assertion of Lemma 5 has been just proved. If $c\rho \neq 0$, $r > 1+\rho$, then

$$\frac{\lambda(r)}{\lambda(r-1)} = \left(\frac{r}{r-1}\right)^{1/2}\frac{r-1-\rho}{r+\rho} \leq \left(\frac{r-1}{r}\right)^{1/2} \leq 1\,.$$

Now since $\lim\limits_{k\to\infty}\lim\limits_{x\to\infty} V_k(x) = \sigma_1$, Lemma 5 is proved.

Proof of Theorem 1. The existence of limiting distributions is given by Theorem 1 of the paper (Levin and Timofeev [9]). As in the proof of (17), the assertion i_2) follows by partial summation. Then Lemma 2 implies i_3). To prove i_4), we shall consider the relation between the characteristic functions $f(t)$ and $g(t)$ of distributions $\nu(v)$ and $\mu(v)$

$$vf\big(t\varphi(v)\big)\mathrm{e}^{it\psi(v)} = \int_0^1 g\big(t\varphi(u)\big)\mathrm{e}^{it\psi(u)}\,du\,,$$

which easily can be obtained by partial summation (see, Levin and Timofeev [9], p. 340). As we have remarked, the limit distributions have two finite moments, hence $f(t)$ and $g(t)$ are twice differentiable. So we obtain

$$vf'\big(t\varphi(v)\big)\varphi(v)\mathrm{e}^{it\psi(v)} + ivf\big(t\varphi(v)\big)\psi(v)\mathrm{e}^{it\psi(v)} =$$
$$= \int_0^1 \Big(g'\big(t\varphi(u)\big)\varphi(v)\mathrm{e}^{it\psi(u)} + ig\big(t\varphi(u)\big)\psi(v)\mathrm{e}^{it\psi(u)}\Big)\,du\,.$$

When $t = 0$, this is the same differential equation (16). Applying it in equality for the second derivatives at the point $t = 0$, we deduce

$$\big(\varphi^2(v)\big)'\big(\sigma(\mu) - a(\mu)^2\big) = \varphi^2(v)\big(\sigma(\nu) + 2a(\mu)^2 - \sigma(\mu)\big)\,.$$

Hence and from i_3) we have the relation

$$\beta\big(\sigma(\mu) - a(\mu)^2\big) = (1-\beta)\big(\sigma(\nu) + 2a(\mu)^2 - \sigma(\mu)\big)\,,$$

which is the same as that in i_4). The condition $\sigma(\mu) > 0$ has been observed in (Levin and Timofeev [9]).

The following Ruzsa's estimate (see, [17])

$$\sigma(\nu_x) \asymp \min_{\lambda\in\mathbb{R}}\left\{\lambda^2 + \sum_{p^k\leq x}\frac{\left(\frac{h(p^k)}{B(x)} - \lambda\log p^k\right)^2}{p^k}\right\} = 1 - 2\alpha_x^2 + o(1)$$

implies $i_5)$.

In order to prove the converse statement, we apply Lemmas 3 and 2. Further we can use the proved part of Theorem.

Example of the function $h(m) = \log m$ shows that $\nu(v) = E_0(v)$ does not imply $\mu(v) = E_a(v)$ for some $a \in \mathbb{R}$. For this function $\mu(v)$ exists, $a(\mu)^2 = 1/2$, and $\sigma(\mu) = 2/3$.

Proof of Theorem 2. The existence of the limit distributions $\nu(v)$ and $\mu(v)$ follows from Theorem 1. Formula (7) has been obtained in Lemma 5. While $a(\mu_x) = -\alpha_x + o(1)$, the first of the formulae (8) follows from $i_2)$ of Theorem 1. The desired expression for σ_2 is presented by (18) when $v = 1$.

Necessity follows from Lemmas 2 and 4.

Proof of Corollary 1. Only the necessity is not evident. By Lemma 2 for an arbitrary sequence $y_1 \to \infty$ we have a subsequence $y \to \infty$ such that $\psi_y(u) \to \psi(u)$ and $\varphi_y(u) \to \varphi(u)$, where $\psi(u)$ and $\varphi(u)$ are continuous functions. The first assumption of (9) implies $h \in \mathcal{H}(0)$. Hence $\psi(u) = 0$ independently of the sequence $y_1 \to \infty$ and, further by Lemma 5, $\sigma(\nu_x) \to 1$. Thus, the desired result follows from Theorem 2, the exponent ρ is determined by the formula (8).

