Parametrika 1.1
Tutorial

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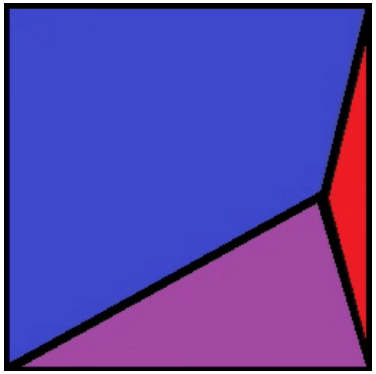
April 3, 2023



Contents

1 Overview

The program Parametrika 1.1 was released in April of 2023. It was written with Python 3.7 using Kivy (`https://kivy.org/`).

The window of the main Program is presented in Fig. 1. The main windows of the modules *Bulk Crystals* and *PP Crystals* are presented in Figs. 2 and 3, respectively. In both modules, one can choose either *Up-conversion* or *Down-conversion* modules. The crystals' database can be edited by pressing *Edit Database*.

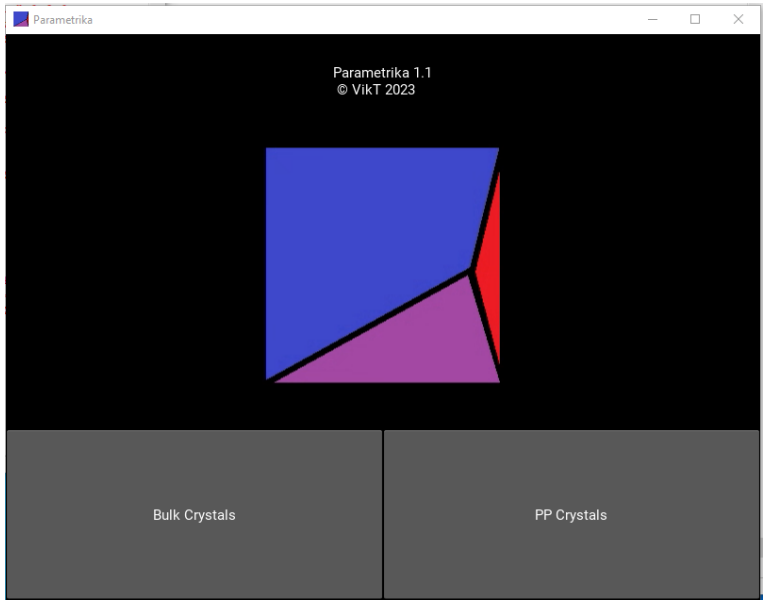


Figure 1: The main window.

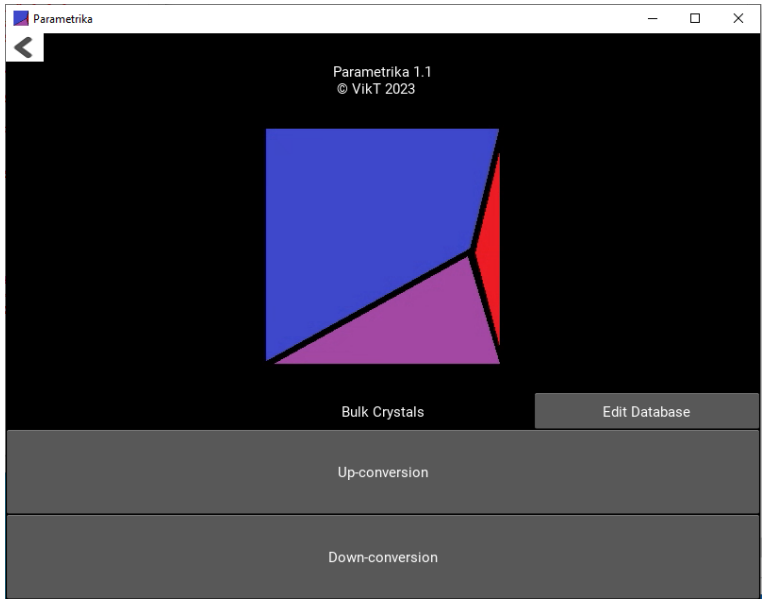


Figure 2: Window of the module *Bulk Crystals*.

Figure 3: Window of the module *PP Crystals*.

2 Module *Bulk Crystals*

2.1 Module *Down-conversion*

2.1.1 Three interacting waves

The phase-matching for optical parametric down-conversion is calculated. Three interacting waves, their angular frequencies and wavelengths:

- *Signal*: ω_1, λ_1.
- *Idler*: ω_2, λ_2.
- *Pump*: ω_3, λ_3.

Conservation law of the photon energy (Fig. 4):

$$\hbar\omega_3 = \hbar\omega_1 + \hbar\omega_2, \tag{1}$$

where $\hbar$ is the reduced Plank constant. $\omega = 2\pi c/\lambda$, where c is speed of light. Therefore:

$$\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}. \tag{2}$$

Phase-matching schemes for the collinear as well as noncollinear interaction types are presented in Fig. 5.

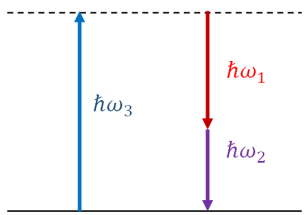


Figure 4: Scheme of photon energies in the optical parametric down-conversion.

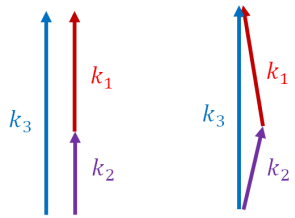


Figure 5: Collinear (left) and noncollinear (right) phase-matching schemes. $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ are the wavevectors of signal, idler and pump waves, respectively.

2.1.2 Choose wavelengths

First, write the wavelengths values in nanometers for signal and pump waves, Fig. 6. Idler wavelength is calculated by the use of Eq. (2).

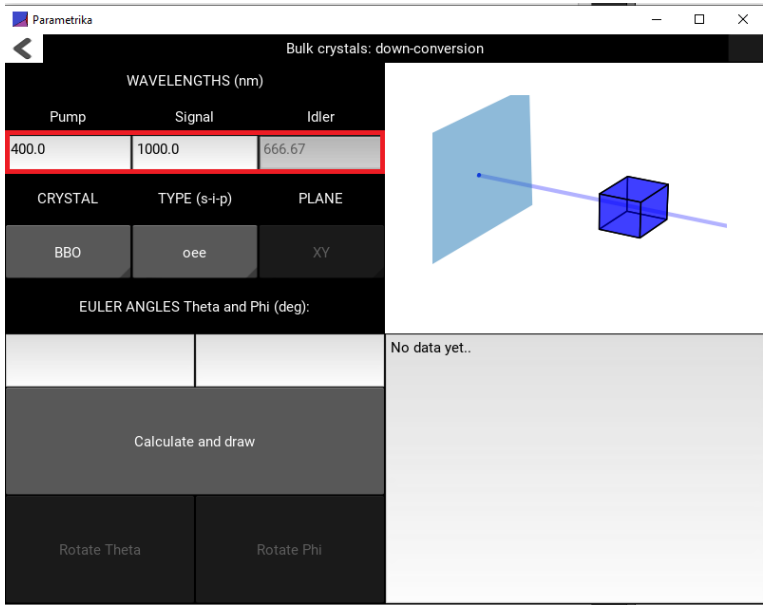


Figure 6: Input wavelengths menu.

2.1.3 Nonlinear crystals

List of nonlinear crystals (Fig. 7):

- *ADP*, ammonium dihydrogen phosphate (uniaxial).
- *BBO*, beta-barium borate (uniaxial).
- *GaSe*, gallium selenide (uniaxial).
- *KDP*, potassium dihydrogen phosphate (uniaxial).
- *KTP*, potassium titanyl phosphate (biaxial).
- *LBO*, lithium triborate (biaxial).
- *LN*, lithium niobate (uniaxial).

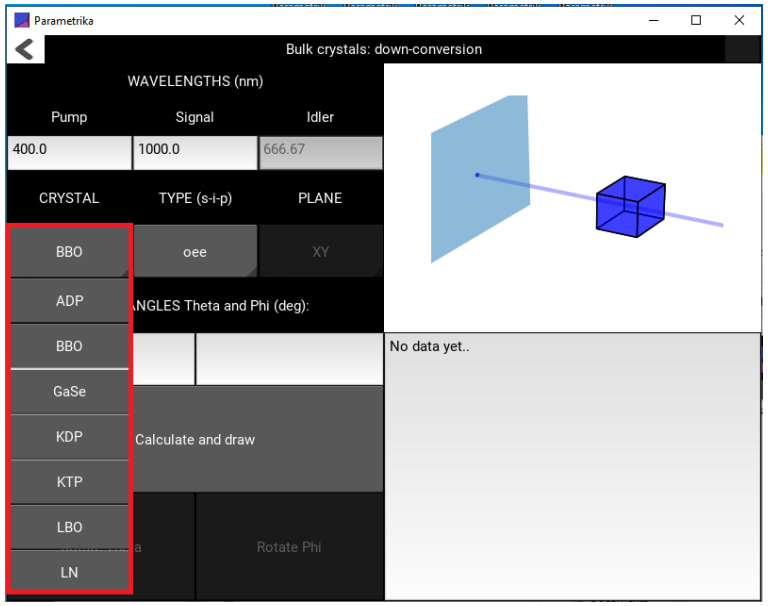


Figure 7: Select crystal drop-down menu.

2.1.4 Interaction type

In the interaction type, the notations are in the following order: signal-idler-pump, e.g. *ooe* means that signal and idler waves are ordinary waves and pump wave is extraordinary wave.

List of interaction types (Fig. 8):

- *ooe*
- *oee*
- *eoe*
- *eeo*
- *eoo*
- *oeo*

For negative uniaxial crystals (*ADP*, *BBO*, *GaSe*, *KDP*, *LN*), the interaction types *eeo*, *eoo*, *oeo* are impossible.

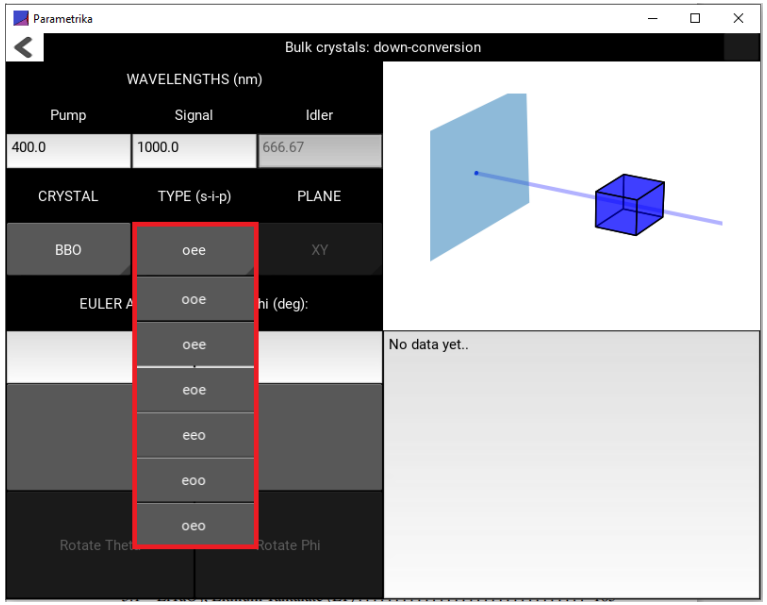


Figure 8: Select type drop-down menu.

2.1.5 Interaction plane

For biaxial crystals (*KTP*, *LBO*), the plane bar is activated. List of planes (Fig. 9):

- *XY*
- *XZ*
- *YZ*

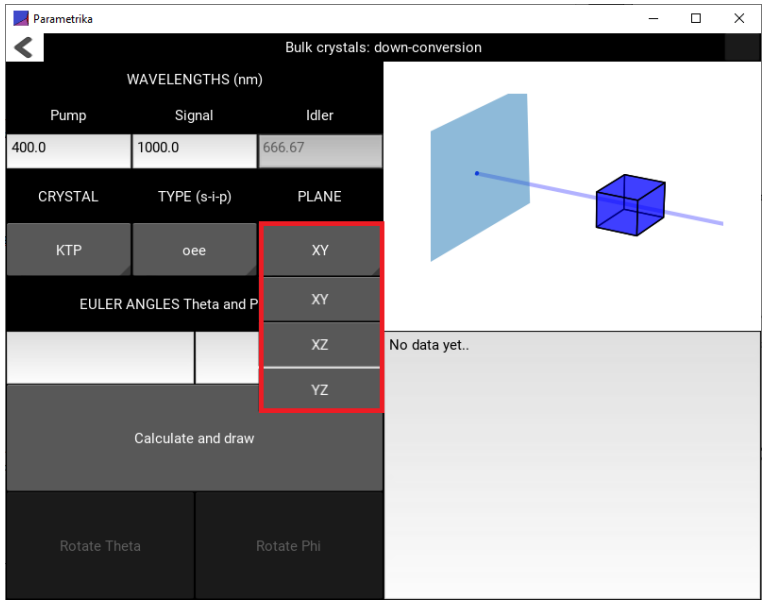


Figure 9: Select plane drop-down menu.

2.1.6 Geometry

The Euler angles θ and φ (*Theta* and *Phi*) are shown in Fig. 10. In the uniaxial crystal (*ADP*, *BBO*, *GaSe*, *KDP*, *LN*), z axis is the *optical axis*. Then, principal refractive indices $n_x = n_y = n_o$ and $n_z = n_e$.

In uniaxial crystals, all possible phase-matching angles are calculated. In biaxial crystals (*KTP*, *LBO*), the phase matching is calculated only in one chosen plane.

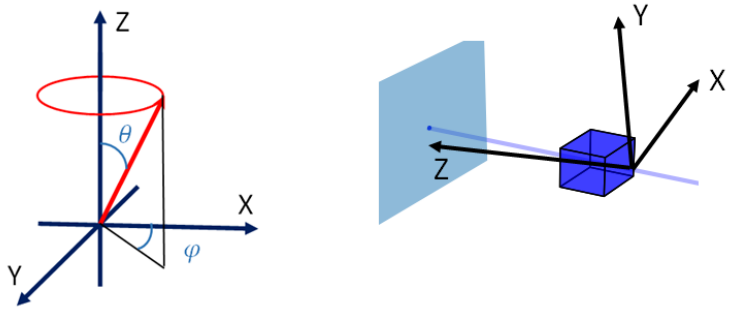


Figure 10: Left: Euler angles θ and φ (*Theta* and *Phi*) in the Cartesian coordiante system x, y, z. Right: coordinate system for uniaxial crystal.

2.1.7 Run!

- To run the program press *Calculate and draw* button, Fig. 11.

