01 Stationary time series

Recap and Additional Key Points

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Stationarity

A process $\{Y_t\}$, $t \in \mathbb{Z}$, is said to be a (weakly) **stationary** process, if:

- $\mathbb{E}(Y_t) = \mu_Y < \infty$, $\forall t \in \mathbb{Z}$;
- $\mathbb{V}\mathrm{ar}(Y_t) = \sigma_Y^2 < \infty$, $\forall t \in \mathbb{Z}$;
- $\mathbb{C}\mathrm{ov}(Y_t, Y_{t+h}) = \sigma_{Y,h} < \infty$, $\forall t \in \mathbb{Z}$, $h > 0$;

White Noise

A process $\{Y_t\}$, $t \in \mathbb{Z}$, is said to be a **White Noise** (WN) process, if:

- $\mathbb{E}(Y_t) = 0$, $\forall t \in \mathbb{Z}$;
- $\mathbb{V}\mathrm{ar}(Y_t) = \sigma_Y^2 < \infty$, $\forall t \in \mathbb{Z}$;
- $\mathbb{C}\mathrm{ov}(Y_t, Y_{t+h}) = 0$, $\forall t \in \mathbb{Z}$, $h > 0$.

Every White Noise process is stationary, however, not every stationary process is a *WN* process.

The Lag (or, the Backshift) operator

The **lag operator** (also known as the **backshift operator**) is defined as $L^k Y_t = Y_{t-k}$, for some integer value k. Note that generally the backshift operator L^k refers to how far back "to the past" are we looking and not as a literal subtraction of an integer k from the index value t. For example, if we had quarterly data at Y_{2010Q3}, then we use the lag operator to refer to a previous quarter $LY_{2010Q3} = Y_{2010Q2}$, $L^2 Y_{2010Q3} = Y_{2010Q1}$, $L^3 Y_{2010Q3} = Y_{2009Q4}$ and so on. Sometimes B^k is used instead of L^k.

The $ARMA(p, q)$ Process

An $ARMA(p, q)$ process, where $p, q \in \mathbb{N}$,can be defined as:

$$\Phi(L)Y_t = \alpha + \Theta(L)\epsilon_t, \quad \epsilon_t \sim WN(0, \ \sigma^2)$$

where the lag polynomials are defined as:

- $\Phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$;
- $\Theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$.

Furthermore:

- If $p = 0$, then $ARMA(0, q) \equiv MA(q)$, where:
  - The $ACF(\tau) \to 0$, as $\tau \to \infty$ (i.e. declines);
  - The $PACF(\tau) = 0$ for $\tau > q$ (i.e. cuts off).
- If $q = 0$, then $ARMA(p, 0) \equiv AR(p)$, where:
  - The $ACF(\tau) = 0$ for $\tau > p$ (i.e. cuts off);
  - The $PACF(\tau) \to 0$, as $\tau \to \infty$ (i.e. declines).
- If $p > 0$ and $q > 0$, then $ARMA(p, q)$ is such that:
  - The $ACF(\tau) \to 0$, as $\tau \to \infty$ (i.e. declines);
  - The $PACF(\tau) \to 0$, as $\tau \to \infty$ (i.e. declines).

The above process can also be written in an expanded form as:

$$Y_t = \alpha + \sum_{i=1}^{p} \phi_p Y_{t-p} + \sum_{j=1}^{q} \theta_q \epsilon_{t-q} + \epsilon_t$$

Intuition of Stationarity for an $AR(1)$ process

Firstly, consider the $AR(1)$ model. In this case, the polynomial is $\Phi(z) = 1 - \phi_1 z$, which results in:

$$Y_t = \phi_1 Y_{t-1} + \epsilon_t$$

If we iterate the above process, we get:

$$\begin{aligned} Y_t &= \epsilon_t + \phi_1(\phi_1 Y_{t-2} + \epsilon_{t-1}) \\ &= \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2(\phi_1 Y_{t-3} + \epsilon_{t-2}) \\ &= ... \\ &= \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + ... + \phi_1^n \epsilon_{t-n} + \phi_1^{n+1} Y_{t-n-1} \end{aligned}$$

