SELBERG'S CENTRAL LIMIT THEOREM ON THE CRITICAL LINE AND THE LERCH ZETA-FUNCTION

RAMŪNAS GARUNKŠTIS, ANDRIUS GRIGUTIS, AND ANTANAS LAURINČIKAS

Abstract. A. Selberg obtained a limit theorem for the Riemann zeta-function $\zeta(1/2+it)$. D. Joyner, A. Laurinčikas, A. Selberg, K. Tsang considered limit theorems for $\zeta(\sigma+it)$ when σ is near to $1/2$. We derive limit theorems for the Lerch zeta-function $L(\lambda,\alpha,1/2+it)$ when the pair of parameters (λ,α) are near to $(1,1)$, $(1/2,1)$, $(1,1/2)$ or $(1/2,1/2)$.

1. Introduction

In the forties of the last century A. Selberg proved that on the critical line $\sigma = 1/2$ suitably normalized logarithm of the Riemann zeta-function is asymptotically normally distributed. Here we formulate his result separately for real and imaginary parts. As usual, denote by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

the standard normal distribution function. We have

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ t \in [0,T] : \frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} < x \right\} = \Phi(x)$$

and the same equality is true if we replace $\log|\zeta(1/2+it)|$ by $\arg\zeta(1/2+it)$. The limit law remains the same, if σ is "near" to the critical line. Let $\beta_T = \log T$ for $1/2 \le \sigma \le 1/2 + 1/\log T$ and $\beta_T = \log(1/(\sigma - 1/2))$ for $1/2 + 1/\log T < \sigma < 1/2 + o(1)$ as $T \to \infty$. Then

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ t \in [0,T] : \frac{\log|\zeta(\sigma+it)|}{\sqrt{2^{-1}\log\beta_T}} < x \right\} = \Phi(x).$$

Similar results, when σ is "near" to the critical line, are known for $\arg\zeta(\sigma+it)$. For these results and their proofs see Bombieri and Hejhal [1], Joyner [3], Laurinčikas [4], Selberg [7], Tsang [8].

The Riemann zeta-function is a special case of the more general Lerch zeta-function

$$L(\lambda,\alpha,s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^s},$$

where $0 < \lambda \le 1$ and $0 < \alpha \le 1$ (for more information see Laurinčikas and Garunkštis [5]). In general, the Lerch zeta-function has no Euler product. The proof of the limit theorem for the Riemann zeta-function relies on the Euler product and it is not clear whether limit theorems can be obtained for all Lerch zeta-functions. Similarly as Selberg's limit theorem remains valid for $\zeta(\sigma+it)$ when σ is near to $1/2$, one expects that the same is true for

$L(\lambda, \alpha, 1/2+it)$ when (λ, α) is near to $(1,1)$, $(1/2,1)$, $(1,1/2)$ or $(1/2,1/2)$. For these four cases we have

$$\begin{aligned} &L(1,1,s)=\zeta(s), \quad L(1,1/2,s)=(2^s-1)\zeta(s), \\ &L(1/2,1,s)=(1-2^{1-s})\zeta(s), \quad \text{and} \quad L(1/2,1/2,s)=2^s L(s,\chi), \end{aligned} \tag{1.1}$$

where χ is a Dirichlet character $\mod 4$ with $\chi(3)=-1$.

First we consider limit theorems for $\log|L(\lambda,\alpha,1/2+it)|$. We use notation

$$\nu_T(\dots)=T^{-1}\text{meas}\{t\in[0,T]:\dots\},$$

where in place of dots a condition satisfied by t is written. By ε we always denote a positive fixed number, which can be as small as we want.