Proof of Corollary 2. Behaving as in the proof of the first Corollary we obtain $\psi(u) = 0$ and $\sigma(\nu_x) \to 1$. To determine $\varphi(u)$, we apply the relation (18) but obtained using the subsequence $y \to \infty$ only. We have

$$1 = \sigma(\mu) \le \lim_{y\to\infty} \sigma(\mu_y) = \int_0^1 \varphi^2(u)\, du \le 1 .$$

Hence $\varphi(u) = 1$, and Corollary is proved.

Proof of Theorem 3. Sufficiency is well–known (Kubilius [5]). To prove necessity, we denote

$$S_x := B^{-2}(x) \sum_{\substack{p,\, q \le x \\ pq > x}} \frac{h(p)h(q)}{pq} .$$

The Cauchy inequality yields the estimates

$$B^{-2}(x) \sum_{\substack{p,\, q\le x,\, pq>x \\ |h(p)|<\varepsilon B(x)}} \frac{h(p)h(q)}{pq} \ll \varepsilon \sum_{p\le x} \frac{1}{p} \Bigg(\sum_{x/p<q\le x} \frac{1}{q} \Bigg)^{1/2} \ll \varepsilon$$

and

$$B^{-2}(x) \sum_{p,\, q\le x,\, pq>x}^{\star} \frac{h(p)h(q)}{pq} \ll \Bigg(\sum_{p\le x}^{\star} \frac{1}{p} \sum_{x/p<q\le x} \frac{1}{q} \Bigg)^{1/2} \ll$$

$$\ll \Bigg(\sum_{p\le\sqrt{x}}^{\star} \log \frac{\log x}{\log x - \log p} + \sum_{\sqrt{x}<p\le x}^{\star} \frac{1}{p} \Bigg)^{1/2} \ll T_x = o(1) ,$$

where as earlier $\star$ stands for the condition $h(p) < -\varepsilon B(x)$. Hence

$$S_x = B^{-2}(x) \sum_{\substack{p,\, q \le x,\, pq > x \\ h(p) > 0,\, h(q) > 0}} \frac{h(p)h(q)}{pq} + o(1) \,.$$

But (see Kubilius [6], [7] or Hildebrand [4]) $\sigma(\nu_x) = 1 - S_x + o(1)$, thus, observing that either of the assertions of Theorem 3 yields $\sigma(\nu_x) = 1 + o(1)$, we obtain $S_x = o(1)$ and, further,

$$\left(\sum_{\substack{\sqrt{x} < p \le x \\ h(p) > 0}} \frac{h(p)}{pB(x)} \right)^2 \le S_x + o(1) = o(1) \,.$$

Now repeatedly changing x to $\sqrt{x}$ and using the condition of Theorem once more we obtain

$$\sum_{\substack{x^\delta < p \le x \\ |h(p)| \ge \varepsilon B(x)}} \frac{1}{p} \ll \delta^{-1} T_x + o_\varepsilon(1) = o_{\varepsilon,\delta}(1)$$

for each $\varepsilon > 0$ and $0 < \delta \le 1$. It means that the function h is of the Kubilius type (see Ruzsa, [18]) and one can apply the Kubilius truncation procedure ([5]). Corresponding limit theorem for the sum of independent random variables ξ_p , $p \le x$, normalized by $A(x)$ and $B(x)$, implies the condition (4). Theorem 3 is proved.

3. Concluding remark

Corollary 3 in (Hildebrand [4]) asserts that $\sigma(\nu_x) = 1 + o(1)$ implies $\psi(u) = 0$, $0 < u \le 1$. That in the special case can be seen from our Lemma 5. The Hildebrand's proof is based upon the equality

$$\int_0^1 \sum_{\substack{p,\, q \le x^t \\ pq > x^t}} \frac{h(p)h(q)}{pq} \, dt = \sum_{\substack{p,\, q \le x \\ pq > x}} \frac{h(p)h(q)}{pq} \int_P^Q dt \,,$$

where $P = \max\left\{ \frac{\log p}{\log x} , \frac{\log q}{\log x} \right\}$ and $Q = \frac{\log p}{\log x} + \frac{\log q}{\log x}$, which is false as the example $h(p) \equiv 1$ shows. It should be $Q = \min\left\{ \frac{\log p}{\log x} + \frac{\log q}{\log x} , 1 \right\}$, but then the approach fails. The mistake was repeated in Supplement of the book (Elliott [2]), therefore we could not use these calculations directly.

In connection to the problems considered in the present paper it would be very desirable to obtain necessary and sufficient conditions for the relation $\sigma(\nu_x) = \sigma + o(1)$. Is it true that $\sup_{h \ne 0} \limsup \sigma(\nu_x) \le 1$?

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