SELBERG'S CENTRAL LIMIT THEOREM ON THE CRITICAL LINE AND THE LERCH ZETA-FUNCTION. II

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Abstract: In this paper, we continue the research on limit theorems for the Lerch zeta-function $L(\lambda, \alpha, 1/2+it)$ when the pair of parameters (λ, α) are near to $(0, 0)$, $(0, 1/2)$, $(0, 1)$, $(1/2, 0)$ and $(1, 0)$.

Key wolds: Central limit theorem on the critical line, Lerch zeta-function.

1. Introduction

Denote by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

the standard normal distribution function. As usual, let $s = \sigma + it$ be a complex variable. For $0 \le \lambda \le 1$ and $0 < \alpha \le 1$, the Lerch zeta-function is given by

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+\alpha)^s} \quad (\sigma > 1),$$

and by analytic continuation elsewhere. For more information see [4].

In [3] we proved the following statement. Let ε be a positive fixed number, as small as we want. Let

i) $$1 - T^{-1-\varepsilon} < \lambda < 1 - e^{-T/\log T} \quad \text{or} \quad \lambda = 1 \quad \text{or} \quad |\lambda - 1/2| < T^{-1-\varepsilon}$$

and

$$1 - T^{-1-\varepsilon} < \alpha \le 1 \quad \text{or} \quad |\alpha - 1/2| < T^{-1-\varepsilon};$$

ii) $$1 - e^{-T/\log T} \le \lambda < 1$$

and

$$1 - T^{-1-\varepsilon} < \alpha \le 1 \quad \text{or} \quad |\alpha - 1/2| < T^{-1-\varepsilon}.$$

If we assume i), then

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ t \in [0, T] : \frac{\log |L(\lambda, \alpha, 1/2 + it)|}{\sqrt{2^{-1} \log\log T}} < x \right\} = \Phi(x). \tag{1}$$

If we assume ii), then

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ t \in [0, T] : \frac{\log |L(\lambda, \alpha, 1/2 + it) - E(\lambda, \alpha, t)|}{\sqrt{2^{-1} \log\log T}} < x \right\} = \Phi(x), \tag{2}$$

where

$$E(\lambda,\alpha,t)=-\left(\frac{2\pi}{t}\right)^{it}\exp\left\{it+\frac{3\pi i}{4}-2\pi i\lambda\alpha+2\pi i\alpha\right\}\frac{e^{-\pi t}}{(1-\{\lambda\})^{1/2-it}}.$$

Similar limit theorems were obtained for $\arg L(\lambda,\alpha,1/2+it)$.

In this paper we prove limit theorems for the Lerch zeta-function $L(\lambda,\alpha,1/2+it)$ when the pair of parameters (λ,α) are near to $(0,0)$, $(0,1/2)$, $(0,1)$, $(1/2,0)$ and $(1,0)$. For a clarity, we set up a table of 9 different pairs of (λ,α):

(0,0)	(1/2,0)	(1,0)
(0,1/2)	(1/2,1/2)	(1,1/2)
(0,1)	(1/2,1)	(1,1)

In [3] we investigated the limit laws when (λ,α) are near to $(1,1)$, $(1,1/2)$, $(1/2,1)$ and $(1/2,1/2)$. In this paper, we deal with 5 remaining cases from the table above.

When (λ,α) are exactly $(1,1)$, $(1,1/2)$, $(1/2,1)$ or $(1/2,1/2)$ we have

$$\begin{aligned}&L(1,1,s)=\zeta(s),\quad L(1,1/2,s)=(2^s-1)\zeta(s),\\&L(1/2,1,s)=(1-2^{1-s})\zeta(s),\quad\text{and}\quad L(1/2,1/2,s)=2^sL(s,\chi),\end{aligned}\tag{3}$$

where $\zeta(s)$ denotes the Riemann zeta-function, $L(s,\chi)$ the Dirichlets L-function and χ is a Dirichlet character $\bmod 4$ with $\chi(3)=-1$. Only in these four cases (excluding periodicity by λ) the Lerch zeta-function has an Euler product. In the forties of the last century A. Selberg proved the following limit theorem

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\left\{t\in[0,T]:\frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}}<x\right\}=\Phi(x)$$

The proof of the limit theorem for the Riemann zeta-function relies on the Euler product and it is not clear whether limit theorems can be obtained for all Lerch zeta-functions, since in general the Lerch zeta-function has no Euler product.