Theorem 1.1. *Let*

$$1-T^{-1-\varepsilon}<\lambda<1-e^{-T/\log T} \quad \text{or} \quad \lambda=1 \quad \text{or} \quad |\lambda-1/2|<T^{-1-\varepsilon}$$

and let

$$1-T^{-1-\varepsilon}<\alpha\le 1 \quad \text{or} \quad |\alpha-1/2|<T^{-1-\varepsilon}.$$

Then

$$\lim_{T\to\infty}\nu_T\left(\frac{\log|L(\lambda,\alpha,1/2+it)|}{\sqrt{2^{-1}\log\log T}}<x\right)=\Phi(x).$$

If the parameter λ is "very" close to 1 then the Lerch zeta-function becomes "very" large. From the approximation by a finite sum (2.1) (see the next section) we have that

$$|L(\lambda,\alpha,1/2+it)|=\frac{e^{-\pi t}}{(1-\{\lambda\})^{1/2}}+O\left(t^{1/4}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\alpha}}\right)$$

uniformly for $0<\lambda,\alpha\le 1$, as $t\to\infty$. We can remove that large term and again obtain central limit theorem when λ is "very" close to one. Let

$$E(\lambda,\alpha)=-(2\pi/t)^{it}\exp\left\{it+\frac{3\pi i}{4}-2\pi i\lambda\alpha+2\pi i\alpha\right\}\frac{e^{-\pi t}}{(1-\{\lambda\})^{1/2-it}}.$$

Theorem 1.2. *Let* $1-e^{-T/\log T}\le\lambda<1$ *and let*

$$1-T^{-1-\varepsilon}<\alpha\le 1 \quad \text{or} \quad |\alpha-1/2|<T^{-1-\varepsilon}.$$

Then

$$\lim_{T\to\infty}\nu_T\left(\frac{\log|L(\lambda,\alpha,1/2+it)-E(\lambda,\alpha)|}{\sqrt{2^{-1}\log\log T}}<x\right)=\Phi(x).$$

Now we turn to limit theorems for $\arg L(\lambda,\alpha,s)$. The Lerch zeta-function $L(\lambda,\alpha,s)$ does not vanish for $\Re s>1+\alpha$ (see [5]). We follow tradition and define $\arg L(\lambda,\alpha,s)$ by continuous displacement from the point $s=2$ (choosing $-\pi<\arg L(\lambda,\alpha,2)\le\pi$) along the straight lines connecting the points $s=2, 2+it$, and $\sigma+it$. If on this way there is a zero, then $\arg L(\lambda,\alpha,s)$ remains undefined. We note that countably many zeros do not affect our measures.

We start with cases when (λ,α) tends to $(1,1)$ and $(1/2,1)$.

Theorem 1.3. *Let*

$$1 - T^{-1-\varepsilon} < \lambda < 1 - e^{-T/\log T} \quad \text{or} \quad \lambda = 1 \quad \text{or} \quad |\lambda - 1/2| < T^{-1-\varepsilon}$$

and let $1 - T^{-1-\varepsilon} < \alpha \leq 1$. *Then*

$$\lim_{T\to\infty} \nu_T \left(\frac{\arg L(\lambda, \alpha, 1/2 + it)}{\sqrt{2^{-1} \log\log T}} < x \right) = \Phi(x).$$

In cases (λ, α) tends to $(1, 1/2)$ and $(1/2, 1/2)$ we consider $L(1, 1/2, s) = (2^s - 1)\zeta(s)$ and $L(1/2, 1/2, s) = 2^s L(s, \chi)$. We have that $\arg\left(2^{1/2+it} - 1\right)$ and $\arg\left(2^{1/2+it}\right)$ are equal to $\frac{\log 2}{2\pi} t + O(1)$. Selberg's limit theorem is also valid for argument of Dirichlet L-functions (Bombieri and Hejhal [1]). Thus we obtain the following theorem.