This suggests that the **solution** Y_t can be given by an infinite sum:

$$Y_t = \sum_{k=0}^{\infty} \phi_1^k \epsilon_{t-k}$$

We can see that the above is the same, as the process described by the Wold's representation theorem. Since the above only has a single parameter ϕ_1, we would need to consider three cases:

- If $|\phi_1| = 1$;
- If $|\phi_1| < 1$;
- If $|\phi_1| > 1$;

If $|\phi_1| = 1$

In such a case $\sum_{k=0}^{\infty} \phi_1^k \epsilon_{t-k}$ would not converge. Hence, we can rule out $|\phi_1| = 1$.

If $|\phi_1| > 1$

Then $\sum_{k=0}^{\infty} \phi_1^k \epsilon_{t-k}$ would not converge. **However**, if we rewrite $Y_t = \phi_1 Y_{t-1} + \epsilon_t$ as:

$$Y_{t-1} = \phi_1^{-1} Y_t - \phi_1^{-1} \epsilon_t$$

or, in terms of t:

$$Y_t = \phi_1^{-1} Y_{t+1} - \phi_1^{-1} \epsilon_{t+1}$$

Then, since $|\phi_1| > 1 \implies |\phi_1^{-1}| < 1$ and we would have the following infinite representation:

$$Y_t = -\sum_{k=1}^{\infty} \phi_1^{-k} \epsilon_{t+k}$$

The above **solution** is frequently regarded as unnatural, since this means that the time series depends on its **future**, which does not make sense in practice.

For this reason, we require Y_t to be a **causal** (or **future-independent**) function of ϵ_t.

If $|\phi_1| < 1$

In this case $\sum_{k=0}^{\infty} \phi_1^k \epsilon_{t-k}$ converges. The **solution** is expressed as:

$$Y_t = \sum_{k=0}^{\infty} \phi_1^k \epsilon_{t-k}, \quad |\phi_1| < 1$$

Furthermore:

- The solution Y_t is **stationary**, since

$$\mathbb{E}(Y_t) = \sum_{k=0}^{\infty} \phi_1^k \mathbb{E}(\epsilon_{t-k}) = 0$$

$$\gamma_Y(h) = \sum_{j=0}^{\infty} \phi_1^j \phi_1^{j+h} \sigma^2 = \frac{\sigma^2 \phi_1^h}{1 - \phi_1^2} < \infty$$

- The solution Y_t is **unique**. To verify this, consider any other **solution** Z_t for $\Phi(L)Z_t = \epsilon_t$ (**with the same coefficients and the same ϵ_t that are in** $\Phi(L)Y_t = \epsilon_t$), which is expressed as:

$$Z_t = \epsilon_t + \phi_1\epsilon_{t-1} + \phi_1^2\epsilon_{t-2} + ... + \phi_1^n\epsilon_{t-n} + \phi_1^{n+1}Z_{t-n-1}$$

  If Z_t is also stationary (otherwise it is not comparable to Y_t), then $\mathbb{E}Z_t^2 < \infty$ and independent of t, so that:

$$\mathbb{E}\left(Z_t - \sum_{k=0}^{n}\phi_1^k\epsilon_{t-k}\right)^2 = \phi_1^{2n+2}\mathbb{E}(Z_{t-n-1})^2 \to 0, \text{ as } n \to \infty$$

  This implies that:

$$Z_t = \sum_{k=0}^{\infty}\phi_1^k\epsilon_{t-k}$$

  Hence it must hold that $Z_t \equiv Y_t$.

Consequently, for $Y_t = \phi_1 Y_{t-1} + \epsilon_t$, if $|\phi_1| < 1$, then Y_t is the **unique stationary solution** that is **causal**.

Equivalently, a **stationary** and **causal** solution is unique.

Relationship Between Stationarity and Lag Polynomial Roots: The $AR(1)$ Process

Now, what does the requirement that $|\phi_1| < 1$ have to do with the polynomial $\Phi(\cdot)$?