First we consider limit theorems for $\log|L(\lambda,\alpha,1/2+it)|$. We use notation

$$\nu_T(\dots)=T^{-1}\text{meas}\{t\in[0,T]:\dots\},$$

where in place of dots we mean some condition satisfied by t. By ε we always denote a positive fixed number, which can be as small as we want.

If the parameter λ is "very" close to 0 or 1 and α is "very" close to 0, then the Lerch zeta-function becomes "very" large. From the approximation by a finite sum (5) (see the next section) we have that

$$|L(\lambda,\alpha,1/2+it)|=\frac{e^{-\pi t}}{(1-\{\lambda\})^{1/2}}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\alpha}}+O\left(t^{1/4}\right)$$

uniformly for $0<\lambda,\alpha\le 1$, as $t\to\infty$, where $\{\lambda\}$ denotes the fractional part of the number λ. Accordingly we can remove these large terms and obtain central limit theorems when λ is "very" close to 0 or 1 and α is "very" close to 0. Let

$$\Lambda(\lambda,\alpha,t)=\left(\frac{2\pi}{t}\right)^{it}\exp\left\{it+\frac{\pi i}{4}-2\pi i\lambda\alpha\right\}\frac{1}{\lambda^{1/2-it}}.$$

and

$$E(\lambda,\alpha,t) = -\left(\frac{2\pi}{t}\right)^{it} \exp\left\{it + \frac{3\pi i}{4} - 2\pi i\lambda\alpha + 2\pi i\alpha\right\} \frac{e^{-\pi t}}{(1-\{\lambda\})^{1/2-it}}.$$

Consider the cases:

a)
$$0 < \lambda \le \frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad 1 - \frac{1}{T(\log T)^{1+\varepsilon}} \le \alpha \le 1,$$

b)
$$0 < \lambda \le \frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad |\alpha - 1/2| \le \frac{1}{T(\log T)^{1+\varepsilon}},$$

c)
$$0 < \lambda \le \frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad 0 < \alpha \le \frac{1}{T(\log T)^{1+\varepsilon}},$$

d)
$$|\lambda - 1/2| \le \frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad 0 < \alpha \le \frac{1}{T(\log T)^{1+\varepsilon}},$$

e)
$$1 - \frac{1}{T(\log T)^{1+\varepsilon}} \le \lambda < 1 - e^{-T/\log T} \quad \text{and} \quad 0 < \alpha \le \frac{1}{T(\log T)^{1+\varepsilon}},$$

f)
$$1 - e^{-T/\log T} \le \lambda < 1 \quad \text{and} \quad 0 < \alpha \le \frac{1}{T(\log T)^{1+\varepsilon}}.$$

Theorem 1. If we assume a) or b) then

$$\lim_{T\to\infty} \nu_T\left(\frac{\log|L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t)|}{\sqrt{2^{-1}\log\log T}} < x\right) = \Phi(x). \tag{4}$$

If we assume c) then

$$\lim_{T\to\infty} \nu_T\left(\frac{\log|L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t) - 1/\alpha^{1/2+it}|}{\sqrt{2^{-1}\log\log T}} < x\right) = \Phi(x).$$

If we assume d) or e) then

$$\lim_{T\to\infty} \nu_T\left(\frac{\log|L(\lambda,\alpha,1/2+it) - 1/\alpha^{1/2+it}|}{\sqrt{2^{-1}\log\log T}} < x\right) = \Phi(x).$$

If we assume f) then

$$\lim_{T\to\infty} \nu_T\left(\frac{\log|L(\lambda,\alpha,1/2+it) - E(\lambda,\alpha) - 1/\alpha^{1/2+it}|}{\sqrt{2^{-1}\log\log T}} < x\right) = \Phi(x).$$

For cases from a) to f), similar theorems can be formulated for $\arg L(\lambda,\alpha,1/2+it)$. We formulate only one example below.

