05 Regression with time lags: Autoregressive Distributed Lag Models

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Introduction

The goal of a researcher working with time series data does not differ too much from that of a researcher working with cross-sectional data: they both aim to develop a regression relating a dependent variable to some explanatory variable.

However, the analyst using time series data will face two problems that the analyst using cross-sectional data will not encounter:

1. One time series variable can influence another with a time lag;
2. If the variable is nonstantionary, a problem known as spurious regression may arise.

One should always keep in mind this general rule: **if you have nonstationary time series variables then you should not include them in a regression model**. The appropriate route is to **transform the variables** before running a regression in order to make them *stationary*. An exception to this rule, which will be presented in a later topic, occurs when the variables in a regression model are non-stationary and *cointegrated*.

In this chapter we will assume all variables in the regression are stationary.

The Distributed Lag Model

We say that the value of the dependent variable, at a given point in time, should depend not only on the value of the explanatory variable at that time period, but also on the values of the explanatory variable in the past. A simple model to incorporate such **dynamic** effects has the form:

$$Y_t = \alpha + \beta_0 X_t + ... + \beta_q X_{t-q} + \epsilon_t$$

Since the effect of the explanatory variable does not happen all at once but rather over several time periods. This model is sometimes referred to as a **distributed** (or weighted) **lag model**. Coefficients can be interpreted as measures of the influence of the explanatory variable on the dependent variable. In this case, we have to be careful with timing.

For instance, we interpret results as 'β_2 measures the effect of the explanatory variable two periods ago on the dependent variable, *ceteris paribus*'.

Selection of Lag Order

When working with distributed lag models, we rarely know *a priori* exactly how many lags we should include. Appropriately, the issue of lag length selection becomes a data-based one where we use statistical means to decide how many lags to include. There are many different approaches to lag length selection in econometrics literature. Here we outline a common one that does not require any new statistical techniques. This method uses *t-tests* for whether $\beta_q = 0$ to decide the length. A common strategy is to:

- Begin with a fairly large lag length, q_{max}, and test whether the coefficients on the maximum lag is equal to zero, i.e. test whether $\beta_{q_{max}} = 0$;
- If it is, drop the highest lag and re-estimate the model with the maximum lag equal to $q_{max} - 1$;
- If you find $\beta_{q_{max}-1} = 0$ in this new regression, then lower the lag order by one and re-estimate the mode;
- Keep on dropping the lag order by one and re-estimating the model until you reject the hypothesis that the coefficient on the longest lag is equal to zero.

Exmaple: The Effect of Bad News on Market Capitalization

The share price of a company can be sensitive to bad news.

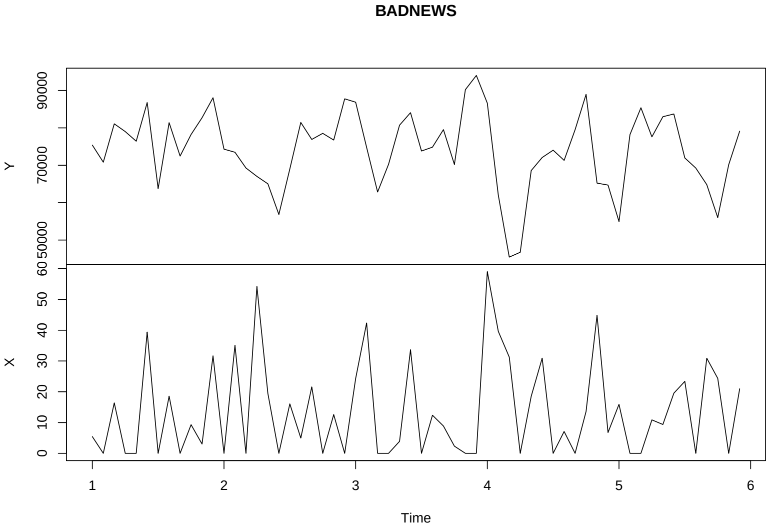
Suppose that Company B is in an industry which is particularly sensitive to the price of oil. If the price of oil goes up, then the profits of Company B will tend to go down and some investors, anticipating this, will sell their shares in Company B driving its price (and market capitalization) down.