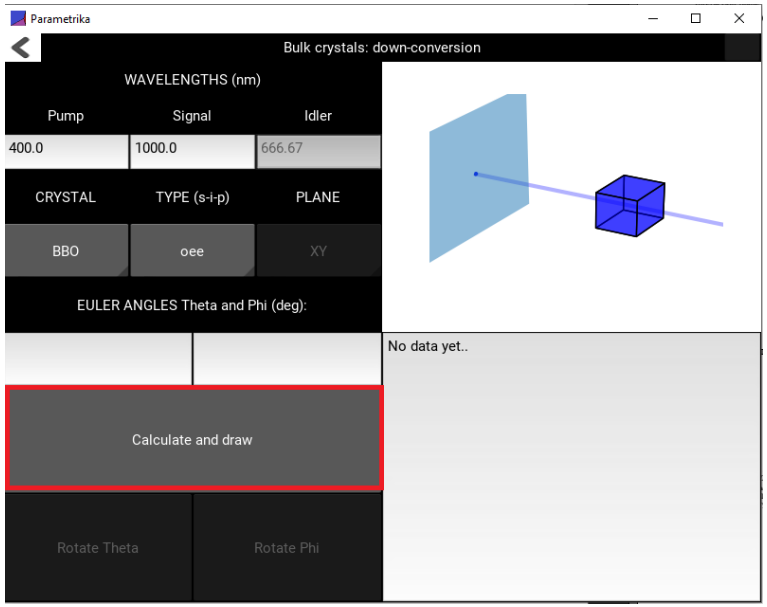


Figure 11: Calculate and draw button.

- If *Euler angles* are initially empty, then the collinear phase-matching is calculated. Otherwise, the program searches for noncollinear phase-matching.
- If the phase-matching is found the buttons *Rotate Theta* and *Rotate Phi* become activated (Fig. 12). By pressing these buttons the crystal is rotated quickly by the step of 5°. The crystal may be also rotated slowly by changing the angles in the corresponding input labels.
- In the biaxial crystals (*KTP*, *LBO*), the rotation only in one plane is allowed.

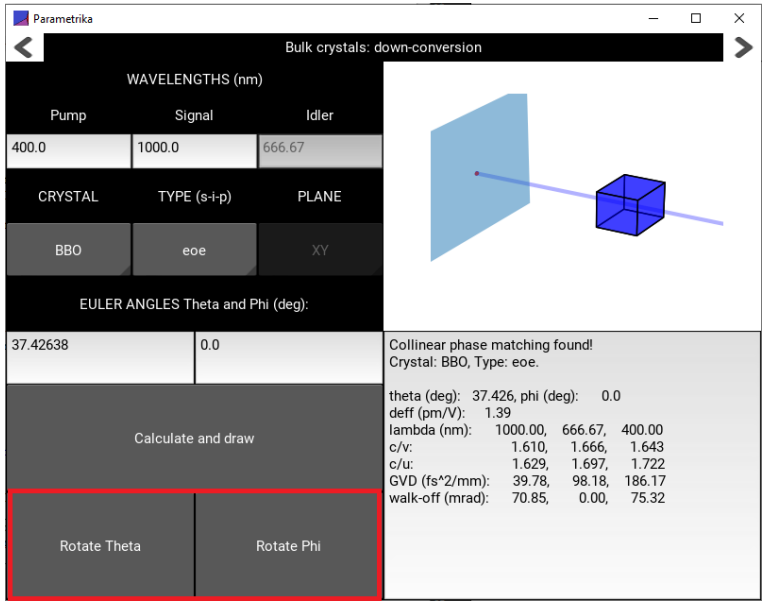


Figure 12: Press these buttons to rotate the crystal.

- Dispersion parameters for all three interacting waves are shown in the output box 1 (Fig. 13).
- The crystal and output waves are visualized in the graphic box 2 (Fig. 13).

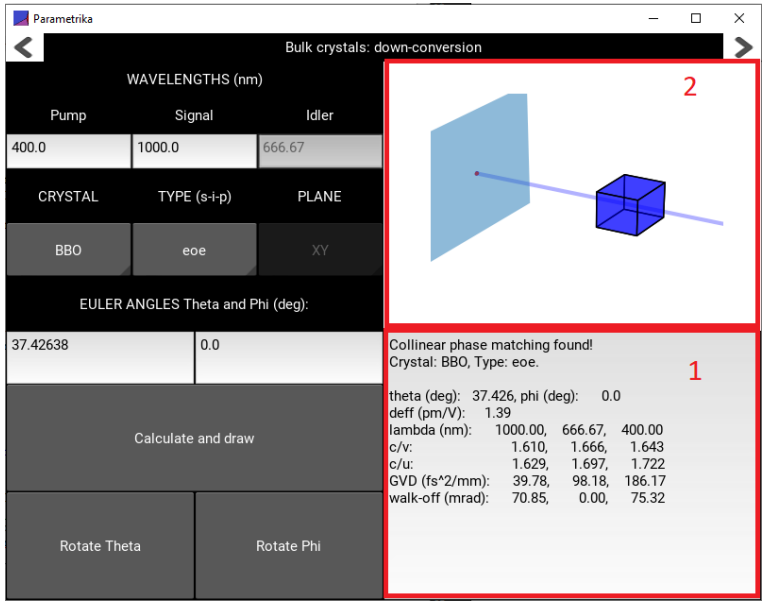


Figure 13: Visualization and information boxes.

2.1.8 3D visualization

- *Uniaxial crystal.* The crystal is cut with respect to the collinear phase-matching angle θ_p (Fig. 14a) and angle φ corresponds to the optimal d_{eff}. The signal and idler cones are visualized in the case of noncollinear phase-matching (Fig. 14b).
- *Biaxial crystal.* The chosen plane is horizontal and the crystal is cut with respect to the collinear phase matching angle. By varying the angle (either *Theta* or *Phi*) the noncollinear phase-matching is calculated (Fig. 14c). Rotation out of the plane is prohibited.
- The walk-off angle is visualized only for uniaxial crystals.

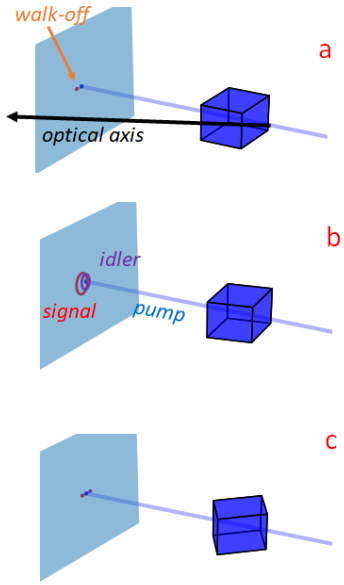


Figure 14: Visualization of (a) collinear phase-matching in uniaxial crystal; (b) noncollinear phase-matching in uniaxial crystal; (c) noncollinear phase-matching in biaxial crystal.

2.1.9 Dispersion parameters

The dispersion parameters are found by the use of the Sellmeier equations from [1].

List of the parameters (Fig. 15):

- c/v: refractive index.
- c/u: fraction of speed of light to the group velocity.
- GVD: group velocity dispersion coefficient.
- *walk-off*: the walk of angle.

The effective nonlinear susceptibility d_{eff} is found by the use of formulas given in [1]. This parameter is wavelength- and angle- dependent.

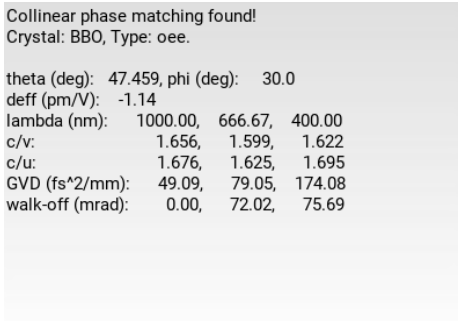


Figure 15: Dispersion parameters in the information box.

2.1.10 Bandwidth estimation window

- After successful calculations in the main window of *Down-conversion* module, second window may be activated by pressing the right arrow button (Fig. 16) for bandwidth calculations.

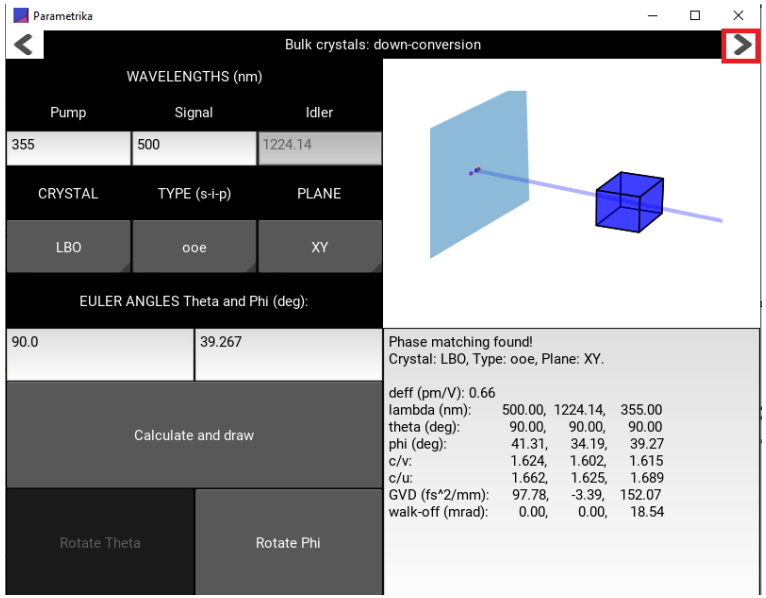


Figure 16: Press right arrow button.

- Choose input parameters 1 and press *RUN* 2 (Fig. 17).
- The user may choose either *signal* of *idler* waves.
- For intensity evaluation, the following input parameters are used: *Energy*, *Pulse duration*, *Beam radius*. For gain band calculation, parameter *Crystal length* is used as well. See Section 4.1.3 for more details.

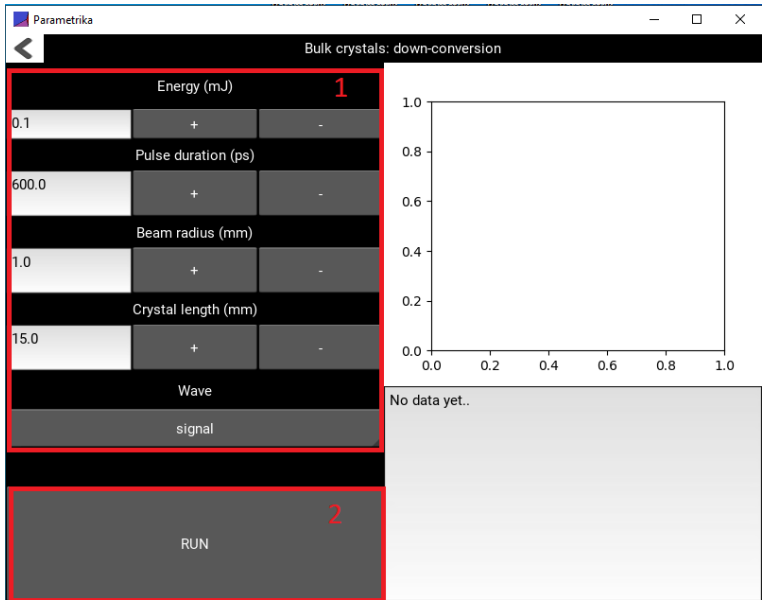


Figure 17: Bandwidth calculation window of module *Down-conversion*.

- Gain band is calculated and presented in a graphical box. In the output box, the calculated gain bandwidth at FWHM is presented as well (Fig. 18).
- The crystal information is given in the output box.

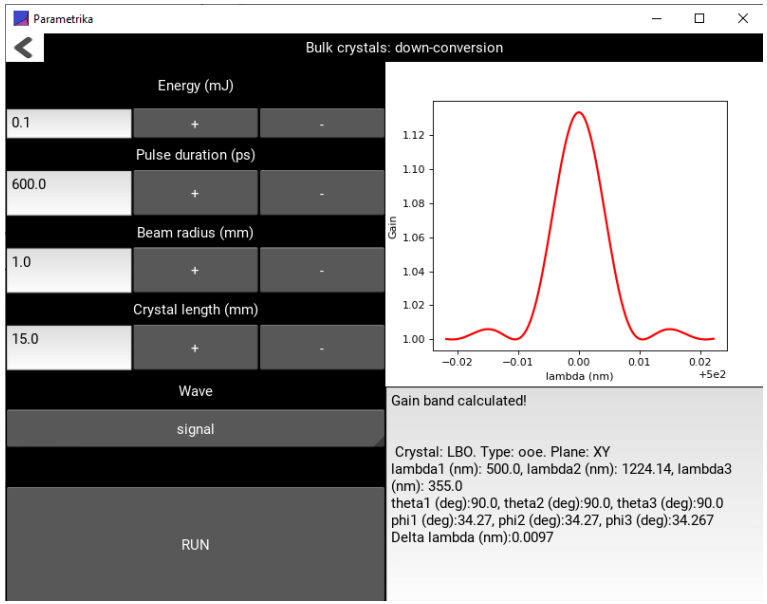


Figure 18: Bandwidth calculation in the module *Down-conversion*.

- Click the left arrow button to return to the main window of *Down-conversion* module (Fig. 19).

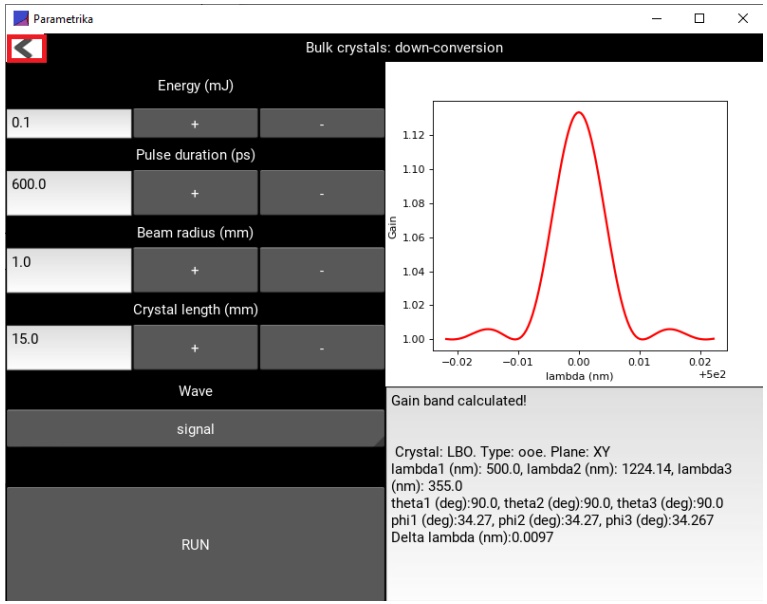


Figure 19: Left arrow button returns to the main window of the *Down-conversion* module.

2.2 Module *Up-conversion*

2.2.1 Three interacting waves

The phase-matching for optical parametric up-conversion is calculated. Three interacting waves, their angular frequencies and wavelengths:

- *Pump 1*: ω_1, λ_1.
- *Pump 2*: ω_2, λ_2.
- *Sum Frequency*: ω_3, λ_3.

Conservation law of the photon energy (Fig. 20):

$$\hbar\omega_1 + \hbar\omega_2 = \hbar\omega_3, \tag{3}$$

where $\hbar$ is the reduced Plank constant. $\omega = 2\pi c/\lambda$, where c is speed of light. Therefore:

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{1}{\lambda_3}. \tag{4}$$

Phase-matching schemes for the collinear as well as noncollinear interaction types are presented in Fig. 21.

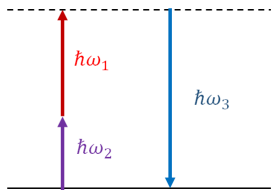


Figure 20: Scheme of photon energies in the optical parametric up-conversion.

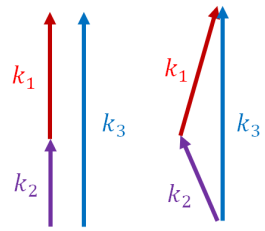


Figure 21: Collinear (left) and noncollinear (right) phase-matching schemes. $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ are the wavevectors of pump 1, pump 2 and sum-frequency waves, respectively.