Theorem 1.4. *Let*

$$1 - T^{-1-\varepsilon} < \lambda < 1 - e^{-T/\log T} \quad \text{or} \quad \lambda = 1 \quad \text{or} \quad |\lambda - 1/2| < T^{-1-\varepsilon}$$

and let $|\alpha - 1/2| < T^{-1-\varepsilon}$. *Then*

$$\lim_{T\to\infty} \nu_T \left(\frac{\arg L(\lambda, \alpha, 1/2 + it) - t \log 2/2\pi}{\sqrt{2^{-1} \log\log T}} < x \right) = \Phi(x).$$

Similarly as before, Theorems 1.3 and 1.4 remains true for $1 - e^{-T/\log T} \leq \lambda < 1$ if we replace $\arg L(\lambda, \alpha, 1/2+it)$ and $\arg L(\lambda, \alpha, 1/2+it) - t \log 2/2\pi$ by $\arg\left(L(\lambda, \alpha, 1/2 + it) - E(\lambda, \alpha)\right)$ and $\arg\left(L(\lambda, \alpha, 1/2 + it) - E(\lambda, \alpha)\right) - t \log 2/2\pi$, accordingly.

Theorems are proved in the next section.

2. Proofs

First we prove several lemmas and later derive theorems.

Lemma 2.1. *For* (λ, α) *equal to* $(1, 1)$, $(1/2, 1)$, $(1, 1/2)$ *or* $(1/2, 1/2)$, *we have that*

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda, \alpha, 1/2 + it)|}{\sqrt{2^{-1} \log\log T}} < x \right) = \Phi(x).$$

Proof. One proof of Selberg's central limit theorem can be found in Bombieri and Hejhal [1]. Their formulation

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |\zeta(1/2 + it)|}{\sqrt{2^{-1} \log\log t}} < x \right) = \Phi(x)$$

is equivalent to

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |\zeta(1/2 + it)|}{\sqrt{2^{-1} \log\log T}} < x \right) = \Phi(x),$$

because of the bound (see Lemma 5 in [1])

$$\int_T^{2T} |\log \zeta(1/2 + it)| dt = O\left(T \sqrt{\log\log T}\right),$$

where $T \to \infty$. They considered general class of zeta functions, which includes Dirichlet L-functions. Thus Lemma 2.1 is true for $(\lambda, \alpha) = (1, 1)$ and $(1/2, 1/2)$.

If $(\lambda, \alpha) = (1/2, 1)$, then the lemma follows by (1.1) and

$$\begin{aligned}\frac{\log|L(1/2,1,1/2+it)|}{\sqrt{2^{-1}\log\log T}} =&\frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} + \frac{\log|1-2^{1/2-it}|}{\sqrt{2^{-1}\log\log T}}\\ =&\frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} + o(1).\end{aligned}$$

The remaining case is analogous. □

Using a limit theorem for the argument of Dirichlet L-functions (see [1]), Lemma 2.1 can be extended to $\arg L(\lambda, \alpha, 1/2 + it)$.

The following lemma will be needed for Lemma 2.3.

Lemma 2.2. *If a sequence of distribution functions $F_n(x)$ converges weakly to continuous distribution function $F(x)$, then this convergence is uniform in x, $-\infty < x < \infty$.*

Proof. The lemma can be found in Petrov [6]. □

If two functions are near to each other then, of course, we expect they are distributed similarly.

Lemma 2.3. *Let $l(T) \to +\infty$ and $h(T) \to +\infty$ for $T \to \infty$. Let f and f_T be two measurable complex functions defined on real numbers. Let f and f_T have only countably many zeros. Assume that*

$$\lim_{T\to\infty} \nu_T \left(\frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x)$$

and, for $t \in [T/h(T), T]$, assume that $|f_T(t) - f(t)| < \exp\left(-l(T)\sqrt{\log\log T}\right)$. Then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log|f_T(t)|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

Proof. If $|f(t)| \neq 0$ then

$$\log|f_T(t)| = \log|f(t)| + \log\left(1 + \frac{|f_T(t)| - |f(t)|}{|f(t)|}\right).$$