Theorem 2. If we assume a), then

$$\lim_{T\to\infty} \nu_T\left(\frac{\arg(L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t))}{\sqrt{2^{-1}\log\log T}} < x\right) = \Phi(x).$$

As an interesting fact we note that $L(0,\alpha,1/2+it) = L(1,\alpha,1/2+it)$, but limit laws differs when λ tends to 0 and 1, compare equalities (1) and (4).

The limit laws (1) and (2) remains true if in the conditions i) and ii) we replace $T^{-1-\varepsilon}$ by $1/T(\log T)^{1+\varepsilon}$. Theorems are proved in the next section.

2. Proofs

First we formulate several lemmas and later derive theorems.

Lemma 3. *For* (λ, α) *equal to* $(0,1)$, $(0,1/2)$, *we have that*

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda,\alpha,1/2+it)|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

Proof. Since $L(0,\alpha,s) = L(1,\alpha,s)$, the proof follows from Lemma 2.1 in [3]. □

Similarly as in [1], Lemma 3 can be extended to $\arg L(\lambda,\alpha,1/2+it)$. The following lemma will be needed also for Theorem 1.

Lemma 4. *If a sequence of distribution functions* $F_n(x)$ *converges weakly to continuous distribution function* $F(x)$, *then this convergence is uniform in* x, $-\infty < x < \infty$.

Proof. The proof of lemma can be found in Petrov [5]. □

We consider how close are two Lerch zeta-functions if their parameters are also close. Recall that

$$E(\lambda,\alpha,t) = -\left(\frac{2\pi}{t}\right)^{it} \exp\left\{ it + \frac{3\pi i}{4} - 2\pi i\lambda\alpha + 2\pi i\alpha \right\} \frac{e^{-\pi t}}{(1-\{\lambda\})^{1/2-it}}$$

and

$$\Lambda(\lambda,\alpha,t) = \left(\frac{2\pi}{t}\right)^{it} \exp\left\{ it + \frac{\pi i}{4} - 2\pi i\lambda\alpha \right\} \frac{1}{\lambda^{1/2-it}}.$$

Lemma 5. *Let* $0 < \lambda_1, \lambda_2, \alpha_1, \alpha_2 \le 1$. *Let* $T \to \infty$ *and* $\max\{|\lambda_1-\lambda_2|, |\alpha_1-\alpha_2|\} \ll T^{-3/4}$. *Then*

$$\begin{aligned} &L(\lambda_1,\alpha_1,1/2+iT) - E(\lambda_1,\alpha_1,T) - \Lambda(\lambda_1,\alpha_1,T) - \alpha_1^{-1/2-iT} \\ &- L(\lambda_2,\alpha_2,1/2+iT) + E(\lambda_2,\alpha_2,T) + \Lambda(\lambda_2,\alpha_2,T) + \alpha_2^{-1/2-iT} \\ &\ll T\left(|\lambda_1-\lambda_2| + |\alpha_1-\alpha_2|\right) + T^{-1/4}. \end{aligned}$$

Proof. The Lerch zeta-function can be approximated by a finite sum. We have (see Garunkštis [2]) that

$$\begin{aligned} (5)\qquad L(\lambda,\alpha,1/2+it) &= \sum_{0\le n\le\sqrt{t/2\pi}} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^{1/2+it}} \\ &+ \left(\frac{2\pi}{t}\right)^{it} \exp\left\{ it + \frac{\pi i}{4} - 2\pi i\lambda\alpha \right\} \Bigg(\sum_{0\le n\le\sqrt{t/2\pi}} \frac{e^{-2\pi i\alpha n}}{(n+\lambda)^{1/2-it}} - \frac{e^{-\pi t+\pi i/2+2\pi i\alpha}}{(1-\{\lambda\})^{1/2-it}} \Bigg) \\ &+ O(t^{-1/4}) \end{aligned}$$

