**Final score and strategy of assesment**

1. Exercises for homeworks evaluated by 30 percent of the final score;
2. Student presentation from the given scientific paper evaluated by 40 percent of the final score;
3. Exam from the theretical part of the course evaluated by 30 percent of the final score;

**Literature**

1. T.M Apostol. Introduction to analytic number theory 1976.
2. E. C. Titchmarch The theory of Riemann Zeta-function 1986.
3. A. Laurinčikas Rymano dzeta funkcijos teorijos pagrindai, 1990 m. VU
4. K. Прахар Расспределение простых чисел. Primzahlverteilung 1957.

**§1. Introduction**

1. **Acient greeks. Pythagor triples around 600 BC.**

The first scientific approach to the study of integers is generally attributed to acient greeks. Greek philosopher Pythagoras founded own shool, memebers of which called the Pythagorens. The Pythagoreans explored the nature of numbers, interested in numbermysticizm or numerology.

They also were interested in right triangles whose sides are integers. Such triangles are now called Pythagorean triples or Pythagorean triangles. Quite simple example of such triple is the right triangle with legs 3 and 4 and hypotenuse 5. The Pythagoreans interested how many triples one could to construct and gave a method for determening it‘s infinetely many. For example

let , the is an odd number greater than 1, then one could define legs of right triangle in such way ; . Hypotenuse . Then, represent the Pythagorean triples.

Triples

;;;,...are examples.

As well, if one side of triangle has even length, one could construct such Pythagorean triples

; ;

For example,

;,...

1. **Euklid around 300 years BC. Infinitely many primes.**

Significant place in number theory intended for the prime numbers.

Prime number, is a number greater than 1 whose only divisors are 1 and the number itself.

Numbers that are not prime are called composite, except 1that is considered neither prime nor composite.

Euklid proved that there are infinitely many primes.

**Theorem 1. (Euclid)**

There are infinitely many primes.

**Proof.**

Supose that <<...<are all of the primes. Let and let be a prime dividing , then can not be any of ,,...,, otherwise would devide the diference which is imposible. So this prime is still another prime, and ,,..., would not be all of the primes.

**End.**

**Note.**

It is a common mistake to think that this proof says the product is a prime. The proof actually only uses the fact that there is a prime dividing this product.

For example is not a prime and divisible from 59.(-primes until 13 including)

1. **Perfect numbers.**

Also Euklid made contribution to another problem posed by Pythagoreans- finding all perfect numbers. Number is called perfect if it is equal to sum of it‘s proper divisors. (i.e. divisors less the number itself.).For example 6 is perfect, because it is equal

6

28 is also perfect, because it divisors are 1,2,4,7,14 and

Euklid find all perfect numbers. He proved that if number has a form

,

where p and are primes, then it is an even perfect number.

6

28

496

The other perfect numbers are 8128,33550336, where and .

When number is not perfect, because is not prime and has divisor 23.

Two thausand year later Euler proved that every even perfect number must be of Euklid‘s type.

Nowadays unsolved problem in mathematics is

**Are there any odd perfect number?**

In nowadays is known if odd perfect number exist it must be very large, greater than

And the other unsolved problem

**Are there infinitely many perfect numbers?**

To answer to this question we should to answer for which prime number is also prime.

1. **Mersen numbers.**

Numbers of the form

where is prime are known as Mersenne numbers.

Nowadays are known 51 prime Mersenne numbers.

Let

then

then

then

then

All these Mersenne numbers are prime. But for

number is composite.

Nowadays the largest prime Mersen number is

. Is also the largest known prime number.

If we write this number it has 24.862.048 digits in 10 base system.

The following problem in number theory is open

**Are there infinitely many Mersen numbers?**

1. **Fermat numbers.**

The numbers , where are called Fermat numbers. So Fermat numbers are of the type

These numbers discovered Piere de Ferma.The 5 first Fermat numbers are prime and Fermat made a conjecture that all such numbers are prime.

In fact, 5 first Fermat number are prime

3,5,17,257, 65537

but if the number

4294967297

is composite. This fact showed by Leonard Euler in 1732.