2.2.2 Choose wavelengths

First, write the wavelengths values in nanometers for signal and pump waves, Fig. 22. Sum frequency wavelength is calculated by the use of Eq. (4).

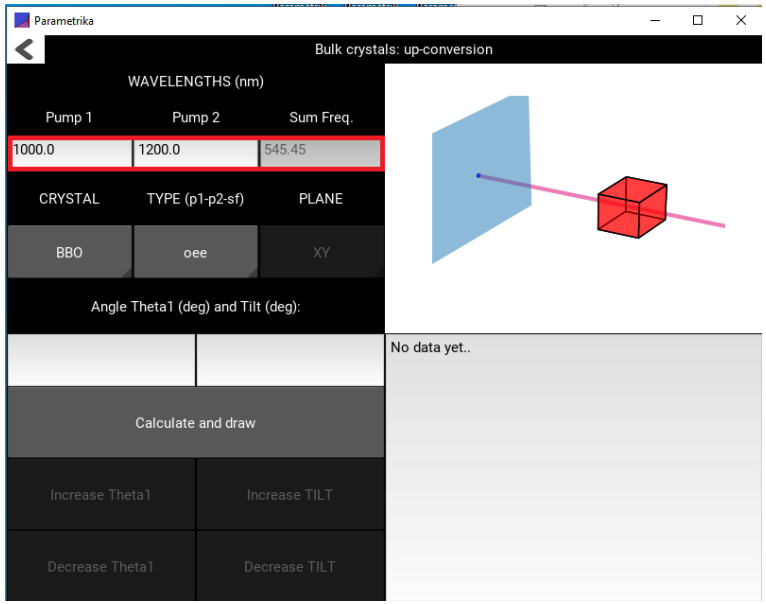


Figure 22: Input wavelengths menu.

2.2.3 Nonlinear crystals

List of nonlinear crystals (Fig. 23):

- *ADP*, ammonium dihydrogen phosphate (uniaxial).
- *BBO*, beta-barium borate (uniaxial).
- *GaSe*, gallium selenide (uniaxial).
- *KDP*, potassium dihydrogen phosphate (uniaxial).
- *KTP*, potassium titanyl phosphate (biaxial).
- *LBO*, lithium triborate (biaxial).
- *LN*, lithium niobate (uniaxial).

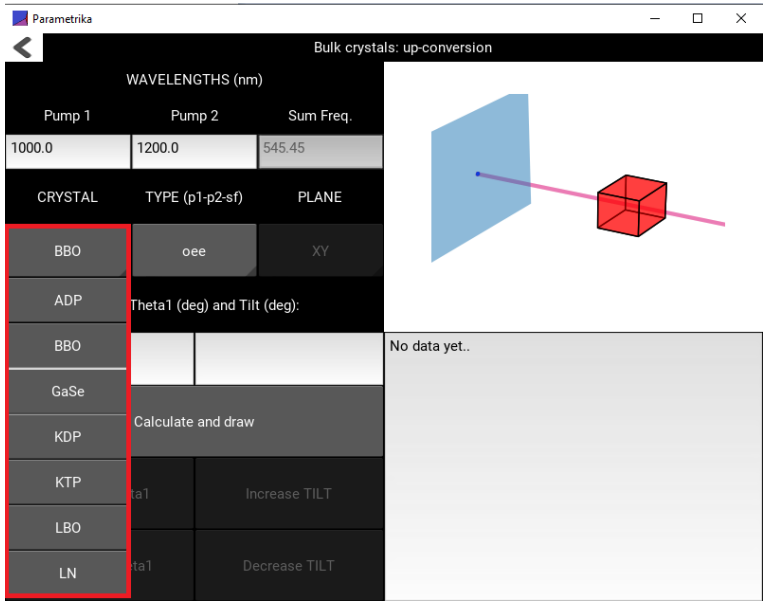


Figure 23: Select crystal drop-down menu.

2.2.4 Interaction type

In the interaction type, the notations are in the following order: pump 1-pump 2-sum frequency, e.g. *ooe* means that both pump waves are ordinary waves and sum frequency wave is extraordinary wave.

List of interaction types (Fig. 24):

- *ooe*
- *oee*
- *eoe*
- *eeo*
- *eoo*
- *oeo*

For negative uniaxial crystals (*ADP*, *BBO*, *GaSe*, *KDP*, *LN*), the interaction types *eeo*, *eoo*, *oeo* are impossible.

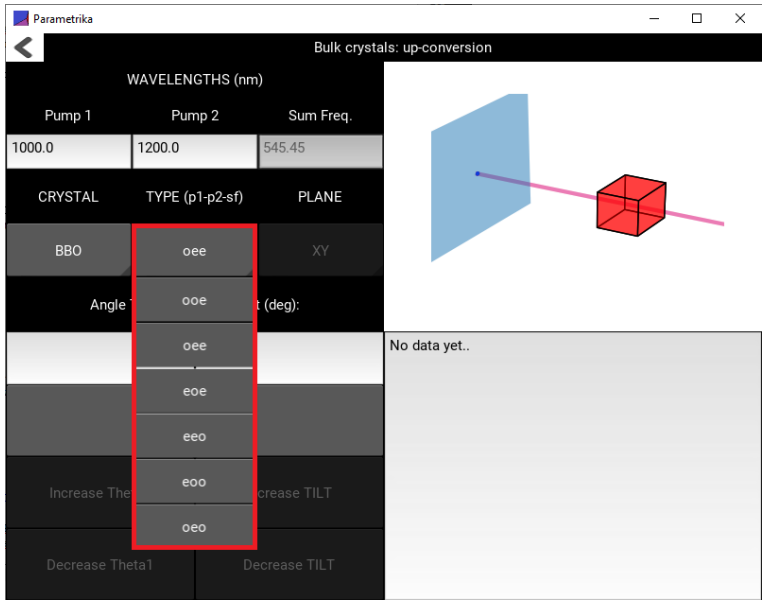


Figure 24: Select type drop-down menu.

2.2.5 Interaction plane

For biaxial crystals (*KTP*, *LBO*), the plane bar is activated. List of planes (Fig. 25):

- *XY*
- *XZ*
- *YZ*

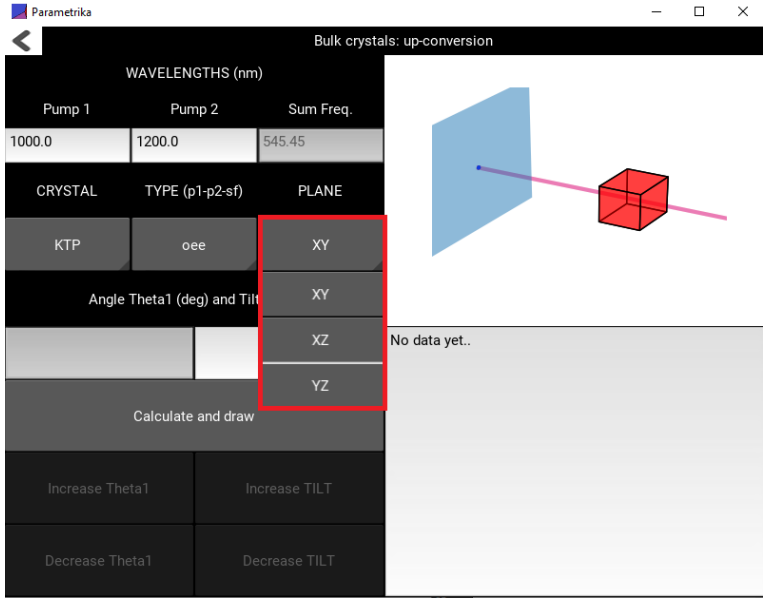


Figure 25: Select plane drop-down menu.

2.2.6 Geometry

The Euler angles θ and φ (*Theta* and *Phi*) are shown in Fig. 26. In the uniaxial crystal (*ADP*, *BBO*, *GaSe*, *KDP*, *LN*), z axis is the *optical axis*. Then, principal refractive indices $n_x = n_y = n_o$ and $n_z = n_e$.

In uniaxial crystals, all possible phase-matching angles are calculated. In biaxial crystals (*KTP*, *LBO*), the phase matching is calculated only in one chosen plane.

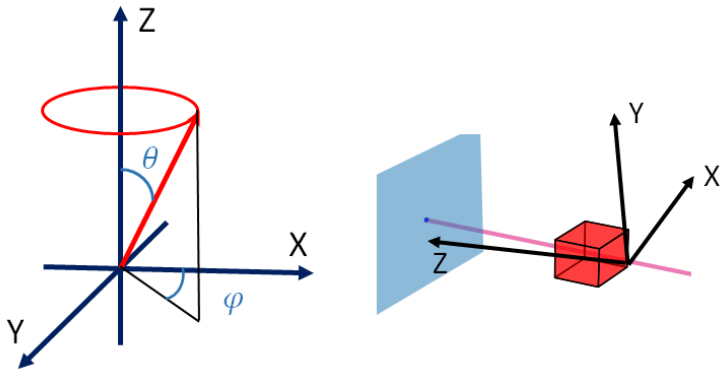


Figure 26: Left: Euler angles θ and φ (*Theta* and *Phi*) in the Cartesian coordiante system x, y, z. Right: coordinate system for uniaxial crystal.

2.2.7 Run!

- To run the program press *Calculate and draw* button, Fig. 27.

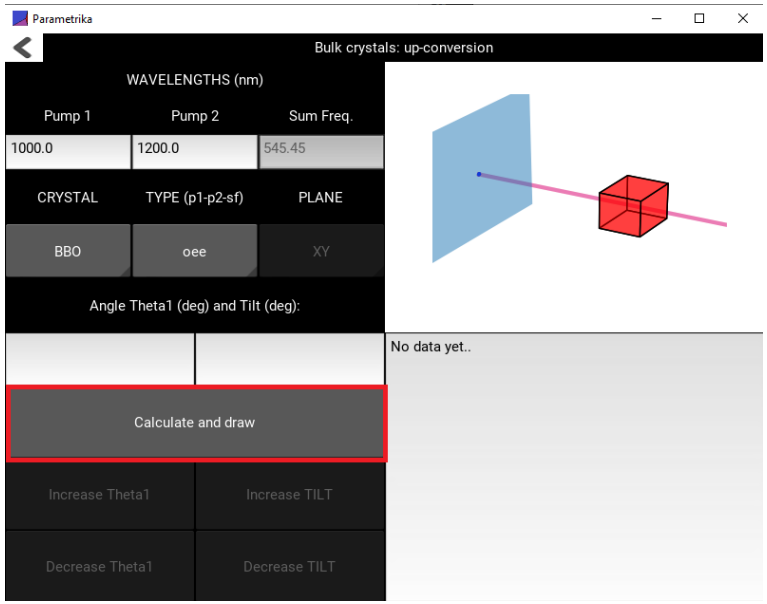


Figure 27: Calculate and draw button.

- If *Angle Theta1* and *Tilt* are initially empty, then the collinear phase-matching is calculated. Otherwise, the program searches for noncollinear phase-matching.
- *Uniaxial crystals.* If the phase-matching is found four buttons: *Increase Theta1*, *Decrease Theta1*, *Increase Tilt* and *Decrease Tilt* become activated (Fig. 28). By pressing these buttons the angles are altered by the step of 1°. The crystal may be also rotated more slowly by changing the angles in the corresponding input labels.

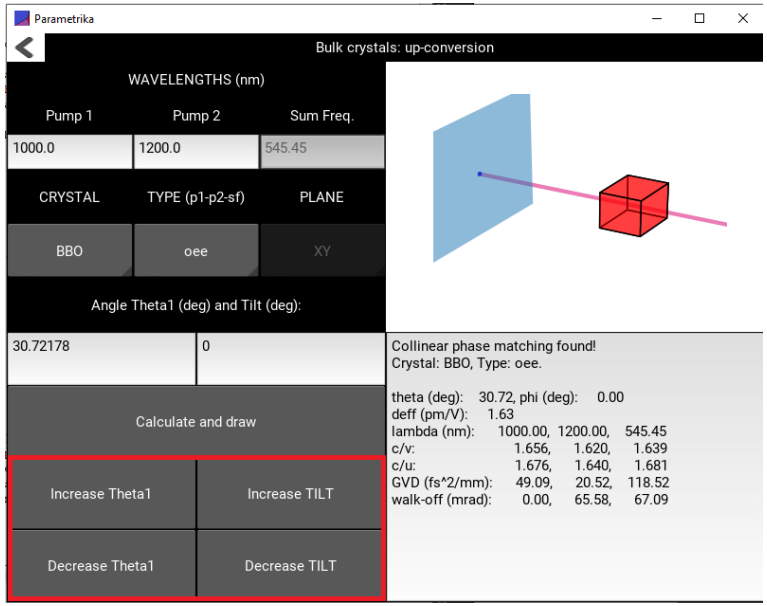


Figure 28: Press these buttons to rotate the crystal and change the tilt angle.

- *Biaxial crystals.* If the phase-matching is found only two buttons: *Increase Tilt* and *Decrease Tilt* become activated (Fig. 29). By pressing these buttons the angles are altered by the step of 1°. The crystal may be also rotated more slowly by changing the angle in the corresponding input label. The rotation outside the chosen plane is prohibited.

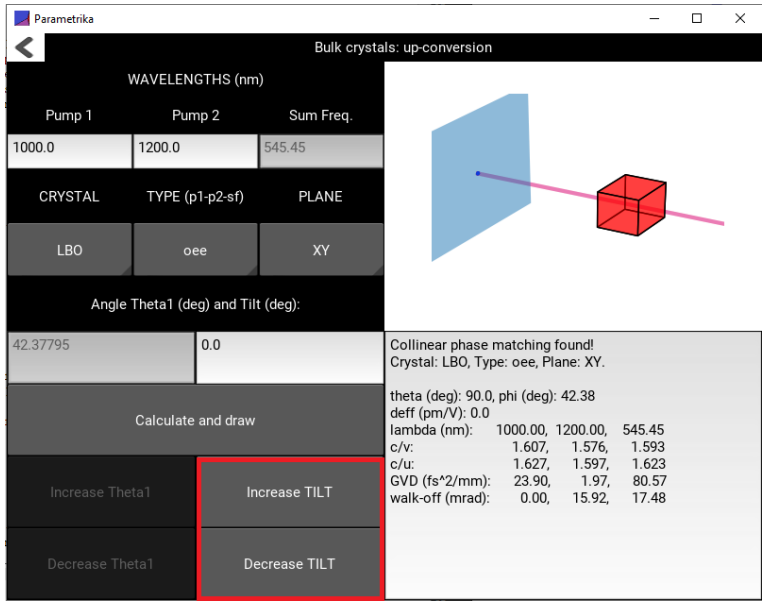


Figure 29: Press these buttons to change the tilt angle.

- Dispersion parameters for all three interacting waves are shown in the output box 1 (Fig. 30).
- The crystal and output waves are visualized in the graphic box 2 (Fig. 30).