From the last equality we see that $\log|f_T(t)|$ is "near" to $\log|f(t)|$ if $|f(t)|$ is not very "small". We expect that there are "not many" t for which $|f(t)|$ is very "small". Accordingly, we divide the interval $[T/h(T), T]$ in to two subsets

$$J_T = \left\{ t \in [T/h(T), T] : \frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} < -l(T) \right\}$$

and

$$I_T = \{ t \in [T/h(T), T] : t \notin J_T \}.$$

By Lemma 2.2 and conditions of the lemma (recall that $\Phi(x)$ denotes the standard normal distribution function) we see that

$$\begin{aligned}\frac{1}{T}\text{meas}\{J_T\} =&\nu_T \left(\frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} < -l(T) \right) + o(1)\\ =&\Phi\left(-l(T)\right) + o(1) = o(1),\end{aligned}$$

as $T \to \infty$. For $t \in I_T$, we have

$$\begin{aligned}\log|f_T(t)| &= \log|f(t)| + \log\left(1 + O\left(\frac{|f_T(t) - f(t)|}{|f(t)|}\right)\right)\\ &= \log|f(t)| + o(1).\end{aligned}$$

Now we can finish the proof. As $T \to \infty$, we obtain that

$$\begin{aligned}&\nu_T\left(\frac{\log|f_T(t)|}{\sqrt{2^{-1}\log\log T}} < x\right)\\ &= \frac{1}{T}\text{meas}\{t \in I_T : \frac{\log|f_T(t)|}{\sqrt{2^{-1}\log\log T}} \le x\} + \frac{1}{T}\text{meas}\{t \in J_T : \frac{\log|f_T(t)|}{\sqrt{2^{-1}\log\log T}} \le x\} + o(1)\\ &= \frac{1}{T}\text{meas}\{t \in I_T : \frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} + o(1) \le x\} + O\left(\frac{1}{T}\text{meas}\{t \in J_T\}\right) + o(1)\\ &= \nu_T\left(\frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} \le x + o(1)\right) + o(1) = \Phi(x) + o(1).\end{aligned}$$

□

We consider how close are two Lerch zeta-functions if their parameters are also close. In the previous section we have introduced the function

$$E(\lambda, \alpha) = -(2\pi/t)^{it}\exp\left\{it + \frac{3\pi i}{4} - 2\pi i\lambda\alpha + 2\pi i\alpha\right\}\frac{e^{-\pi t}}{(1 - \{\lambda\})^{1/2-it}}.$$

Lemma 2.4. *Let δ be a positive fixed number and $\lambda_1, \lambda_2, \alpha_1, \alpha_2 > \delta$. Let $t \to \infty$ and $\max\{|\lambda_1 - \lambda_2|, |\alpha_1 - \alpha_2|\} \ll t^{-3/4}$. Then*

$$\begin{aligned}&L(\lambda_1, \alpha_1, 1/2 + it) - E(\lambda_1, \alpha_1) - L(\lambda_2, \alpha_2, 1/2 + it) + E(\lambda_2, \alpha_2)\\ &\quad \ll t\left(|\lambda_1 - \lambda_2| + |\alpha_1 - \alpha_2|\right) + t^{-1/4}.\end{aligned}$$

Proof. The Lerch zeta-function can be approximated by a finite sum. We have (see Garunkštis [2]) that

$$\begin{aligned}(2.1)\qquad L(\lambda, \alpha, 1/2 + it) &= \sum_{0 \le n \le \sqrt{t/2\pi}} \frac{e^{2\pi i\lambda n}}{(n + \alpha)^{1/2+it}}\\ &+ \left(\frac{2\pi}{t}\right)^{it}\exp\left\{it + \frac{\pi i}{4} - 2\pi i\lambda\alpha\right\}\Bigg(\sum_{0 \le n \le \sqrt{t/2\pi}} \frac{e^{-2\pi i\alpha n}}{(n + \lambda)^{1/2-it}} - \frac{e^{-\pi t + \pi i/2 + 2\pi i\alpha}}{(1 - \{\lambda\})^{1/2-it}}\Bigg)\\ &+ O(t^{-1/4})\end{aligned}$$

