**§3. On the sum of reciprocal of primes**

Reciprocal number for the natural number is called a number .

In this lecture we will consider the infinite sum of reciprocal of primes i .e. the series of the type

Is this series converges or not? From real analysis we know that harmonic series

diverges. The set of primes is more rearly than the set of natural numbers. So, may be it converges?

Now let consider series of reciprocals of the primes. That is the series

The following theorem is about it.

**Theorem 6.**

The series

divergent.

**Proof. (J. Clarcson)**

We assume the series converges and obtain a contradiction. If the series

converges there is an integer such that

as the tail of converges series.

Let , and consider the numbers 1+, for .

None one of these numbers 1+ are not divisible by any primes ,,...,. Therefore, all the prime factors of 1+ occur among the primes , ,... ,and by the fundamental theorem of arithmetic

for some , where ,,..., (and the are not necessarily distinct, allowing primes to appear more than once in the factorization).

Then the term

must appear in the expansion of

Therefore for each we have

since the sum on the right includes among its terms all the terms on the left.

But the right-hand side of this inequality is dominated by the convergent geometric series

Therefore the series

has bounded partial sums and hence converges.

But this is a contradiction because

and series

diverges as harmonic series.

**End.**

So the series of reciprocal of primes

divergent.

The proof of Clarkson is in question about rate of divergency. The following theorem is due to **L. Euler**

**Theorem 7. (L. Euler)**

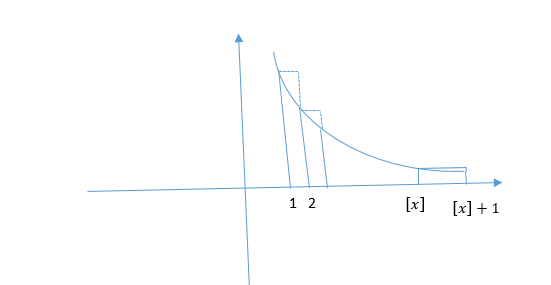
If there exists a stricly positive real number such that

**Proof.**

Let consider a product (Euler product)

A stroke in above sum means that sum is taken over such which prime factors didn‘t exceed and . ( number in the sum could occur value more than )

Therefore by the integral test



and

Then for let‘s take the natural logarithm of each side

Using Taylor series expansion for as well as the sum of converging series

Hence

Therefore

Because the series

convergent therefore it‘s sum is a finite number.

**End.**

In the inequality

one could more precisely to estimate the constant .

It is not dificcult to see that series

equal 1 by telescoping principle.

So, we could write

In fact is known that

where is the Meissel-Mertens constant.

**§4. Prime-counting function**

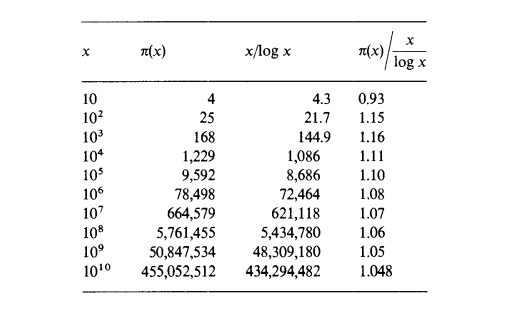
Another aproach in [mathematic](https://en.wikipedia.org/wiki/Mathematics)s, to prime numbers is the prime-counting function . That is is the [function](https://en.wikipedia.org/wiki/Function_(mathematics)) counting the number of [primes](https://en.wikipedia.org/wiki/Prime_number) less than or equal to some [real number](https://en.wikipedia.org/wiki/Real_number) *x*. It is denoted by *π*(*x*) and unrelated to the [number *π*](https://en.wikipedia.org/wiki/Pi).

Thus,

Sometimes also is used such definition

Of great interest in [number theory](https://en.wikipedia.org/wiki/Number_theory) is the [growth rate](https://en.wikipedia.org/wiki/Asymptotic_analysis) of the prime-counting function *π*(*x*). It was [conjectured](https://en.wikipedia.org/wiki/Conjecture) in the end of the 18th century by [Gauss](https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss) and by [Legendre](https://en.wikipedia.org/wiki/Adrien-Marie_Legendre) to be approximately

Here is a brief table of this function and its comparison with , where is the natural logarithm of :



By examining the table like this for , Gauss and Legendre proposed independently that for large the ratio

was nearly 1 and they conjuctured tha this ratio would approach to 1as approaches to .

Both Gauss and Legendre attempted to prove his statement but did not succeed. The problem of deciding the truth or falsehood of this conjucture attracted the attention of mathematicians for nearly 100 years.

In 1851 the Russian mathematician Chebyshev made an important step foward by proving that if the ratio did tend to a limit, the this limit must be 1. However he was unable to prove that the ratio does tend to a limit.

In 1859 Riemann attaccked the problem with analytic methods, using a formula discovered by Euler in 1737 which relates the prime numbers to the function

for real . But Riemann was unable to completely solve this problem before his death in 1866.

Only thirty years later using analytic tools introduced by Riemann i.e. in 1896 J. Hadamar and C.J. de la Valee Poussin independently and simultaneously suceeeded in proving that

This remarkable result is called the prime number theorem and its proof was one of crowning achievements of analytic number theorey.

Only in 1949 Selberg and Erdosh caused a sensation in the mathematical world when they discovered an elementary proof of the prime number theorem.

**§5. Eratoshenes Legendre sieve**

The main goal of this paragraph to prove that the ratio of value of prime counting function with tends to zero, if growing to infinity.

For this we will need to find primes from the list

is the first prime. The method will be use is called Eratoshenes-Legendre sieve.

If from this list we remove all primes and it‘s multiples then all non deleted numbers from this list will be a primes and will satisfy the condition

Eratoshenes sieve is an algoritm for listing primes, but not a method for counting them. Counting version is usually called the Eratoshenes-Legendre sieve. It is as follows

In the left side we have number of primes, satisfing the condition

In the right side we have an algoritm to indentifying these primes.

-here is the largest positive integer less than positive real number . From all positive integer we remove 1, because is not a prime but also not composite.

A term for each prime below square root of is equal the number of multiples of this prime below , i.e. how many of integers less or equal have this prime in their decomposition.

Also when we have a number less or equal than that is divisible by two diferent primes less or equal to , it will be counted twice in our sum over the , so we remove this number two times. To compensate for that we add the sum over the terms .

Now we have a similar problem in that if we have an integer in our range that is divisible by three diferent primes, then in the first sum we counted and subtract this number three times, the secod sum counts this number and add it times, so to compensate we have the third sum over positive integers divisible by three diferent primes.

Therefore if an integer is divisible by primes, then

Sum it removes times

Sum add it times

etcetera

Sum removed it times

And we get that such integer totaly removed

i.e only one time.

**Homework**

**7.**

Prove that the partial sums of the reciprocals of the primes never equal an integer.

**8.** Find information aboutBertrand‘s postulate and using it to provethat if is an integer, then

is not an integer.