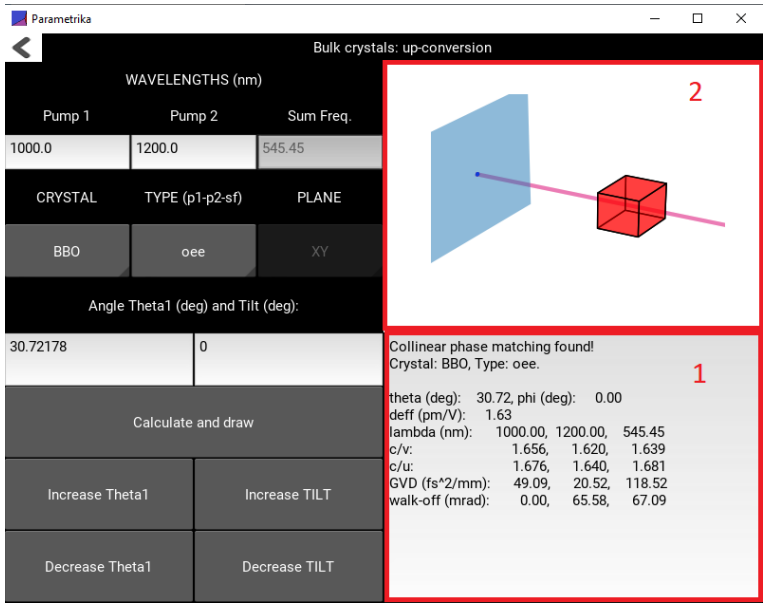


Figure 30: Visualization and information boxes.

2.2.8 3D visualization

- *Uniaxial crystal.* The crystal is cut with respect to the collinear phase-matching angle θ_p and angle φ corresponds to the optimal d_{eff} (Fig. 31a). In the case of the noncollinear phase-matching, the tilt angle is the angle between the *pump 1* and *pump 2* waves (Fig. 31b).

- *Biaxial crystal.* The chosen plane is horizontal and the crystal is cut with respect to the collinear phase matching angle. By varying the tilt angle (*Increase Tilt*, *Decrease Tilt*) the noncollinear phase-matching is calculated (Fig. 31c). Rotation out of the plane is prohibited.

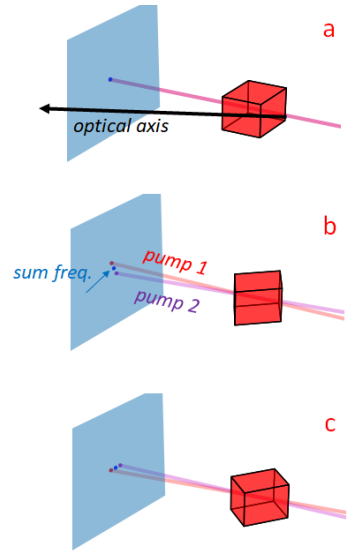


Figure 31: Visualization of (a) collinear phase-matching in uniaxial crystal; (b) noncollinear phase-matching in uniaxial crystal; (c) noncollinear phase-matching in biaxial crystal.

2.2.9 Dispersion parameters

The dispersion parameters are found by the use of the Sellmeier equations from [1].

List of the parameters (Fig. 32):

- c/v: refractive index.
- c/u: fraction of speed of light to the group velocity.
- GVD: group velocity dispersion coefficient.
- *walk-off*: the walk of angle.

The effective nonlinear susceptibility d_{eff} is found by the use of formulas given in [1]. This parameter is wavelength- and angle- dependent.

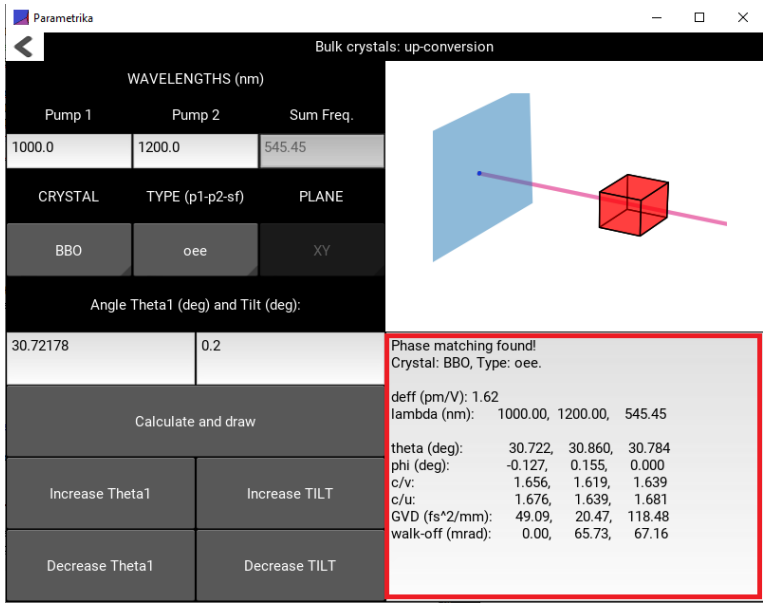


Figure 32: Dispersion parameters in the information box.

3 Module *PP Crystals*

3.1 Module *Down-conversion*

3.1.1 Three interacting waves

The quasi-phasematching for optical parametric down-conversion in the periodically poled crystal is calculated. Three interacting waves, their angular frequencies and wavelengths:

- *Pump*: ω_3, λ_3.
- *Signal*: ω_1, λ_1.
- *Idler*: ω_2, λ_2.

Conservation law of the photon energy (Fig. 33):

$$\hbar\omega_3 = \hbar\omega_1 + \hbar\omega_2, \tag{5}$$

where $\hbar$ is the reduced Plank constant. $\omega = 2\pi c/\lambda$, where c is speed of light. Therefore:

$$\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}. \tag{6}$$

In the periodically poled crystal, quasi-phasematching condition reads:

$$\frac{2\pi n_3}{\lambda_3} - \frac{2\pi n_1}{\lambda_1} - \frac{2\pi n_2}{\lambda_2} = \frac{2\pi}{\Lambda}. \tag{7}$$

Here, n and Λ are the refractive index and lattice period, respectively. Lattice wavenumber $k_g = \frac{2\pi}{\Lambda}$. Phase-matching scheme is depicted in Fig. 34.

Refractive index is a wavelength and temperature function $n(\lambda, T)$.

The user should provide *Pump wavelength* λ_3 and *Temperature* T.

Either *Signal wavelength* λ_1 or *Lattice period* Λ should be provided, then the remaining can be calculated.

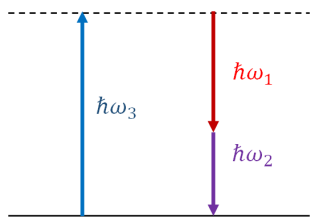


Figure 33: Scheme of photon energies in the optical parametric down-conversion.

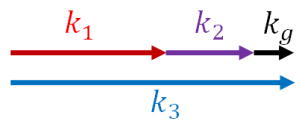


Figure 34: Collinear quasi-phasematching in the periodically poled crystal. $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ are the wavevectors of signal, idler and pump waves, respectively. $\mathbf{k}_g$ is the lattice wavevector.

3.1.2 Nonlinear crystals

List of nonlinear crystals (Fig. 35):

- *PPLN-cm*, periodically poled congruent lithium niobate, (uniaxial).
- *PPLN-sm*, periodically poled stoichiometric lithium niobate, (uniaxial).
- *PPKTP*, periodically poled potassium titanyl phosphate (biaxial).

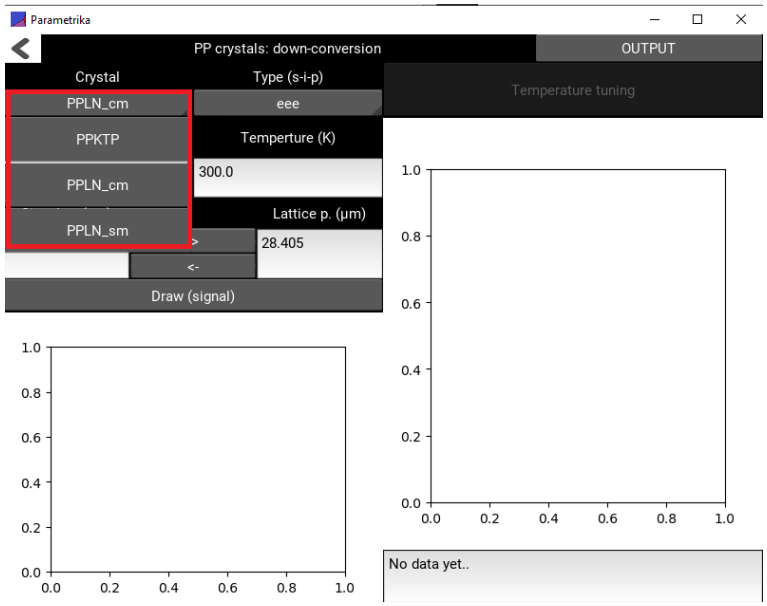


Figure 35: Select crystal drop-down menu.

3.1.3 Interaction type

Uniaxial crystals. In the interaction type, the notations are in the following order: signal-idler-pump, e.g. *ooe* means that signal and idler waves are ordinary waves and pump wave is extraordinary wave.

List of interaction types for uniaxial crystals (Fig. 36):

- *eee*
- *ooe*
- *oeo*
- *eoo*

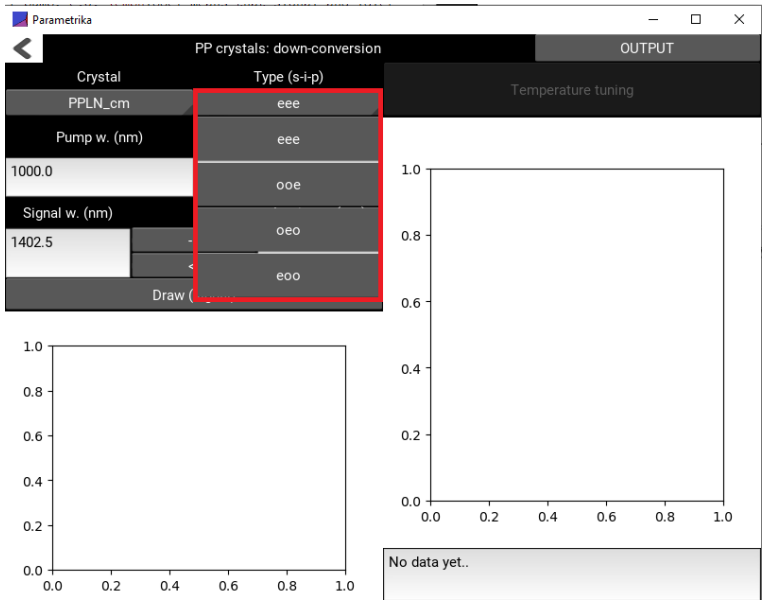


Figure 36: Select type drop-down menu. Uniaxial crystals.

Biaxial crystals. In the interaction type, the notations are in the following order: signal-idler-pump. For example, the interaction type *ZZZ* means, that the refractive indices of all three interacting waves are the principal refractive indices $n_z(\lambda, T)$.

List of the interaction types in biaxial crystals (Fig. 37):

- *ZZZ*
- *YZY*
- *YYZ*
- *XZX*
- *XXZ*

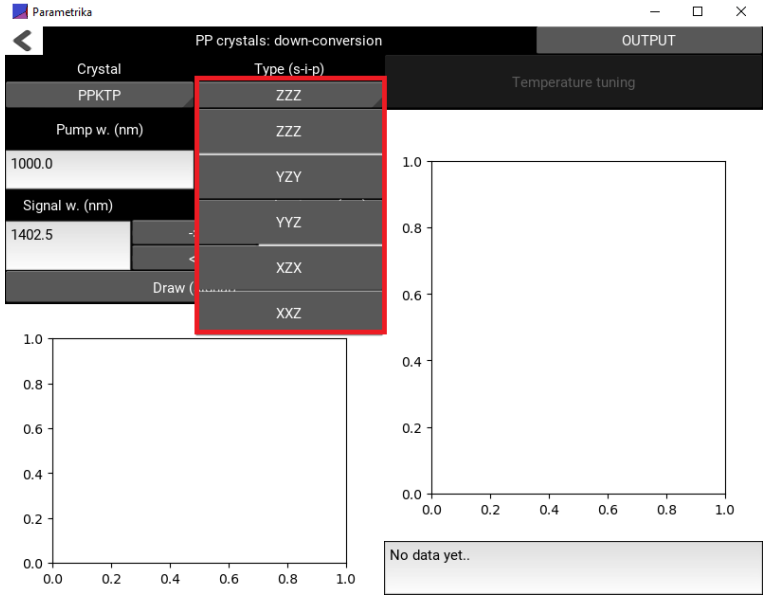


Figure 37: Select type drop-down menu. Biaxial crystals.

3.1.4 Pump wavelength and temperature

Pump wavelength and temperature should be provided in *Pump w.* and *Temperature* boxes, respectively (Fig. 38).

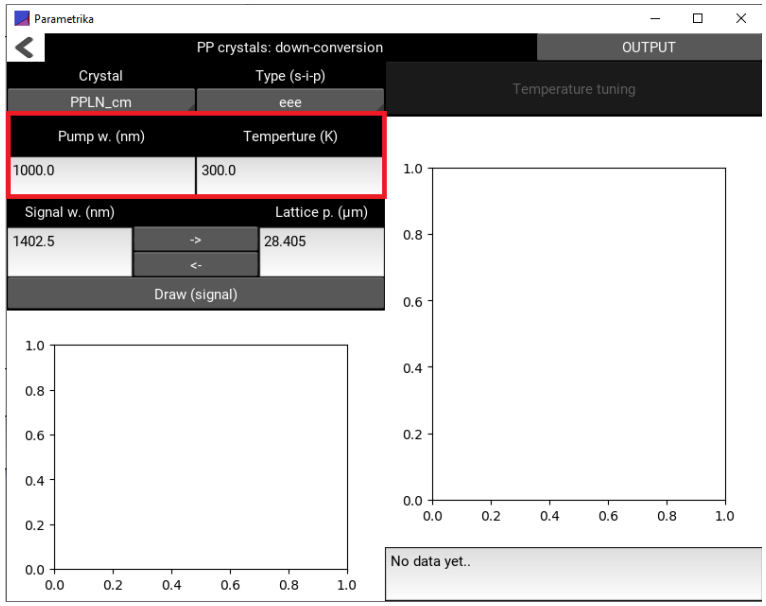


Figure 38: *Pump w.* and *Temperature* edit boxes.

3.1.5 Signal wavelength and lattice period

One of the edit boxes, either *Signal w.* or *Lattice p.*, should be filled. Then, by clicking either right or left arrow (Fig. 39) the remaining parameter is calculated: either lattice period or signal wavelength.

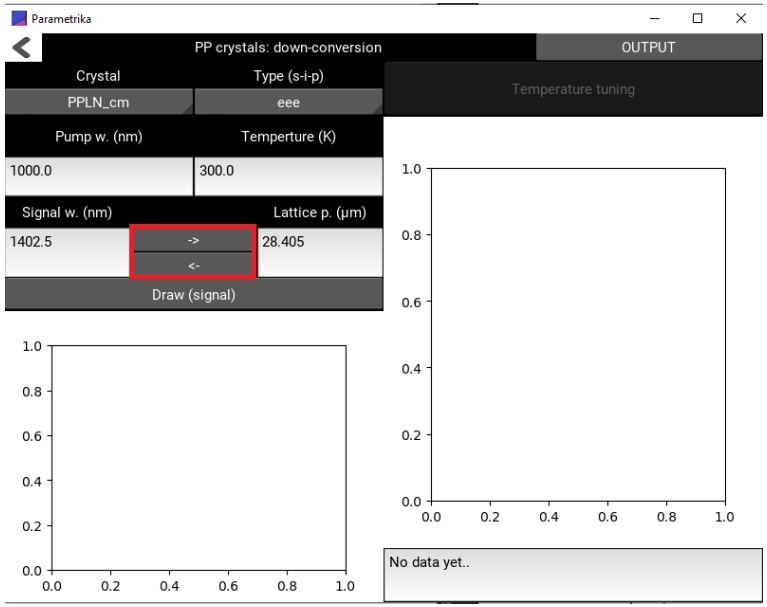


Figure 39: Calculate lattice period or signal wavelength.