uniformly in λ and α. Thus

$$
\begin{aligned}
&L(\lambda_1,\alpha_1,1/2+it)-E(\lambda_1,\alpha_1)-L(\lambda_2,\alpha_2,1/2+it)+E(\lambda_2,\alpha_2)\\
&\ll\left|\sum_{0\le n\le\sqrt{t}}\frac{e^{2\pi i\lambda_1 n}(n+\alpha_2)^{1/2+it}-e^{2\pi i\lambda_2 n}(n+\alpha_1)^{1/2+it}}{((n+\alpha_1)(n+\alpha_2))^{1/2+it}}\right|\\
&\quad+\left|\sum_{0\le n\le\sqrt{t}}\frac{e^{-2\pi i(\alpha_1 n+\alpha_1\lambda_1)}(n+\lambda_2)^{1/2-it}-e^{-2\pi i(\alpha_2 n+\alpha_2\lambda_2)}(n+\lambda_1)^{1/2-it}}{((n+\lambda_1)(n+\lambda_2))^{1/2-it}}\right|\\
&\quad+O(t^{-1/4}).
\end{aligned}
\tag{2.2}
$$

We consider the first sum in formula (2.2).

$$
\begin{aligned}
&\sum_{0\le n\le\sqrt{t}}(n+\alpha_1)^{-1/2-it}e^{2\pi i\lambda_1 n}\left(1-e^{2\pi in(\lambda_2-\lambda_1)}\left(1+\frac{\alpha_1-\alpha_2}{n+\alpha_2}\right)^{1/2+it}\right)\\
&\ll\sum_{0\le n\le\sqrt{t}}(n+\delta)^{-1/2}\left((n+1)|\lambda_2-\lambda_1|+\frac{t|\alpha_2-\alpha_1|}{n+\delta}+t|\lambda_2-\lambda_1||\alpha_2-\alpha_1|\right)\\
&\ll t^{3/4}|\lambda_2-\lambda_1|+t|\alpha_2-\alpha_1|+t^{5/4}|\lambda_2-\lambda_1||\alpha_2-\alpha_1|.
\end{aligned}
$$

Similarly, we derive that the second sum in formula (2.2) is

$$
\begin{aligned}
\ll &t^{3/4}|\alpha_2-\alpha_1|+t|\lambda_2-\lambda_1|+t^{1/4}|\alpha_2\lambda_2-\alpha_1\lambda_1|\\
&+t^{5/4}|\lambda_2-\lambda_1||\alpha_2-\alpha_1|+t|\alpha_2\lambda_2-\alpha_1\lambda_1||\lambda_2-\lambda_1|.
\end{aligned}
$$

The lemma is proved. □

Proof of Theorems 1.1 and 1.2 follows by Lemmas 2.1, 2.3, and 2.4.

□

Proof of Theorems 1.3 and 1.4 is similar to the proof of Theorem 1.1. Note that Lemmas 2.1 and 2.3 can be easily rewritten for $\arg L(\lambda,\alpha,1/2+it)$ (see the note after the proof of Lemma 2.1).

□

References

[1] E. Bombieri and D.A. Hejhal, *On the distribution of zeros of linear combinations of Euler products*, Duke Math. J. **80** (1995), 821–865

[2] R. Garunkštis, *Approximation of the Lerch zeta-function*, Lith. Math. J., 44 (2)(2004), 176–180,

[3] D. Joyner, *Distribution theorems of L-functions*, Longman Scientific, Harlow 1986

[4] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer Academic Publishers, Dordrecht/Boston/London 1994

[5] A. Laurinčikas and R. Garunkštis, *The Lerch zeta-function*, Kluwer Academic Publishers 2002

[6] V.V. Petrov, *Limit Theorems for Sums of Independent Random Variables*, Nauka, Moscow 1987 (in Russian)

[7] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, in Proceedings of the Amalfi Conference on Analytic Number Theory, September 1989, Università di Salerno, Salerno, 1992, 367–385; Collected Papers, Vol. II, Springer-Vedag, Berlin, 1991, 47–63

[8] K. Tsang, *The distribution of the values of the Riemann zeta-function*, Ph. D. Thesis, Princeton University 1984

Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, 03225 Vilnius, Lithuania

E-mail address: ramunas.garunkstis@mif.vu.lt

E-mail address: andrius.grigutis@mif.vu.lt

E-mail address: antanas.laurincikas@maf.vu.lt