If we were to calculate the root of $\Phi(z) = 1 - \phi_1 z$, we would see that $|\phi_1| < 1$ directly results in the root $|z| = \dfrac{1}{|\phi_1|} > 1$.

Alternatively, this means that $\Phi(z)$ **does not have ANY roots** for $|z| \leq 1$.

Equivalently, the inverse $\Phi(z)^{-1}$:

$$\sum_{k=0}^{\infty} \phi_1^k z^k = \frac{1}{1 - \phi_1 z} = \Phi(z)^{-1}$$

is a convergent power series for $|\phi| < 1$ and, **for the inverse**, $|z| \leq 1$.

We can summarize the above to the general case.

Stationarity Generalization: The $AR(p)$ model

Let us consider an $AR(p)$ model:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \epsilon_t, \quad \epsilon_t \sim WN(0, \sigma^2)$$

and the polynomial: $\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - ... - \phi_p z^p$, $z \in \mathbb{C}$. We then rewrite Y_t with the lag operator in the above polynomial $\Phi(L)Y_t = \epsilon_t$.

As before, we want a similar property for the roots of $\Phi(z)$.

Then:

- If $\exists z \in \mathbb{C} : |z| = 1$ such that $\Phi(z) = 0$, then a stationary solution does **not** exist (see the $|\phi_1| = 1$ case for the $AR(1)$).
- Y_t is the **unique stationary solution** if and only if $\Phi(z) \neq 0, \ \forall z \in \mathbb{C} : |z| = 1$. (In practice it is difficult to verify an inequality.)
- Y_t is the **unique stationary solution that is CAUSAL** if and only if all of the roots of the polynomial $\Phi(z)$ lie outside the unit **disk**, i.e. $\Phi(z) \neq 0, \ \forall z \in \mathbb{C} : |z| \leq 1$.
  In other words, $\Phi(z) = 0, \ z \in \mathbb{C} : |z| > 1$**.** (We can check this in practice.)

The $ARMA(p, q)$ Process: Defining a Solution

For simplicity, assume that $\alpha = 0$. To **solve** the $ARMA(p, q)$ model equation, we need to express it as Y_t (without the lag polynomial).

Consequently, we say that Y_t is the **solution** to $\Phi(L)Y_t = \Theta(L)\epsilon_t$, if:

$$Y_t = \frac{\Theta(L)}{\Phi(L)}\epsilon_t$$

- Since we already assume that ϵ_t WN, for the above equality to make sense, we **require** $\Theta(L)/\Phi(L)$ to be convergent.
- From Wold's representation theorem this is equivalent to:

$$Y_t = \sum_{i=0}^{\infty} \psi_j \epsilon_{t-j}$$

where $\psi_0 = 1$ and $\sum_{i=0}^{\infty} \psi_j^2 < \infty$.

Question: How does Wold's theorem apply to the *ARMA* process?

If we set $\Psi(L) = \Theta(L)/\Phi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + ...$, then we can express $\Psi(z)\Phi(z) = \Theta(z)$:

$$(1-\phi_1 z-\phi_2 z^2-...-\phi_p z^p)(\psi_0+\psi_1 z+\psi_2 z^2+...) = 1+\theta_1 z+\theta_2 z^2+...+\theta_q z^q$$

Equating the coefficients of z^j on both sides for $j = 0, 1, ...$ gives us:

$$1 = \psi_0, \quad \theta_1 = \psi_1 - \psi_0\phi_1, \quad \theta_2 = \psi_2 - \psi_1\phi_1 - \psi_0\phi_2, \quad ...$$

In general, for $j = 0, 1, ..., q$, we have that:

$$\theta_j = \psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k}, \text{ where } \psi_{j-k} = 0, \text{ if } j - k < 0$$

and $\theta_j = 0$, for $j > q$.

Question: Does $\sum_{i=0}^{\infty} \psi_j^2 < \infty$ hold for an $ARMA$ process?