However, this effect might not happen immediately. For instance, if Company B holds large inventories produced with cheap oil, it can sell these and maintain its profits for a while. But when new production is required, the higher oil price will lower profits.

Furthermore, the effect of the oil price jump might not last forever, since Company B also has some flexibility in its production process and can gradually adjust to higher oil prices. Hence, news about the oil price should affect the market capitalization of Company B, but the effect might not happen immediately and might not last too long.

Say we have data collected on a monthly basis over five years (i.e., 60 months) on the following variables:

- Y market capitalization of Company B ($000)
- X the price of oil (dollars per barrel) above the benchmark price



Since this is time series data and it is likely that previous months news about the oil price will affect current market capitalization, it is necessary to include lags of X in the regression. Below are present OLS estimates of the coefficients in a distributed lag model in which market capitalization is allowed to depend on present news about the oil price and news up to $q_{max} = 4$ months ago. That is:

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + ... + \beta_4 X_{t-4} + \epsilon_t$$

```
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept) 91173.3150  1949.8502 46.7591   0.0000
## L(X, 0:4)0   -131.9943    47.4361 -2.7826   0.0076
## L(X, 0:4)1   -449.8597    47.5566 -9.4595   0.0000
## L(X, 0:4)2   -422.5183    46.7778 -9.0324   0.0000
## L(X, 0:4)3   -187.1041    47.6409 -3.9274   0.0003
## L(X, 0:4)4    -27.7710    47.6619 -0.5827   0.5627
```

Just looking at the coefficient values, what can we conclude about the effect of news about the oil price on Company B's market capitalization?

Increasing the oil price by one dollar per barrel in a given month is associated with:

1. An immediate reduction in market capitalization of \$ 131'994, *ceteris paribus*.
2. A reduction in market capitalization of \$ 449'860 on month later, *ceteris paribus*.

and so on. To provide some intuition about what the *ceteris paribus* condition implies in this context, note that, for example, we can also express the second statement as: 'Increasing the oil price by one dollar in a given month will tend to reduce the market capitalization in the following month by \$ 449'860, **assuming that no other change in the oil price occurs**'.

Since the *p-value* corresponding to the explanatory variable X_{t-4} is **greater** than 0.05, we cannot reject the null hypothesis that $\beta_4 = 0$ at the 5% level of significance. Accordingly we drop this variable from the model and re-estimate the lag length equal to 3, yielding the following results:

```
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept) 90402.2210  1643.1828 55.0165   0.0000
## L(X, 0:3)0   -125.9000    46.2405 -2.7227   0.0088
## L(X, 0:3)1   -443.4918    45.8816 -9.6660   0.0000
## L(X, 0:3)2   -417.6089    45.7332 -9.1314   0.0000
## L(X, 0:3)3   -179.9043    46.2520 -3.8896   0.0003
```

The *p-value* for testing $\beta_3 = 0$ is 0.0003, which is much less than 0.05. We therefore conclude that the variable X_{t-3} does indeed belong in the distributed lag model. Hence $q = 3$ is the lag length we select for this model.

In a formal report, we would present this table of results. Since these results are similar to those discussed above, we will not repeat their interpretation.

Dynamic Models with Stationary Variables

In regression analysis, researches are typically interested in measuring the effect of an explanatory variable (or variableS) on a dependent variable. However, this goal is complicated when the researcher uses time series data since an explanatory variable may influence a dependent variable with a time lag.

This often necessitates the inclusion of lags of the explanatory variable in the regression. Furthermore, the dependent variable may be correlated with lags of itself, suggesting that lags of the dependent variable should also be included in the regression.

These considerations motive the commonly used **autoregressive distributed lag** (ADL) model:

$$Y_t = \alpha + \delta t + \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \beta_0 X_t + ... + \beta_q X_{t-q} + \epsilon_t$$

In this model:

- The dependent variable Y depends on p lags of itself;
- Y also depends on the current value of an explanatory variable X as well as q lags of X.;
- The model also allows for a deterministic trend t.