uniformly in λ and α, $0 < \lambda, \alpha \leq 1$. Thus

$$\begin{aligned} & L(\lambda_1, \alpha_1, 1/2 + iT) - E(\lambda_1, \alpha_1, T) - \Lambda(\lambda_1, \alpha_1, T) - \alpha_1^{-1/2-iT} \\ & - L(\lambda_2, \alpha_2, 1/2 + iT) + E(\lambda_2, \alpha_2, T) + \Lambda(\lambda_2, \alpha_2, T) + \alpha_2^{-1/2-iT} \\ & \ll \left| \sum_{1 \leq n \leq \sqrt{T}} \frac{e^{2\pi i \lambda_1 n}(n+\alpha_2)^{1/2+iT} - e^{2\pi i \lambda_2 n}(n+\alpha_1)^{1/2+iT}}{((n+\alpha_1)(n+\alpha_2))^{1/2+iT}} \right| \\ & + \left| \sum_{1 \leq n \leq \sqrt{T}} \frac{e^{-2\pi i(\alpha_1 n + \alpha_1 \lambda_1)}(n+\lambda_2)^{1/2-iT} - e^{-2\pi i(\alpha_2 n + \alpha_2 \lambda_2)}(n+\lambda_1)^{1/2-iT}}{((n+\lambda_1)(n+\lambda_2))^{1/2-iT}} \right| \\ & + O(T^{-1/4}) := A + B + O(T^{-1/4}). \end{aligned} \tag{6}$$

We consider the first sum in formula (6).

$$\begin{aligned} A &= \sum_{1 \leq n \leq \sqrt{T}} (n+\alpha_1)^{-1/2-iT} e^{2\pi i \lambda_1 n} \left(1 - e^{2\pi i n(\lambda_2 - \lambda_1)} \left(1 + \frac{\alpha_1 - \alpha_2}{n + \alpha_2} \right)^{1/2+iT} \right) \\ &\ll \sum_{1 \leq n \leq \sqrt{T}} (n+\alpha_1)^{-1/2} \left(1 - \exp\{2\pi i n(\lambda_2 - \lambda_1)\} \exp\left\{ \left(\frac{1}{2} + iT \right) \log\left(1 + \frac{\alpha_1 - \alpha_2}{n+\alpha_2} \right) \right\} \right). \end{aligned}$$

By Taylor expansion of functions e^x and $\log x$ we obtain

$$A \ll \sum_{1 \leq n \leq \sqrt{T}} (n+\alpha_1)^{-1/2} \left(n|\lambda_2 - \lambda_1| + \frac{T|\alpha_2 - \alpha_1|}{n + \alpha_2} + T|\lambda_2 - \lambda_1||\alpha_2 - \alpha_1| \right).$$

The bounds

$$\sum_{1 \leq n \leq \sqrt{T}} n^{1/2} \ll T^{3/4}, \quad \sum_{1 \leq n \leq \sqrt{T}} n^{-3/2} < \infty \quad \text{and} \quad \sum_{1 \leq n \leq \sqrt{T}} n^{-1/2} \ll T^{1/4}$$

leads to

$$A \ll T^{3/4}|\lambda_2 - \lambda_1| + T|\alpha_2 - \alpha_1| + T^{5/4}|\lambda_2 - \lambda_1||\alpha_2 - \alpha_1|.$$

Similarly, we derive that the second sum in formula (6) is

$$\begin{aligned} B \ll & T^{3/4}|\alpha_2 - \alpha_1| + T|\lambda_2 - \lambda_1| + T^{1/4}|\alpha_2\lambda_2 - \alpha_1\lambda_1| \\ & + T^{5/4}|\lambda_2 - \lambda_1||\alpha_2 - \alpha_1| + T|\alpha_2\lambda_2 - \alpha_1\lambda_1||\lambda_2 - \lambda_1|. \end{aligned}$$

The lemma is proved. □

Proof of Theorem 1

We proof only the case a), where (λ, α), depending on T, is close to $(0, 1)$. Remaining cases are analogous. Recall that $L(0, 1, 1/2 + it) = \zeta(1/2 + it)$.

If $|\zeta(1/2 + it)| \neq 0$, then

$$\begin{aligned} & \log |L(\lambda, \alpha, 1/2 + it) - \Lambda(\lambda, \alpha, t)| \\ & = \log |\zeta(1/2 + it)| + \log \left(1 + \frac{|L(\lambda, \alpha, 1/2 + it) - \Lambda(\lambda)| - |\zeta(1/2 + it)|}{|\zeta(1/2 + it)|} \right). \end{aligned}$$