3.1.6 Run!

- Choose input parameters and press *Draw (signal and idler)* button (Fig. 40).
- The *Signal w.* window should be filled before pressing *Draw (signal and idler)* button.
- The graph of the dependence $\lambda(\Lambda)$ is drawn, red line. The black dot notes the values given in *Signal w.* and *Lattice p.* boxes.
- This graph is also drawn after pressing the right and left arrows, Fig. 39.

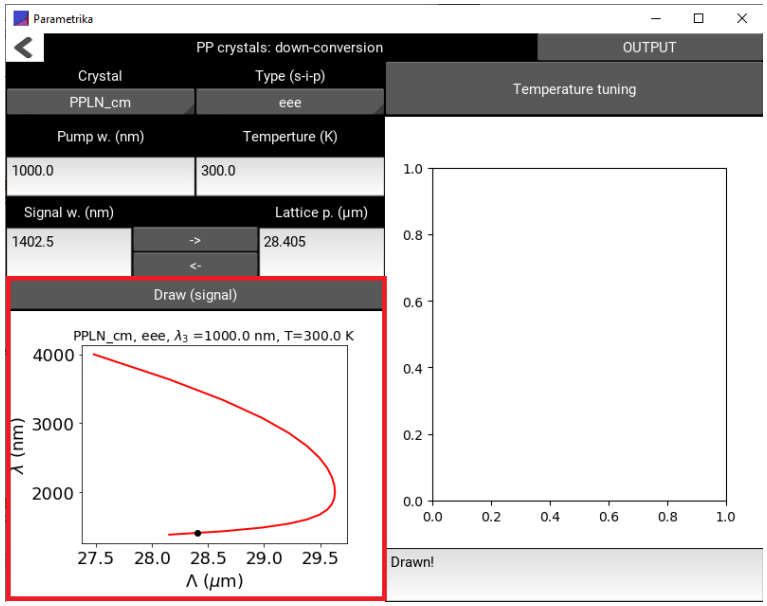


Figure 40: Run the calculations and draw $\lambda(\Lambda)$ graph.

- After successful calculations, the *Temperature tuning* button is activated (Fig. 41).
- Press this button and the signal wavelength dependence on temperature $\lambda_1(T)$ graph will be drawn, black line. The blue dot notes the values given in the *Temperature* and *Signal w.* boxes.

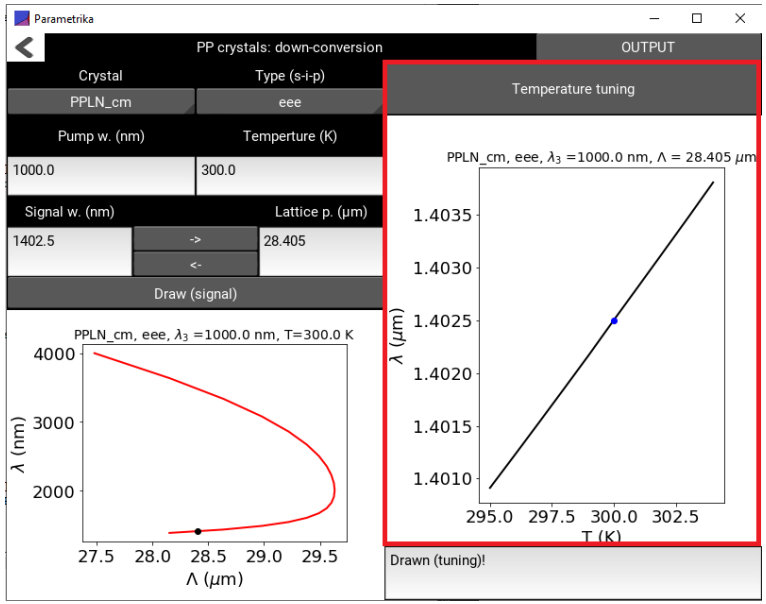


Figure 41: Temperature tunning button and graph.

3.1.7 Output data

- To see the output data push the *Output* button, Fig. 42.

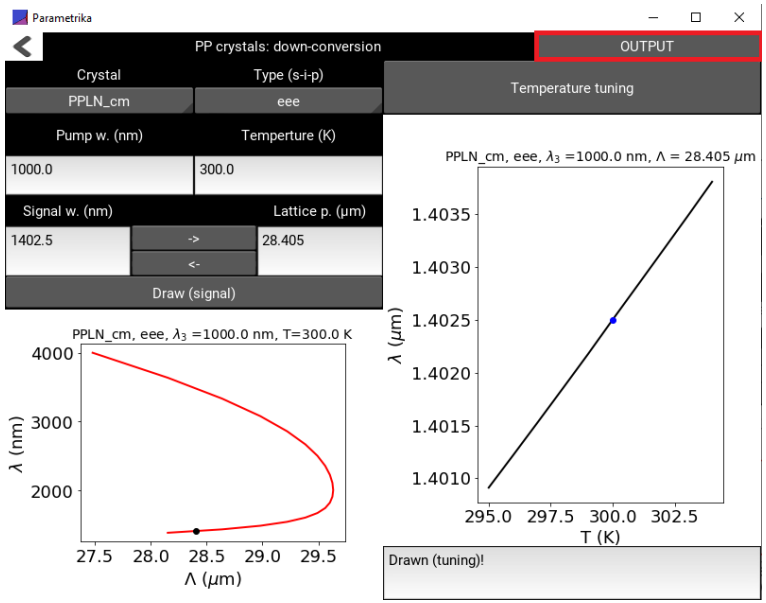


Figure 42: *Output* button.

- In the new window, push *UPDATE and PRINT* button, Fig. 43.

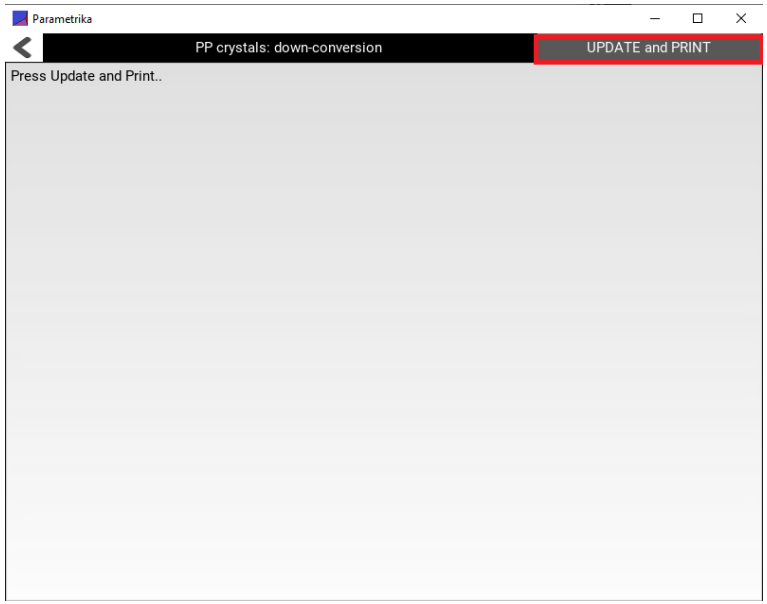


Figure 43: *UPDATE and PRINT* button.

- Output data is presented in the output window, Fig. 44.
- The data of $\lambda(\Lambda)$ graph (Fig. 40) is presented in the output window, Fig. 44. It can be copied and pasted in MS Excel data sheet. In Excel, check the left column and perform *Data* → *Text to Columns*.
- The dispersion parameters are shown in the output window, Fig. 44:
  - c/v: refractive index.
  - c/u: fraction of speed of light to the group velocity.
  - GVD: group velocity dispersion coefficient.
- After changing input parameters in the main window of *PP Crystal Down-conversion* module press *UPDATE and PRINT* button again.

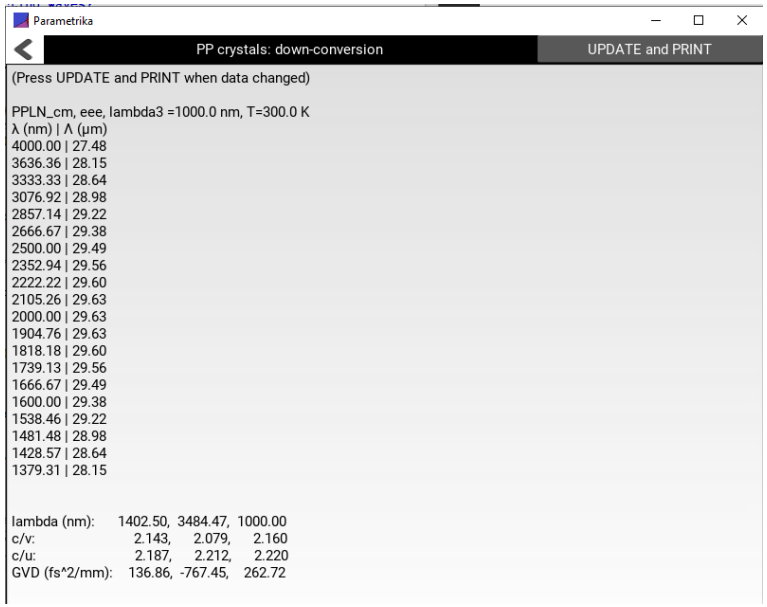


Figure 44: Output data.

3.2 Module Up-conversion

3.2.1 Three interacting waves

The quasi-phasematching for optical parametric up-conversion in the periodically poled crystal is calculated. Three interacting waves, their angular frequencies and wavelengths:

- *Pump 1*: ω_1, λ_1.
- *Pump 2*: ω_2, λ_2.
- *Sum Frequency*: ω_3, λ_3.

Conservation law of the photon energy (Fig. 45):

$$\hbar\omega_1 + \hbar\omega_2 = \hbar\omega_3, \tag{8}$$

where $\hbar$ is the reduced Plank constant. $\omega = 2\pi c/\lambda$, where c is speed of light. Therefore:

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{1}{\lambda_3}. \tag{9}$$

In the periodically poled crystal, quasi-phasematching condition reads:

$$\frac{2\pi n_3}{\lambda_3} - \frac{2\pi n_1}{\lambda_1} - \frac{2\pi n_2}{\lambda_2} = \frac{2\pi}{\Lambda}. \tag{10}$$

Here, n and Λ are the refractive index and lattice period, respectively. Lattice wavenumber $k_g = \frac{2\pi}{\Lambda}$. Phase-matching scheme is depicted in Fig. 46.

Refractive index is a wavelength and temperature function $n(\lambda, T)$.

The user should provide wavelengths of *Pump 1* λ_1 and *Pump 2* λ_2. The wavelength of *Sum freq.* λ_3 is calculated by the use of Eq. (9).

The calculations are performed for the temperature $T = 300$ K.

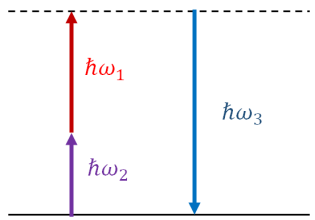


Figure 45: Scheme of photon energies in the optical parametric up-conversion.

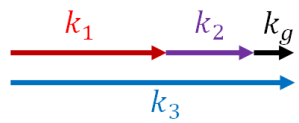


Figure 46: Collinear quasi-phasematching in the periodically poled crystal. $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ are the wavevectors of pump 1, pump 2 and sum frequency waves, respectively. $\mathbf{k}_g$ is the lattice wavevector.

3.2.2 Nonlinear crystals

List of nonlinear crystals (Fig. 47):

- *PPLN-cm*, periodically poled congruent lithium niobate, (uniaxial).
- *PPLN-sm*, periodically poled stoichiometric lithium niobate, (uniaxial).
- *PPKTP*, periodically poled potassium titanyl phosphate (biaxial).

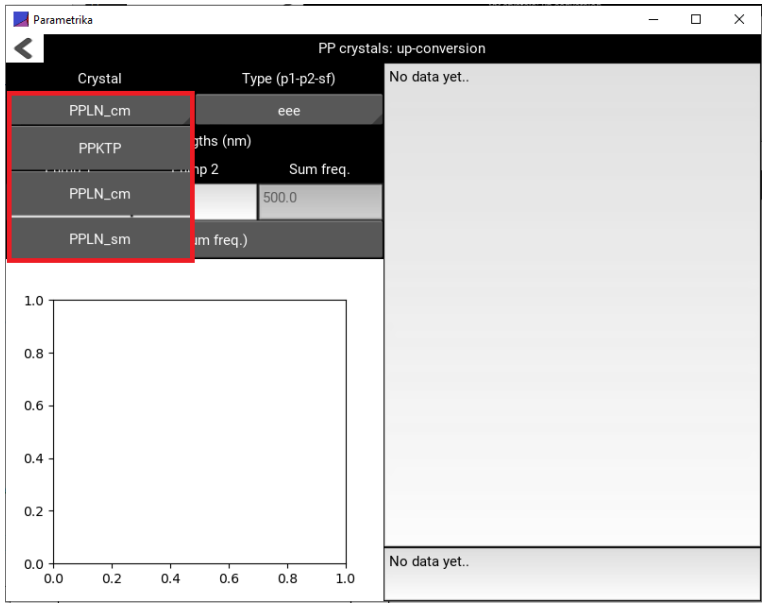


Figure 47: Select crystal drop-down menu.

3.2.3 Interaction type

Uniaxial crystals. In the interaction type, the notations are in the following order: pump 1-pump 2-sum frequency, e.g. *ooe* means that buth pump waves are ordinary waves and sum frequency wave is extraordinary wave.

List of interaction types for uniaxial crystals (Fig. 48):

- *eee*
- *ooe*
- *oeo*
- *eoo*

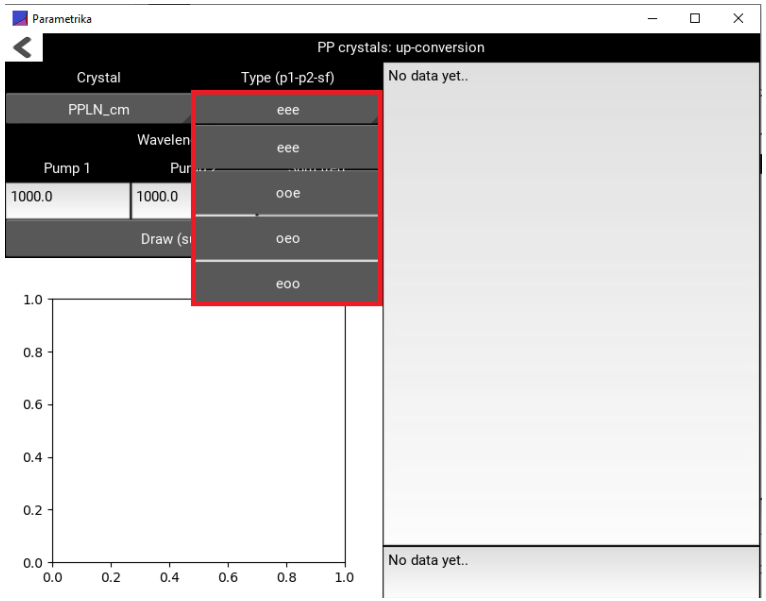


Figure 48: Select type drop-down menu. Uniaxial crystals.