Since the model contains p lags of Y and q lags of X, we denote it by $ADL(p, q)$.

In this chapter, we focus on the case where there is only *one* explanatory variable X. Note however, that we could equally allow for many explanatory variables in the analysis.

Let us consider two stationary variables Y_t and X_t and assume that it holds that:

$$Y_t = \alpha + \phi Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1} + \epsilon_t, \quad 0 < \phi < 1$$

As an illustration, we can think of Y_t as 'company sales' and X_t as 'advertising', both in month t. If we assume that ϵ_t is a white noise process, independent of X_t, Y_t and X_{t-1} and Y_{t-1}, the above relation can be estimated by the use of ordinary least squares.

The interesting element in this equation is that it describes the dynamic effects of a change in X_t upon current and future values of Y_t.

Taking the partial derivatives, we can derive that the immediate response is given by $\partial Y_t/\partial X_t = \beta_0$. Sometimes this is referred to as the **impact** (or **short-run**) multiplier. An increase in X with one unit has an immediate impact on Y of β_0 units.

The effect after one period is:

$$\partial Y_{t+1}/\partial X_t = \phi \partial Y_t/\partial X_t + \beta_1 = \phi\beta_0 + \beta_1$$

Note: this can also be derived in a more explicit way:

$$\begin{aligned}
Y_{t+1} &= \alpha + \phi Y_t + \beta_0 X_{t+1} + \beta_1 X_t + \epsilon_{t+1} \\
&= \alpha + \phi(\alpha + \phi Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1} + \epsilon_t) + \beta_0 X_{t+1} + \beta_1 X_t + \epsilon_{t+1} \\
&= \alpha(1+\phi) + \phi^2 Y_{t-1} + \beta_0 X_{t+1} + \boldsymbol{(\phi\beta_0 + \beta_1) X_t} + \phi\beta_1 X_{t-1} + \phi\epsilon_t + \epsilon_{t+1}
\end{aligned}$$

Similarly, after two periods:

$$\partial Y_{t+2}/\partial X_t = \phi \partial Y_{t+1}/\partial X_t = \phi(\phi\beta_0 + \beta_1)$$

and so on. This shows that after the first period, the effect is *decreasing* if $|\phi| < 1$.

Imposing this so-called stability condition allows us to determine the long-run effect of a unit change in X_t. It is given by the **long-run multiplier** (or equilibrium multiplier):

$$\beta_0+(\phi\beta_0+\beta_1)+\phi(\phi\beta_0+\beta_1)+... = \beta_0+(1+\phi+\phi^2+...)(\phi\beta_0+\beta_1) = \frac{\beta_0+\beta_1}{1-\phi}$$

This says that if advertising X_t increases with one unit for <u>one moment</u>, the expected cumulative increase (or decrease) in sales is given by $(\beta_0+\beta_1)/(1-\phi)$.

On the other hand, if the increase in X_t is <u>permanent</u>, the long-run multiplier also has the interpretation of the expected long-run permanent increase in Y_t. The long-run equilibrium relation between Y and X can be seen (imposing $Y_{t-1} = Y_t = Y_{t+1}... = Y$, $X_{t-1} = X_t = X_{t+1}... = X$, $\epsilon_t = \epsilon_{t+1} = ... = 0$, or taking the expectations of both sides, which, under stationarity, give $\mathbb{E}(Y_t) = \mathbb{E}(Y_{t-1}) = Y$ and $\mathbb{E}(X_t) = \mathbb{E}(X_{t-1}) = X$) to be:

$$Y = \alpha + \phi Y + \beta_0 X + \beta_1 X$$

or

$$Y = \frac{\alpha}{1-\phi} + \frac{\beta_0+\beta_1}{1-\phi}X$$

which presents an alternative derivation of the long-run multiplier.

We shall write $Y = \alpha/(1-\phi) + (\beta_0 + \beta_1)/(1-\phi)X$ concisely as $Y = \tilde{\alpha} + \tilde{\beta}X$, with obvious definitions of $\tilde{\alpha}$ and $\tilde{\beta}$. Thus, if X changes to a new constant X', Y will finally change to $Y' = \tilde{\alpha} + \tilde{\beta}X'$ (but it will take some time!).