From the last equality we see that $\log|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda,\alpha,t)|$ is "near" to $\log|\zeta(1/2+it)|$ if $|\zeta(1/2+it)|$ is not very "small". We expect that there are "not many" t for which $|\zeta(1/2+it)|$ is very "small". For this reason we choose some monotone function $K(T)$, which satisfies the following conditions:

$$K(T)\to+\infty \quad \text{as} \quad T\to+\infty \quad \text{and} \quad K(T)\ll\sqrt{\log\log T}.$$

Accordingly, we divide the interval $[0,T]$ in to two intervals: $[0,T/K(T))$ and $[T/K(T),T]$. The second interval we divide in to two subsets

$$J_T=\left\{t\in[T/K(T),T]:\frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}}<-K(T)\right\}$$

and

$$I_T=\left\{t\in[T/K(T),T]:t\notin J_T\right\}.$$

By Lemma 4 (recall that $\Phi(x)$ denotes the standard normal distribution function) we see that

$$\begin{aligned}\frac{1}{T}\mathrm{meas}\{J_T\}=&\nu_T\left(\frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}}<-K(T)\right)+o(1)\\ =&\Phi\left(-K(T)\right)+o(1)=o(1),\end{aligned}$$

as $T\to\infty$.

For $t\in I_T$, we have

$$\begin{aligned}&\log|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda,\alpha,t)|\\ &=\log|\zeta(1/2+it)|+\log\left(1+O\left(\frac{|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda)-\zeta(1/2+it)|}{|\zeta(1/2+it)|}\right)\right)\end{aligned}$$

Since $t\in I_T$, by Lemma 5 we see that

$$\begin{aligned}&\frac{|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda)-\zeta(1/2+it)|}{|\zeta(1/2+it)|}\\ &\ll\exp\left(K(T)\sqrt{\log\log T}\right)\Big|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda)-\zeta(1/2+it)\Big|\\ &\ll\exp\left(K(T)\sqrt{\log\log T}\right)\left(\exp\left(-\frac{\pi T}{K(T)}\right)(1-\{\lambda\})^{-1/2}+T\lambda+T|\alpha-1|+\left(\frac{K(T)}{T}\right)^{1/4}\right)\\ &\ll\log T\left(\exp\left(-T/\log\log T\right)+(\log T)^{-1-\varepsilon}+\left(\frac{\log\log T}{T}\right)^{1/4}\right)\ll o(1).\end{aligned}$$

And finally, for $t\in I_T$, we have

$$\log|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda,\alpha,t)|=\log|\zeta(1/2+it)|+o(1).$$

Now we can finish the proof. As $T \to \infty$, we obtain that

$$\nu_T \left(\frac{\log |L(\lambda, \alpha, 1/2 + it) - \Lambda(\lambda, \alpha, t)|}{\sqrt{2^{-1} \log \log T}} < x \right)$$
$$= \frac{1}{T} \text{meas}\{t \in [0, T/K(T)) : \frac{\log |L(\lambda, \alpha, 1/2 + it) - \Lambda(\lambda, \alpha, t)|}{\sqrt{2^{-1} \log \log T}} \le x\}$$
$$+ \frac{1}{T} \text{meas}\{t \in I_T : \frac{\log |L(\lambda, \alpha, 1/2 + it) - \Lambda(\lambda, \alpha, t)|}{\sqrt{2^{-1} \log \log T}} \le x\}$$
$$+ \frac{1}{T} \text{meas}\{t \in J_T : \frac{\log |L(\lambda, \alpha, 1/2 + it) - \Lambda(\lambda, \alpha, t)|}{\sqrt{2^{-1} \log \log T}} \le x\} + o(1)$$
$$= \frac{1}{T} \text{meas}\{t \in I_T : \frac{\log |\zeta(1/2 + it)|}{\sqrt{2^{-1} \log \log T}} + o(1) \le x\} + O\left(\frac{1}{T} \text{meas}\{t \in J_T\} \right) + o(1)$$
$$= \nu_T \left(\frac{\log |\zeta(1/2 + it)|}{\sqrt{2^{-1} \log \log T}} \le x + o(1) \right) + o(1) = \Phi(x) + o(1).$$

□

Proof of Theorem 2 is similar to the proof of Theorem 1. Note that Lemma 3 and proof of Theorem 1 can be rewritten for $\arg L(\lambda, \alpha, 1/2 + it)$ (see the note after the proof of Lemma 3).

□

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