Biaxial crystals. In the interaction type, the notations are in the following order: pump 1-pump 2-sum frequency. For example, the interaction type *ZZZ* means, that the refractive indices of all three interacting waves are the principal refractive indices $n_z(\lambda, T)$.

List of the interaction types in biaxial crystals (Fig. 49):

- *ZZZ*
- *YZY*
- *YYZ*
- *XZX*
- *XXZ*

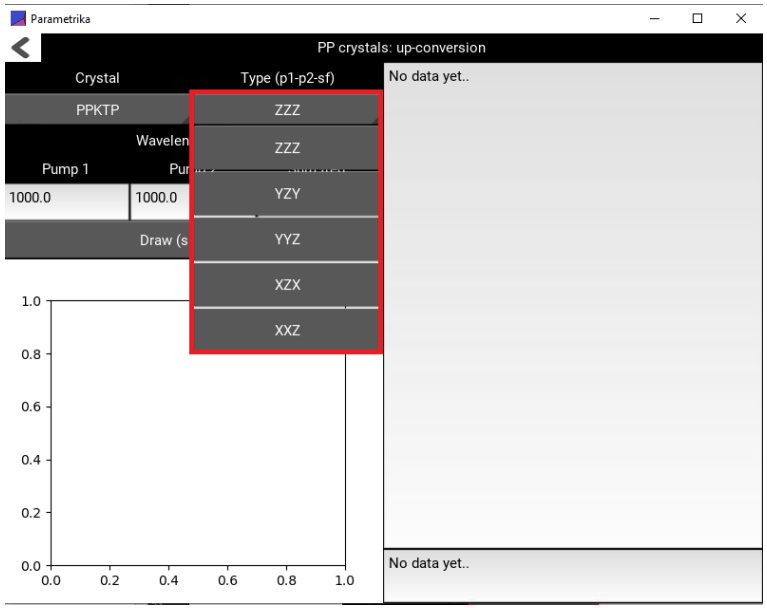


Figure 49: Select type drop-down menu. Biaxial crystals.

3.2.4 Pump wavelengths

Pump wavelengths should be provided in *Pump 1* and *Pump 2* edit boxes, respectively (Fig. 50).

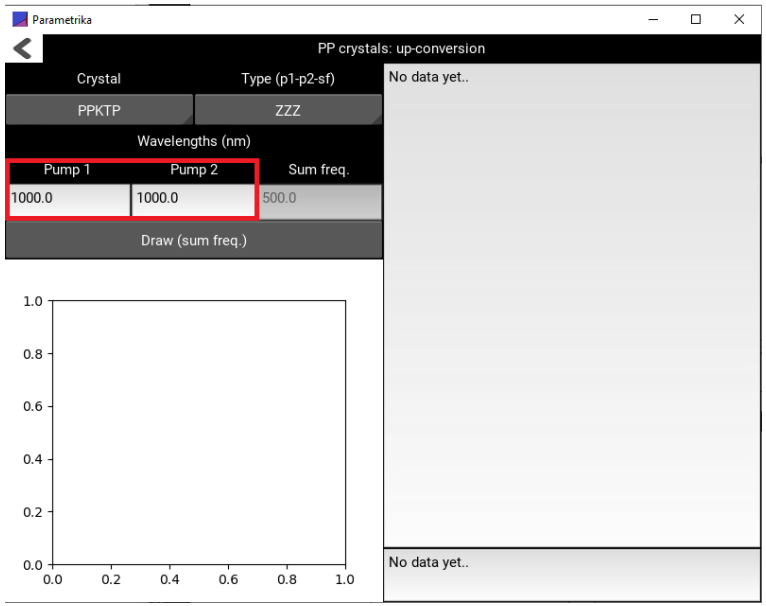


Figure 50: *Pump 1* and *Pump 2* edit boxes.

3.2.5 Run!

- Choose input parameters and press *Draw (sum freq.)* button (Fig. 51).
- The *Pump 1* and *Pump 2* boxes should be filled before pressing *Draw (sum freq.)* button.

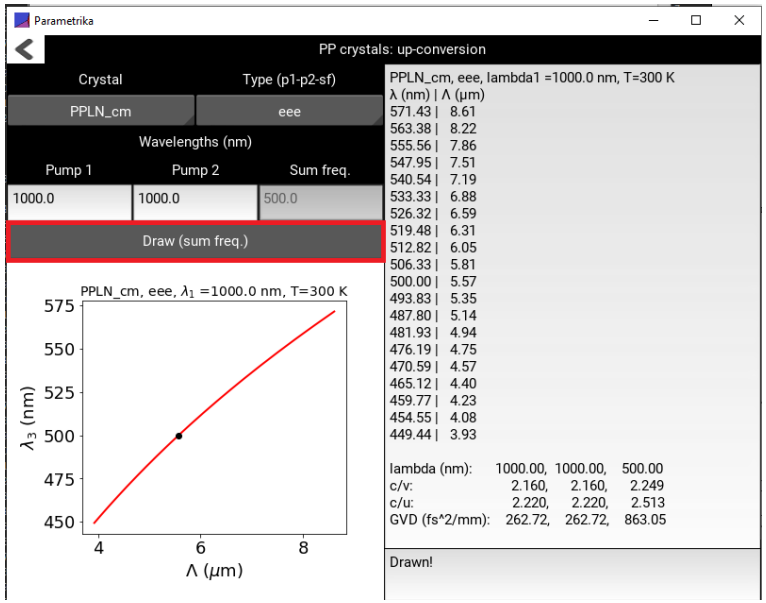


Figure 51: Run the calculations.

- If the calculations are successful the $\lambda_3(\Lambda)$ graph is drawn by the red line 1, Fig. 52. The black dot corresponds to the input data.
- The output data is presented in the output box 2, Fig. 52.
- Information is also presented in the information box 3, Fig. 52.

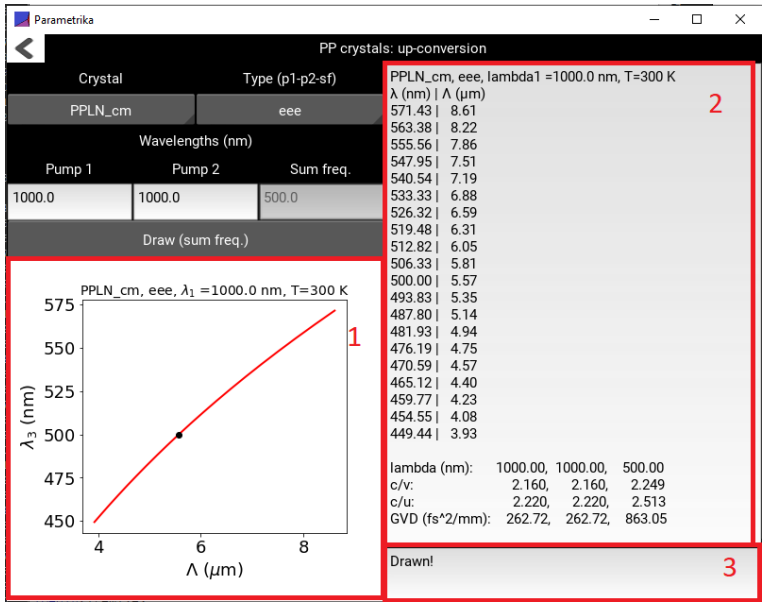


Figure 52: $\lambda_3(\Lambda)$ graph and output data.

3.2.6 Output

- Output data is presented in the output window, Fig. 53.
- The data of $\lambda_3(\Lambda)$ graph is presented in the output window, Fig. 53. It can be copied and pasted in MS Excel data sheet. In Excel, check the left column and perform *Data* → *Text to Columns*.
- The dispersion parameters are shown in the output window, Fig. 53:
  - c/v: refractive index.
  - c/u: fraction of speed of light to the group velocity.
  - GVD: group velocity dispersion coefficient.

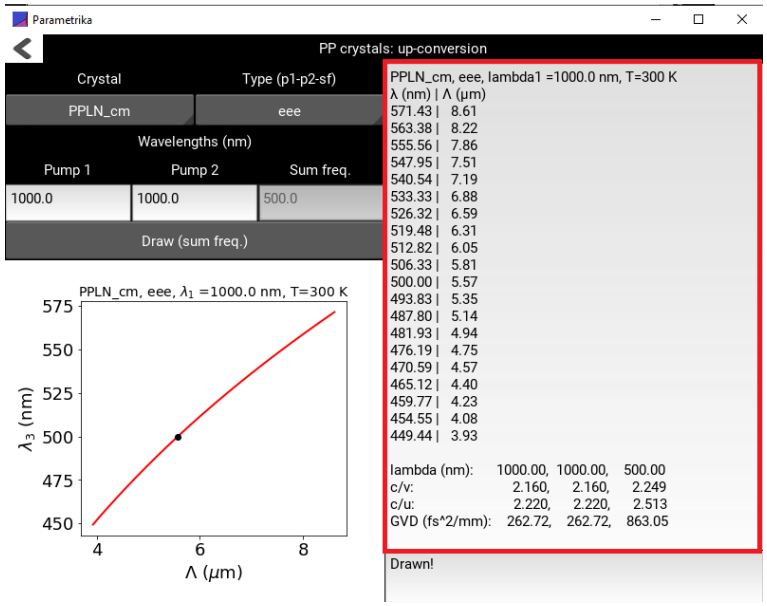


Figure 53: Output data.

4 What's inside? Formulas

4.1 Bulk crystals. Down-conversion

4.1.1 Notations

- Indices 1,2,3 stand for signal, idler and pump waves, respectively.
- $n_o(\lambda)$ and $n_e(\lambda)$ are the principle refractive indices of the uniaxial crystal.
- $n_x(\lambda)$, $n_y(\lambda)$ and $n_z(\lambda)$ are the principle refractive indices of the biaxial crystal.
- θ and ϕ are the Euler angles.

4.1.2 Phase-matching

Uniaxial crystal. Collinear phase-matching

Type ooe

The phase-matching angle $\theta_p = \theta_3$ is found solving the following equation numerically:

$$2k_1k_3(\theta_3) + (k_2^2 - k_3^2(\theta_3) - k_1^2) = 0, \tag{11}$$

where

$$k_1 = \frac{n_o(\lambda_1)}{\lambda_1},\ k_2 = \frac{n_o(\lambda_2)}{\lambda_2},\ k_3(\theta_3) = \frac{n^{(e)}(\lambda_3, \theta_3)}{\lambda_3} \tag{12}$$

and

$$\frac{1}{[n^{(e)}(\lambda_3, \theta_3)]^2} = \frac{\cos^2(\theta_3)}{n_o^2(\lambda_3)} + \frac{\sin^2(\theta_3)}{n_e^2(\lambda_3)}. \tag{13}$$

Type oee

The phase-matching angle $\theta_p = \theta_3$ is found solving the following equation numerically:

$$2k_1k_3(\theta_3) + (k_2^2(\theta_3) - k_3^2(\theta_3) - k_1^2) = 0, \tag{14}$$

where

$$k_1 = \frac{n_o(\lambda_1)}{\lambda_1},\ k_2(\theta_3) = \frac{n^{(e)}(\lambda_2, \theta_3)}{\lambda_2},\ k_3(\theta_3) = \frac{n^{(e)}(\lambda_3, \theta_3)}{\lambda_3} \tag{15}$$

and

$$\frac{1}{[n^{(e)}(\lambda_{2,3}, \theta_3)]^2} = \frac{\cos^2(\theta_3)}{n_o^2(\lambda_{2,3})} + \frac{\sin^2(\theta_3)}{n_e^2(\lambda_{2,3})}. \tag{16}$$

Type eoe

The phase-matching angle $\theta_p = \theta_3$ is found solving the following equation numerically:

$$2k_1(\theta_3)k_3(\theta_3) + (k_2^2 - k_3^2(\theta_3) - k_1^2(\theta_3)) = 0, \tag{17}$$

where

$$k_1(\theta_3) = \frac{n^{(e)}(\lambda_1, \theta_3)}{\lambda_1},\ k_2 = \frac{n_o(\lambda_2)}{\lambda_2},\ k_3(\theta_3) = \frac{n^{(e)}(\lambda_3, \theta_3)}{\lambda_3} \tag{18}$$

and

$$\frac{1}{[n^{(e)}(\lambda_{1,3}, \theta_3)]^2} = \frac{\cos^2(\theta_3)}{n_o^2(\lambda_{1,3})} + \frac{\sin^2(\theta_3)}{n_e^2(\lambda_{1,3})}. \tag{19}$$

Types eeo, eoo, oeo

These types of phase-matching are not calculated since all the crystals in the list are negative: $n_e < n_o$.

Uniaxial crystal. Noncollinear phase-matching

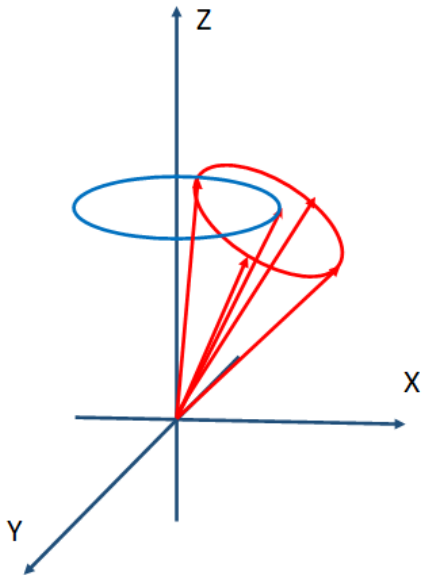


Figure 54: Signal wave cone (red) in the case of noncollinear interaction in the uniaxial crystal. Blue line notes possible directions of the pump wave.

- Euler angles θ_3 and φ_3 are given (taken from inputs *Theta* and *Phi*).
- Define $\beta = \pi/2 - \theta_3$.
- Involve into consideration angle γ, that is varied from 0 to 2π, that will give a ring-type profiles of the signal and idler waves at the output, Fig. 54.
- The goal is to obtain the series of angles (θ_1,φ_1), (θ_2,φ_2).