Nowadays no more prime Fermat numbers except first 5 are known.

Nowadays unsolved problems in mathematics are the following

**Is composite for all**

**Are there infinitely many prime Fermat numbers**

**Are there infinitely many composite Fermat numbers?**

1. **Irregularity distribution of primes in oder of natural numbers .**

Many problems in number theory arise around the prime numbers. In history of number theory had been spent a lot of time to find suitable formula to generate prime numbers. In nowadays it is unknown simple formula which producing all prime numbers. The reason for this is irregularity distribution of primes in oder of natural numbers.

In oder from 1 to 100 one could find 25 prime numbers, so prime numbers occupied 25 percents of naturfal numbers in order to 100. But less that 10 milion are about 6 .5 percents of natural numbers, less than 10 billion are 4,5 percent of primes. Although all these primes are not known individually. In tables of prime we could find long gaps without any prime.

For example the oder of numbers

**,** , ,...,

has not any prime number. All these numbers are composite. If sufficiently large such gaps could be very long.

1. **Prime twins.**

From the other hand one could indicate the consecutive prime such 3 and 5, or 11 and 13, or 101,103. Such pairs are known as twin primes. Many mathematicians think that there are infinitely many such pairs, but no one has been able to prove this as yet.

So,

**Are there infinitely many prime twins?**

Is also unsolved mathematical problem.

1. **Analytic formula generating prime numbers.**

If we talk about prime, are known some formulae, that do yield many primes. For example, the expresion

Gives a prime for

Or

Which gives a prime for .

Or polynomial

Gives primes for the values

But it is unknow polynomial which give a prime for all . In 1752 Goldbach proved that no polynomial with integer coefficients which produce only prime numbers.

**Theorem 1. (1752 Goldbach)**

If is a polynomial with an integer coefficients , such than is prime for all natural . Then is a constant.

**Proof.**

Fix , if is prime number, then ,

Therefore,

for all . Since is prime for all , then

Then polynomial has infinitely many roots , .

But , so it should have no more than roots, or to be a constant.

**End.**

**§2. The fundamental arithmetic theorem**

**Theorem 2. (The fundamental theorem of arithmetic)**

Every integer  can be represented as a product of prime factors in only one way, apart from the oder of the factors.

**Proof.**

If is prime, than proof is end. Suppose, that is composite. From the first theorem we have

, prime number. If is also prime it is the end of proof. So let is composite. In such way we get a decreasing finite oder of natural numbers and the last number in this oder should be prime.

Therefore we could write as a product of primes

Now we will prove the uniqueness. Suppose that number we represent as product of primes in two ways

Since devides the product it must devide at least one factor. Relabel

so that . Then since both and are primes. Then,

If , we get that

And we get that

So we could keep that .

**End.**

**Note.**

In factorization of an integer prime number may occur more than once. If the distinct factors of are and if occurs as a factor times, we can write

Or more briefly

This is called the factorization of into prime powers.

The second theorem is called Eratosthenes sieve and sugest a rule to find the prime numbers.

**Theorem 3. (Eratosthenes sieve)**

If in the sequence of natural numbers

2,3,4,...

one will delete first few prime numbers and it‘s multiples, then first non-deleted number in this sequence will be a prime. If in this sequence one will delete all prime numbers and it‘s multiples, then the all non-deleted numbers in this sequence will be primes and will satisfy the condition

.

**Exercises for homework**

**1.**

Prove, that, is a composite number, if composite.

**2.**

Prove, that,if is a prime, then is a prime.

**3.**

Show that if number is prime, for , then .

**4.**

Prove that number

is composite if is a composite odd number.

**5.**

Prove that is a composite number, if is an integer.

**6.**

Let , ,..., are the prime numbers less or equal .

1. Prove there is unique representation for any as follows

where and , ,...,.

Show that . Hence show that there are infinitely many primes.

**7.**

Let  and are the prime twins. Prove, that composite number

satisfying an inequality divisible by 6.