As we have already seen from the $AR(p)$ case, we can calculate the inverse of $\Phi(L)$, if all of its roots are outside the unit circle.

Consequently, the lag polynomial:

$$\frac{\Theta(z)}{\Phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j$$

is absolutely convergent on the unit circle, so $\sum_{j=0}^{\infty} |\psi_j| < \infty$. This also ensures that $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

Note: Similarly to before, if some roots are inside the unit circle then the $ARMA(p, q)$ process is not causal, but can be expressed as a combination of past and **future** values of ϵ. As this situation does not make much sense in econometrics, we usually rule it out.

Common roots in the $ARMA(p, q)$ process

If the roots of $\Phi(L)$ and $\Theta(L)$ are the same - they cancel each other out. The roots that cancel between the AR and the MA parts are not identifiable, so cannot be estimated.

To see why this is, remember that a stationary and invertible ARMA process can be expressed as an infinite MA by multiplying by the inverse of the characteristic lag polynomial of the AR part:

$$Y_t = \frac{\Theta(L)}{\Phi(L)}\epsilon_t$$

For example, consider an $ARMA(1, 1)$ process with $\phi(L) = 1 - 0.5L$ and $\theta(L) = 1 - 0.5L$. This means that:

$$Y_t = \frac{1 - 0.5L}{1 - 0.5L}\epsilon_t = \epsilon_t$$

In other words, Y_t is a WN process. If we were to simulate such a process - we would find that the estimated parameters are not significantly different from zero (and any automated $ARMA$ order selection process would, *usually*, suggest $p = q = 0$ as the best order in terms of $AICc/BIC$).

As another example, an $ARMA(1, 2)$ process with $\phi(L) = 1 - 0.5L$ and $\theta(L) = (1 - 0.5L)(1 + 0.5L)$ would result in an $MA(1)$ process:

$$Y_t = \frac{(1 - 0.5L)(1 + 0.5L)}{1 - 0.5L}\epsilon_t = (1 + 0.5L)\epsilon_t$$

Looking it the other way around, we could express $Y_t = (1 + 0.5L)\epsilon_t$ as various different $ARMA(1, 2)$ processes by using the common roots. For example, by taking $\phi(L) = 1 - \beta L$ and $\theta(L) = (1 - \beta L)(1 + 0.5L)$ with $|\beta| < 1$.

In practice, if the true underlying $ARMA(p, q)$ process has common AR and MA roots - we will never be able to identify it. However, after removing the same roots, the resulting process, which we **can identify**, will be equivalent to the true one.

Consequently, in practical applications we assume that $\Phi(L)$ and $\Theta(L)$ do not have any common roots. Otherwise there are (infinitely) many possible combinations, where two roots cancel each other out.

Note: Similarly to the ideas for the stationarity conditions for $\Phi(L)$, we can define **invertibility** conditions for $\Theta(L)$. Invertibility allows ϵ_t to be expressed in terms of Y_s, $s \leq t$.

Consequently, when we talk about an $ARMA(p, q)$ process, $\Phi(L)Y_t = \Theta(L)\epsilon_t$:

- We assume that the polynomials $\Phi(z)$ and $\Theta(z)$ have no common roots.
- A unique **stationary** solution Y_t exists if and only if $\Phi(z)$ has no roots **on the unit circle**, i.e. $\Phi(L) \neq 0$, $\forall |z| = 1$.
- A unique **stationary** solution Y_t that is **causal** exists if and only if $\Phi(z)$ has no roots **inside the unit disk**, i.e. $\Phi(z) \neq 0, \ \forall z \in \mathbb{C} : |z| \leq 1$;
- Y_t is **invertible** if $\Theta(z) \neq 0, \ \forall z \in \mathbb{C} : |z| \leq 1$;

Finally, an $ARMA$ representation is not unique. For example, we can write a stationary $AR(1)$ process as an $MA(\infty)$ and an $MA(1)$ and as $AR(\infty)$. In practice, this also means that we can approximate the infinite processes with some finite order.