Type ooe

First, calculate the noncolinear angle α between the pump and signal waves:

$$\alpha = -\arccos\left(-\frac{k_2^2 - k_3^2(\theta_3) - k_1^2}{2k_1k_3(\theta_3)}\right). \tag{20}$$

k_1, k_2 and $k_3(\theta_3)$ are found from Eq. (12). Then, for each γ find signal wave angle θ_1:

$$\theta_1 = \arccos\left(\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\gamma)\cos(\beta)\right). \tag{21}$$

Find signal wave angle φ_1:

$$\varphi_1 = \pm\arccos\left(\frac{\cos(\alpha)\cos(\beta) - \sin(\alpha)\cos(\gamma)\sin(\beta)}{\sin(\theta_1)}\right) + \varphi_3. \tag{22}$$

Find idler wave angle θ_2:

$$\theta_2 = \arccos\left(\frac{k_3(\theta_3)\cos(\theta_3) - k_1\cos(\theta_1)}{k_2}\right). \tag{23}$$

Find idler wave angle φ_2:

$$\varphi_2 = \arcsin\left(\frac{k_3(\theta_3)\sin(\theta_3)\sin(\varphi_3) - k_1\sin(\theta_1)\sin(\varphi_1)}{k_2\sin(\theta_2)}\right). \tag{24}$$

Type eoe

First, define the noncollinear angle between the signal and pump waves $\alpha(\theta_1)$:

$$\alpha(\theta_1) = -\arccos\left(-\frac{k_2^2 - k_3^2(\theta_3) - k_1^2(\theta_1)}{2k_1(\theta_1)k_3(\theta_3)}\right). \tag{25}$$

$k_1(\theta_1)$, k_2 and $k_3(\theta_3)$ are found from Eq. (18). Then, numericaly calculate signal wave angle θ_1 for each γ from the equation:

$$\cos(\theta_1) = \cos(\alpha(\theta_1))\sin(\beta) + \sin(\alpha(\theta_1))\cos(\gamma)\cos(\beta). \tag{26}$$

Find signal wave angle φ_1:

$$\varphi_1 = \pm\arccos\left(\frac{\cos(\alpha(\theta_1))\cos(\beta) - \sin(\alpha(\theta_1))\cos(\gamma)\sin(\beta)}{\sin(\theta_1)}\right) + \varphi_3. \tag{27}$$

Find idler wave angle θ_2:

$$\theta_2 = \arccos\left(\frac{k_3(\theta_3)\cos(\theta_3) - k_1(\theta_1)\cos(\theta_1)}{k_2}\right). \tag{28}$$

Find idler wave angle φ_2:

$$\varphi_2 = \arcsin\left(\frac{k_3(\theta_3)\sin(\theta_3)\sin(\varphi_3) - k_1(\theta_1)\sin(\theta_1)\sin(\varphi_1)}{k_2\sin(\theta_2)}\right). \tag{29}$$

Type oee

First, define the noncollinear angle between the idler and pump waves $\alpha(\theta_2)$:

$$\alpha(\theta_2) = -\arccos\left(-\frac{k_1^2 - k_3^2(\theta_3) - k_2^2(\theta_2)}{2k_2(\theta_2)k_3(\theta_3)}\right). \tag{30}$$

k_1, $k_2(\theta_2)$ and $k_3(\theta_3)$ are found from Eq. (15). Then, numericaly calculate idler wave angle θ_2 for each γ from the equation:

$$\cos(\theta_2) = \cos(\alpha(\theta_2))\sin(\beta) + \sin(\alpha(\theta_2))\cos(\gamma)\cos(\beta). \tag{31}$$

Find idler wave angle φ_2:

$$\varphi_2 = \pm\arccos\left(\frac{\cos(\alpha(\theta_2))\cos(\beta) - \sin(\alpha(\theta_2))\cos(\gamma)\sin(\beta)}{\sin(\theta_2)}\right) + \varphi_3. \tag{32}$$

Find signal wave angle θ_1:

$$\theta_1 = \arccos\left(\frac{k_3(\theta_3)\cos(\theta_3) - k_2(\theta_2)\cos(\theta_2)}{k_1}\right). \tag{33}$$

Find signal wave angle φ_1:

$$\varphi_1 = \arcsin\left(\frac{k_3(\theta_3)\sin(\theta_3)\sin(\varphi_3) - k_2(\theta_2)\sin(\theta_2)\sin(\varphi_2)}{k_1\sin(\theta_1)}\right). \tag{34}$$

Types eeo, eoo, oeo

These types of phase-matching are not calculated since all the crystals in the list are negative: $n_e < n_o$.

Biaxial crystal. Collinear phase-matching

For three different planes, we label the refractive indices $n_o(\lambda)$, $n_e(\lambda)$ and $n_p(\lambda)$ as follows:

- **XY plane**. $n_o(\lambda) = n_y(\lambda)$, $n_e(\lambda) = n_x(\lambda)$, $n_p(\lambda) = n_z(\lambda)$.
- **XZ plane**. $n_o(\lambda) = n_z(\lambda)$, $n_e(\lambda) = n_x(\lambda)$, $n_p(\lambda) = n_y(\lambda)$.
- **YZ plane**. $n_o(\lambda) = n_z(\lambda)$, $n_e(\lambda) = n_y(\lambda)$, $n_p(\lambda) = n_x(\lambda)$.

Type ooe

The phase-matching angle $\theta_p = \theta_3$ is found solving the following equation numerically:

$$2k_1k_3(\theta_3) + (k_2^2 - k_3^2(\theta_3) - k_1^2) = 0, \tag{35}$$

where

$$k_1 = \frac{n_p(\lambda_1)}{\lambda_1},\ k_2 = \frac{n_p(\lambda_2)}{\lambda_2},\ k_3(\theta_3) = \frac{n^{(e)}(\lambda_3, \theta_3)}{\lambda_3} \tag{36}$$

and

$$\frac{1}{[n^{(e)}(\lambda_3, \theta_3)]^2} = \frac{\cos^2(\theta_3)}{n_o^2(\lambda_3)} + \frac{\sin^2(\theta_3)}{n_e^2(\lambda_3)}. \tag{37}$$

Type oee

The phase-matching angle $\theta_p = \theta_3$ is found solving the following equation numerically:

$$2k_1k_3(\theta_3) + (k_2^2(\theta_3) - k_3^2(\theta_3) - k_1^2) = 0, \tag{38}$$

where

$$k_1 = \frac{n_p(\lambda_1)}{\lambda_1},\ k_2(\theta_3) = \frac{n^{(e)}(\lambda_2, \theta_3)}{\lambda_2},\ k_3(\theta_3) = \frac{n^{(e)}(\lambda_3, \theta_3)}{\lambda_3} \tag{39}$$

and

$$\frac{1}{[n^{(e)}(\lambda_{2,3}, \theta_3)]^2} = \frac{\cos^2(\theta_3)}{n_o^2(\lambda_{2,3})} + \frac{\sin^2(\theta_3)}{n_e^2(\lambda_{2,3})}. \tag{40}$$

Type eoe

The phase-matching angle $\theta_p = \theta_3$ is found solving the following equation numerically:

$$2k_1(\theta_3)k_3(\theta_3) + (k_2^2 - k_3^2(\theta_3) - k_1^2(\theta_3)) = 0, \tag{41}$$

where

$$k_1(\theta_3) = \frac{n^{(e)}(\lambda_1, \theta_3)}{\lambda_1},\ k_2 = \frac{n_p(\lambda_2)}{\lambda_2},\ k_3(\theta_3) = \frac{n^{(e)}(\lambda_3, \theta_3)}{\lambda_3} \tag{42}$$

and

$$\frac{1}{[n^{(e)}(\lambda_{1,3}, \theta_3)]^2} = \frac{\cos^2(\theta_3)}{n_o^2(\lambda_{1,3})} + \frac{\sin^2(\theta_3)}{n_e^2(\lambda_{1,3})}. \tag{43}$$

Type eeo

The phase-matching angle $\theta_p = \theta_1$ is found solving the following equation numerically:

$$2k_1(\theta_p)k_3 + (k_2^2(\theta_p) - k_3^2 - k_1^2(\theta_p)) = 0, \tag{44}$$

where

$$k_1(\theta_p) = \frac{n^{(e)}(\lambda_1, \theta_p)}{\lambda_1},\ k_2(\theta_p) = \frac{n^{(e)}(\lambda_2, \theta_p)}{\lambda_2},\ k_3 = \frac{n_p(\lambda_3)}{\lambda_3} \tag{45}$$

and

$$\frac{1}{[n^{(e)}(\lambda_{1,2}, \theta_p)]^2} = \frac{\cos^2(\theta_p)}{n_o^2(\lambda_{1,2})} + \frac{\sin^2(\theta_p)}{n_e^2(\lambda_{1,2})}. \tag{46}$$

Type eoo

The phase-matching angle $\theta_p = \theta_1$ is found solving the following equation numerically:

$$2k_1(\theta_p)k_3 + (k_2^2 - k_3^2 - k_1^2(\theta_p)) = 0, \tag{47}$$

where

$$k_1(\theta_p) = \frac{n^{(e)}(\lambda_1, \theta_p)}{\lambda_1},\ k_2 = \frac{n_p(\lambda_2)}{\lambda_2},\ k_3 = \frac{n_p(\lambda_3)}{\lambda_3} \tag{48}$$

and

$$\frac{1}{[n^{(e)}(\lambda_1, \theta_p)]^2} = \frac{\cos^2(\theta_p)}{n_o^2(\lambda_1)} + \frac{\sin^2(\theta_p)}{n_e^2(\lambda_1)}. \tag{49}$$

Type oeo

The phase-matching angle $\theta_p = \theta_2$ is found solving the following equation numerically:

$$2k_1k_3 + (k_2^2(\theta_p) - k_3^2 - k_1^2) = 0, \tag{50}$$

where

$$k_1 = \frac{n_p(\lambda_1)}{\lambda_1},\ k_2(\theta_p) = \frac{n^{(e)}(\lambda_2, \theta_p)}{\lambda_2},\ k_3 = \frac{n_p(\lambda_3)}{\lambda_3} \tag{51}$$

and

$$\frac{1}{[n^{(e)}(\lambda_2, \theta_p)]^2} = \frac{\cos^2(\theta_p)}{n_o^2(\lambda_2)} + \frac{\sin^2(\theta_p)}{n_e^2(\lambda_2)}. \tag{52}$$

Biaxial crystal. Noncollinear phase-matching

First, convert the input Euler angles *Theta* and *Phi* to angle θ_p by the following rules:

- **XY plane**. θ_p takes the *Phi* value.
- **XZ plane**. θ_p takes the *Theta* value.
- **YZ plane**. θ_p takes the *Theta* value.

Goal: calculate phase-matching angles θ_{p1}, θ_{p2} and θ_{p3} for signal, idler and pump waves, respectively. Then, convert them to the propagation angles by the following rule:

- **XY plane**. $\theta_{1,2,3} = \frac{\pi}{2}$, $\varphi_{1,2,3} = \theta_{p1,2,3}$.
- **XZ plane**. $\theta_{1,2,3} = \theta_{p1,2,3}$, $\varphi_{1,2,3} = 0$.
- **YZ plane**. $\theta_{1,2,3} = \theta_{p1,2,3}$, $\varphi_{1,2,3} = \frac{\pi}{2}$.

For three different planes, we label the refractive indices $n_o(\lambda)$, $n_e(\lambda)$ and $n_p(\lambda)$ as follows:

- **XY plane**. $n_o(\lambda) = n_y(\lambda)$, $n_e(\lambda) = n_x(\lambda)$, $n_p(\lambda) = n_z(\lambda)$.
- **XZ plane**. $n_o(\lambda) = n_x(\lambda)$, $n_e(\lambda) = n_z(\lambda)$, $n_p(\lambda) = n_y(\lambda)$.
- **YZ plane**. $n_o(\lambda) = n_y(\lambda)$, $n_e(\lambda) = n_z(\lambda)$, $n_p(\lambda) = n_x(\lambda)$.

To make the notations shorter, we write n_{e1} instead of $n_e(\lambda_1)$ and so on.

Type ooe

The noncollinear angles α_1 and α_2 are found from the equations:

$$\alpha_1 = \arccos\left(-\frac{k_2^2 - k_3^2(\theta_p) - k_1^2}{2k_1k_3(\theta_p)}\right), \tag{53}$$

$$\alpha_2 = -\arccos\left(-\frac{k_1^2 - k_3^2(\theta_p) - k_2^2}{2k_2k_3(\theta_p)}\right), \tag{54}$$

where k_1, k_2 and $k_3(\theta_p)$ are found from Eq. (36).

Calculate the output angles:

$$\theta_{p1} = \theta_p + \alpha_1,\ \theta_{p2} = \theta_p + \alpha_2,\ \theta_{p3} = \theta_p. \tag{55}$$

Type oee

The noncollinear angle $\alpha_2$is found numerically from the equation:

$$2k_2(\theta_p + \alpha_2)k_3(\theta_p)\cos(\alpha_2) + k_1^2 - k_3^2(\theta_p) - k_2^2(\theta_p + \alpha_2) = 0. \tag{56}$$

Find $\theta_{p2} = \theta_p + \alpha_2$. Then, calculate noncollinear angle α_1:

$$\alpha_1 = -\arccos\left(-\frac{k_2^2(\theta_{p2}) - k_3^2(\theta_p) - k_1^2}{2k_1k_3(\theta_p)}\right). \tag{57}$$

Here, k_1, $k_2(\theta_{p2})$ and $k_3(\theta_p)$ are found from Eq. (39).

Calculate the output angles:

$$\theta_{p1} = \theta_p + \alpha_1,\ \theta_{p2} = \theta_p + \alpha_2,\ \theta_{p3} = \theta_p. \tag{58}$$

Type eoe

The noncollinear angle α_1 is found numerically from the equation:

$$2k_1(\theta_p + \alpha_1)k_3(\theta_p)\cos(\alpha_1) + k_2^2 - k_3^2(\theta_p) - k_1^2(\theta_p + \alpha_1) = 0. \tag{59}$$

Find $\theta_{p1} = \theta_p + \alpha_1$. Then, calculate noncollinear angle α_2:

$$\alpha_2 = -\arccos\left(-\frac{k_1^2(\theta_{p1}) - k_3^2(\theta_p) - k_2^2}{2k_2k_3(\theta_p)}\right). \tag{60}$$

Here, $k_1(\theta_{p1})$, k_2 and $k_3(\theta_p)$ are found from Eq. (42).

Calculate the output angles:

$$\theta_{p1} = \theta_p + \alpha_1,\ \theta_{p2} = \theta_p + \alpha_2,\ \theta_{p3} = \theta_p. \tag{61}$$

Type eeo

Find noncollinear angle α_1 between wavvectors $\mathbf{k}_1$ and $\mathbf{k}_3$ from the equation:

$$k_2(\theta_p - \alpha_{2x}(\alpha_1)) - k_{2x}(\alpha_1) = 0, \tag{62}$$

where

$$k_{2x}^2(\alpha_1) = k_1^2(\theta_p + \alpha_1) + k_3^2 - 2k_1(\theta_p + \alpha_1)k_3\cos(\alpha_1) \tag{63}$$

and α_{2x} is found from

$$k_{2x}(\alpha_1)\cos(\alpha_{2x}) + k_1(\theta_p + \alpha_1)\cos(\alpha_1) = k_3. \tag{64}$$

We use Eq. (45) to calculate $k_1(\theta_1)$, $k_2(\theta_2)$ and k_3.

Calculate the output angles:

$$\theta_{p1} = \theta_p + \alpha_1,\ \theta_{p2} = \theta_p - \alpha_{2x}(\alpha_1),\ \theta_{p3} = \theta_p. \tag{65}$$

Type eoo

Find noncollinear angle α_1 between wavvectors $\mathbf{k}_1$ and $\mathbf{k}_3$ from the equation:

$$k_2 - k_{2x}(\alpha_1) = 0, \tag{66}$$

where

$$k_{2x}^2(\alpha_1) = k_1^2(\theta_p + \alpha_1) + k_3^2 - 2k_1(\theta_p + \alpha_1)k_3\cos(\alpha_1) \tag{67}$$

and further α_{2x} is found from

$$k_{2x}(\alpha_1)\cos(\alpha_{2x}) + k_1(\theta_p + \alpha_1)\cos(\alpha_1) = k_3. \tag{68}$$

We use Eq. (48) to calculate $k_1(\theta_1)$, k_2 and k_3.