There is an alternative way to formulate the autoregressive distributed lag model by subtracting Y_{t-1} from both sides of:

$$Y_t = \alpha + \phi Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1} + \epsilon_t$$

Some rewriting gives us:

$$\Delta Y_t = \alpha - (1-\phi) Y_{t-1} + \beta_0 \Delta X_t + (\beta_0 + \beta_1) X_{t-1} + \epsilon_t$$

or

$$\Delta Y_t = \beta_0 \Delta X_t - (1-\phi) \left[Y_{t-1} - \tilde{\alpha} - \tilde{\beta} X_{t-1} \right] + \epsilon_t$$

This formulation:

$$\Delta Y_t = \beta_0 \Delta X_t - (1-\phi)\left[Y_{t-1} - \tilde{\alpha} - \tilde{\beta} X_{t-1}\right] + \epsilon_t$$

is an example of an **error-correction model** (ECM).

It says that the change in Y_t is due to the current change in X_t plus an error-correction term: if Y_{t-1} is above the equilibrium value corresponding to X_{t-1}, that is, if the 'disequilibrium error' in the square brackets is positive, then a 'go to equilibrium' mechanism generates additional negative adjustment in Y_t.

The speed of adjustment is determined by $1-\phi$, which is the adjustment parameter. Note that stability assumption ensures that $0 < 1-\phi < 1$. Therefore only a part of any disequilibrium is made up for in the current period.

Notice that without prior knowledge of the long-run parameters, we cannot estimate the above ECM in its current form. This is because without knowing $\tilde{\alpha}$ and $\tilde{\beta}$, we cannot construct the disequilibrium error $Y_{t-1} - \tilde{\alpha} - \tilde{\beta} X_{t-1}$. In the absence of such knowledge, to direct;y estimate the ECM, we must first multiply out the term in parenthesis to obtain:

$$\Delta Y_t = (1-\phi)\tilde{\alpha} + \beta_0 \Delta X_t - (1-\phi) Y_{t-1} + (1-\phi)\tilde{\beta} X_{t-1} + \epsilon_t$$

and ΔY_t can now be OLS-regressed on ΔX_t, Y_{t-1} and X_{t-1}, obtaining estimates of all short-run and long-run parameters.

We can further generalize. For example, if:

$$Y_t = \alpha + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \beta_0 X_t + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \epsilon_t$$

then, the ECM is:

$$\Delta Y_t = -\phi_2 \Delta Y_{t-1} + \beta_0 \Delta X_{t-1} - \beta_2 \Delta X_{t-2} - (1 - \phi_1 - \phi_2) \left[Y_{t-1} - \tilde{\alpha} - \tilde{\beta} X_{t-1} \right] + \epsilon_t$$

Note that the original model must be rewritten in differences plus a disequilibrium error. To estimate this model, it is again necessary to express it by multiplying out the term in parenthesis.

It is possible for more than two variables to enter into an equilibrium relationship. For example:

$$Y_t = \alpha + \beta_0 X_t + \beta_1 X_{t-1} + \gamma_0 Z_t + \gamma_1 Z_{t-1} + \phi Y_t + \epsilon_t$$

This equation then can be transformed to:

$$\Delta Y_t = \beta_0 \Delta X_t + \gamma_0 \Delta Z_t - (1 - \phi)\left[Y_{t-1} - \tilde{\alpha} - \tilde{\beta} X_{t-1} - \tilde{\gamma} Z_{t-1}\right] + \epsilon_t$$

All the ECM's may be consistently estimated via OLS **provided all the predictors are stationary**.

As long as it can be assumed that the error term ϵ_t is a *white noise* process, or - more generally - *is stationary* and independent of $X_t, X_{t-1}, ...$ and $Y_{t-1}, Y_{t-2}, ...$, the ADL models can be estimated *consistently* by ordinary least squares (OLS). Problems may arise, however, if, along with Y_t and X_t, the implied ϵ_t is also non-stationary.

This will be discussed in the next topic.