Calculate the output angles:

$$\theta_{p1} = \theta_p + \alpha_1,\ \theta_{p2} = \theta_p - \alpha_{2x}(\alpha_1),\ \theta_{p3} = \theta_p. \tag{69}$$

Type oeo

Find noncollinear angle α_2 between wavvectors $\mathbf{k}_2$ and $\mathbf{k}_3$ from the equation:

$$k_1 - k_{1x}(\alpha_2) = 0, \tag{70}$$

where

$$k_{1x}^2(\alpha_2) = k_2^2(\theta_p + \alpha_2) + k_3^2 - 2k_2(\theta_p + \alpha_2)k_3\cos(\alpha_2) \tag{71}$$

and further α_{1x} is found from

$$k_{1x}(\alpha_2)\cos(\alpha_{1x}) + k_2(\theta_p + \alpha_2)\cos(\alpha_2) = k_3. \tag{72}$$

We use Eq. (51) to calculate k_1, $k_2(\theta_2)$ and k_3.

Calculate the output angles:

$$\theta_{p1} = \theta_p - \alpha_{1x}(\alpha_2),\ \theta_{p2} = \theta_p + \alpha_2,\ \theta_{p3} = \theta_p. \tag{73}$$

4.1.3 Gain band

Gain band formulas:

$$P = 1 + \Gamma^2\frac{\sinh^2\left(\sqrt{B}L\right)}{B},\ B > 0. \tag{74}$$

$$P = 1 + \Gamma^2\frac{\sin^2\left(\sqrt{|B|}L\right)}{|B|},\ B \le 0. \tag{75}$$

Here, L is the crystal length,

$$\Gamma = \sqrt{\sigma_1\sigma_2}a_0 \tag{76}$$

and

$$B = \Gamma^2 - \Delta k^2/4,\ \Delta k = k_3 - k_1 - k_2,\ k = \frac{2\pi n}{\lambda}. \tag{77}$$

Nonlinear interaction coefficients:

$$\sigma_{1,2} = \omega_{1,2}\frac{d_{eff}}{cn_{1,2}}. \tag{78}$$

Pump amplitude:

$$a_0 = \sqrt{\frac{2I}{cn_3\varepsilon_0}}. \tag{79}$$

ε_0 is the vacuum parmittivity. Intensity:

$$I = E\frac{4\sqrt{\ln 2}}{\tau\rho^2\pi^{3/2}}. \tag{80}$$

E is the energy, τ is the pulse duration and ρ is the beam radius. Gaussian profiles are assumed.

4.2 Bulk crystals. Up-conversion

4.2.1 Notations

- Indices 1,2,3 stand for pump 1, pump 2 and sum frequency waves, respectively.
- $n_o(\lambda)$ and $n_e(\lambda)$ are the principle refractive indices of the uniaxial crystal.
- $n_x(\lambda)$, $n_y(\lambda)$ and $n_z(\lambda)$ are the principle refractive indices of the biaxial crystal.
- θ and ϕ are the Euler angles.
- α is a tilt angle between pump 1 and pump 2 waves.

4.2.2 Phase-matching

Uniaxial crystal. Collinear phase-matching

Equations (11)–(19) are utilized to calculate the collinear phase-matching in uniaxial crystal.

Uniaxial crystal. Noncollinear phase-matching

- Euler angle θ_1 and tilt angle α are given.
- Angle φ_3 is calculated at the maximum d_{eff} value for collinear phase-matching at given wavelengths.
- The goal is to find the remaining Euler angles: φ_1, θ_2, φ_2 and θ_3.

To make the notations shorter, we write n_{e1} instead of $n_e(\lambda_1)$ and so on.

Type ooe

First, calculate the wavenumber $k_3(\theta_3)$ from the formula:

$$k_3^2(\theta_3) = k_1^2 + k_2^2 + 2k_1k_2\cos(\alpha). \tag{81}$$

k_1 and k_2 are found from Eq. (12). Then, find angle θ_3 from:

$$\cos^2(\theta_3) = \frac{1/n_3^{(e)2} - 1/n_{e3}^2}{1/n_{o3}^2 - 1/n_{e3}^2}, \tag{82}$$

where $n_3^{(e)} = k_3(\theta_3)\lambda_3$.

Calculate θ_2:

$$\theta_2 = \arccos\left(\frac{k_3(\theta_3)\cos(\theta_3) - k_1\cos(\theta_1)}{k_2}\right). \tag{83}$$

Find angle difference $\Delta\varphi_1 = \varphi_3 - \varphi_1$:

$$\cos(\Delta\varphi_1) = \frac{1}{2}\frac{k_3^2(\theta_3)\sin^2(\theta_3) + k_1^2\sin^2(\theta_1) - k_2^2\sin^2(\theta_2)}{k_1k_3(\theta_3)\sin(\theta_1)\sin(\theta_3)}. \tag{84}$$

Then, calculate $\varphi_1 = \varphi_3 - \Delta\varphi_1$. Next, find angle difference $\Delta\varphi_2 = \varphi_2 - \varphi_3$:

$$\cos(\Delta\varphi_2) = \frac{1}{2}\frac{k_3^2(\theta_3)\sin^2(\theta_3) + k_2^2\sin^2(\theta_2) - k_1^2\sin^2(\theta_1)}{k_2k_3(\theta_3)\sin(\theta_2)\sin(\theta_3)}. \tag{85}$$

Then, calculate $\varphi_2 = \varphi_3 + \Delta\varphi_2$.

Type eoe

First, calculate the wavenumber $k_3(\theta_3)$ from the formula:

$$k_3^2(\theta_3) = k_1^2(\theta_1) + k_2^2 + 2k_1(\theta_1)k_2\cos(\alpha). \tag{86}$$

$k_1(\theta_1)$ and k_2 are found from Eq. (18). Then, calculate θ_3 from Eq. (82).

Calculate θ_2:

$$\theta_2 = \arccos\left(\frac{k_3(\theta_3)\cos(\theta_3) - k_1(\theta_1)\cos(\theta_1)}{k_2}\right). \tag{87}$$

Find angle difference $\Delta\varphi_1 = \varphi_3 - \varphi_1$:

$$\cos(\Delta\varphi_1) = \frac{1}{2}\frac{k_3^2(\theta_3)\sin^2(\theta_3) + k_1^2(\theta_1)\sin^2(\theta_1) - k_2^2\sin^2(\theta_2)}{k_1(\theta_1)k_3(\theta_3)\sin(\theta_1)\sin(\theta_3)}. \tag{88}$$

Then, calculate $\varphi_1 = \varphi_3 - \Delta\varphi_1$. Next, find angle difference $\Delta\varphi_2 = \varphi_2 - \varphi_3$:

$$\cos(\Delta\varphi_2) = \frac{1}{2}\frac{k_3^2(\theta_3)\sin^2(\theta_3) + k_2^2\sin^2(\theta_2) - k_1^2(\theta_1)\sin^2(\theta_1)}{k_2k_3(\theta_3)\sin(\theta_2)\sin(\theta_3)}. \tag{89}$$

Then, calculate $\varphi_2 = \varphi_3 + \Delta\varphi_2$.

Type oee

Find angle θ_2 solving numerically equation:

$$k_2(\theta_2)\cos(\theta_2) - (k_3(\theta_3)\cos(\theta_3) - k_1\cos(\theta_1)) = 0, \tag{90}$$

where k_1 and $k_2(\theta_2)$ are found from Eq. (15). Here,

$$\cos^2(\theta_3) = \frac{1/n_3^{(e)2} - 1/n_{e3}^2}{1/n_{o3}^2 - 1/n_{e3}^2}, \tag{91}$$

$n_3^{(e)} = k_3(\theta_3)\lambda_3$ and

$$k_3^2(\theta_3) = k_1^2 + k_2^2(\theta_2) + 2k_1k_2(\theta_2)\cos(\alpha). \tag{92}$$

Find θ_2 and then, from Eqs. (91,92) θ_3.

Find angle difference $\Delta\varphi_1 = \varphi_3 - \varphi_1$:

$$\cos(\Delta\varphi_1) = \frac{1}{2}\frac{k_3^2(\theta_3)\sin^2(\theta_3) + k_1^2\sin^2(\theta_1) - k_2^2(\theta_2)\sin^2(\theta_2)}{k_1k_3(\theta_3)\sin(\theta_1)\sin(\theta_3)}. \tag{93}$$

Then, calculate $\varphi_1 = \varphi_3 - \Delta\varphi_1$. Next, find angle difference $\Delta\varphi_2 = \varphi_2 - \varphi_3$:

$$\cos(\Delta\varphi_2) = \frac{1}{2}\frac{k_3^2(\theta_3)\sin^2(\theta_3) + k_2^2(\theta_2)\sin^2(\theta_2) - k_1^2\sin^2(\theta_1)}{k_2(\theta_2)k_3(\theta_3)\sin(\theta_2)\sin(\theta_3)}. \tag{94}$$

Then, calculate $\varphi_2 = \varphi_3 + \Delta\varphi_2$.

Uniaxial crystals. Types eeo, eoo, oeo

These types of phase-matching are not calculated since all the crystals in the list are negative: $n_e < n_o$.

Biaxial crystal. Collinear phase-matching

Equations (35)–(52) are utilized to calculate the collinear phase-matching in biaxial crystal.

Biaxial crystal. Noncollinear phase-matching

In the case of up-conversion in the biaxial crystal, only the tilt angle $\alpha = \theta_{p2} - \theta_{p1}$ between the pump 1 and pump 2 waves is taken. In different planes, the angle θ_p is treated as follows:

- **XY plane**. θ_p is the Euler angle φ.
- **XZ plane**. θ_p is the Euler angle θ.
- **YZ plane**. θ_p is the Euler angle θ.

Goal: calculate phase-matching angles θ_{p1}, θ_{p2} and θ_{p3} for signal, idler and pump waves, respectively. Then, convert them to the propagation angles by the following rule:

- **XY plane**. $\theta_{1,2,3} = \frac{\pi}{2}$, $\varphi_{1,2,3} = \theta_{p1,2,3}$.
- **XZ plane**. $\theta_{1,2,3} = \theta_{p1,2,3}$, $\varphi_{1,2,3} = 0$.
- **YZ plane**. $\theta_{1,2,3} = \theta_{p1,2,3}$, $\varphi_{1,2,3} = \frac{\pi}{2}$.

For three different planes, we label the refractive indices $n_o(\lambda)$, $n_e(\lambda)$ and $n_p(\lambda)$ as follows:

- **XY plane**. $n_o(\lambda) = n_y(\lambda)$, $n_e(\lambda) = n_x(\lambda)$, $n_p(\lambda) = n_z(\lambda)$.
- **XZ plane**. $n_o(\lambda) = n_x(\lambda)$, $n_e(\lambda) = n_z(\lambda)$, $n_p(\lambda) = n_y(\lambda)$.
- **YZ plane**. $n_o(\lambda) = n_y(\lambda)$, $n_e(\lambda) = n_z(\lambda)$, $n_p(\lambda) = n_x(\lambda)$.

To make the notations shorter, we write n_{e1} instead of $n_e(\lambda_1)$ and so on.

Type ooe

First, calculate $k_3(\theta_{p3})$ from

$$k_3(\theta_{p3}) = \left(k_1^2 + k_2^2 + 2k_1k_2\cos(\alpha)\right)^{1/2}, \tag{95}$$

where k_1 and k_2 are calculated from Eq. (36). Then, find θ_{p3} from

$$\cos^2(\theta_3) = \frac{1/n_3^{(e)2} - 1/n_{e3}^2}{1/n_{o3}^2 - 1/n_{e3}^2}, \tag{96}$$

where $n_3^{(e)} = k_3(\theta_{p3})\lambda_3$.

Nest, note $x = \cos(\theta_{p1})$ and solve quadratic equation:

$$ax^2 + bx + c = 0, \tag{97}$$

where

$$a = k_3^2(\theta_{p3}), \tag{98}$$

$$b = -2k_3(\theta_{p3})\cos(\theta_{p3})\left(k_1 + k_2\cos(\alpha)\right), \tag{99}$$

$$c = k_3^2(\theta_{p3})\cos^2(\theta_{p3}) - k_2^2\sin^2(\alpha). \tag{100}$$

We take solution of Eq. (97):

$$\cos(\theta_{p1}) = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{101}$$

calculate θ_{p1} and then, θ_{p2}:

$$\theta_{p2} = \theta_{p1} + \alpha. \tag{102}$$

Type oee

First, find θ_{p2} solving numerically the equation:

$$k_2(\theta_{p2})\cos(\theta_{p2}) - \left[k_3(\theta_{p3})\cos(\theta_{p3}) - k_1\cos(\theta_{p2} - \alpha)\right] = 0, \tag{103}$$

where $k_1 = n_p(\lambda_1)/\lambda_1$,

$$k_2(\theta_{p2}) = \frac{n_2^{(e)}}{\lambda_2}, \tag{104}$$

$$n_2^{(e)} = \frac{1}{\left(\cos^2(\theta_{p2})/n_{o2}^2 + \sin^2(\theta_{p2})/n_{e2}^2\right)^{1/2}}, \tag{105}$$

$$k_3(\theta_{p3}) = \left(k_1^2 + k_2^2(\theta_{p2}) + 2k_1k_2(\theta_{p2})\cos(\alpha)\right)^{1/2} \tag{106}$$

and θ_{p3} is a function of θ_{p2}:

$$\cos^2(\theta_{p3}) = \frac{1/(\lambda_3 k_3(\theta_{p3}))^2 - 1/n_{e3}^2}{1/n_{o3}^2 - 1/n_{e3}^2}. \tag{107}$$

Calculate θ_{p2}, then express θ_{p3} from Eq. (107). Finally, find θ_{p1}:

$$\theta_{p1} = \theta_{p2} - \alpha. \tag{108}$$

Type eoe

First, find θ_{p1} solving numerically the equation:

$$k_1(\theta_{p1})\cos(\theta_{p1}) - [k_3(\theta_{p3})\cos(\theta_{p3}) - k_2\cos(\theta_{p1}+\alpha)] = 0, \tag{109}$$

where $k_2 = n_p(\lambda_2)/\lambda_2$,

$$k_1(\theta_{p1}) = \frac{n_1^{(e)}}{\lambda_1}, \tag{110}$$

$$n_1^{(e)} = \left(\cos^2(\theta_{p1})/n_{o1}^2 + \sin^2(\theta_{p1})/n_{e1}^2\right)^{-1/2}, \tag{111}$$

$$k_3(\theta_{p3}) = \left(k_1^2(\theta_{p1}) + k_2^2 + 2k_1(\theta_{p1})k_2\cos(\alpha)\right)^{1/2} \tag{112}$$

and θ_{p3} is a function of θ_{p1}:

$$\cos^2(\theta_{p3}) = \frac{1/(\lambda_3 k_3(\theta_{p3}))^2 - 1/n_{e3}^2}{1/n_{o3}^2 - 1/n_{e3}^2}. \tag{113}$$

Calculate θ_{p1}, then express θ_{p3} from Eq. (113). Finally, find θ_{p2}:

$$\theta_{p2} = \theta_{p1} + \alpha. \tag{114}$$

Type eeo

First, find θ_{p1} solving numerically the equation:

$$k_3^2 - \left[k_1^2(\theta_{p1}) + k_2^2(\theta_{p1}+\alpha) + 2k_1(\theta_{p1})k_2(\theta_{p1}+\alpha)\cos(\alpha)\right] = 0, \tag{115}$$

where

$$k_1(\theta_{p1}) = \frac{n_1^{(e)}}{\lambda_1}, \tag{116}$$

$$n_1^{(e)} = \left(\cos^2(\theta_{p1})/n_{o1}^2 + \sin^2(\theta_{p1})/n_{e1}^2\right)^{-1/2} \tag{117}$$

and

$$k_2(\theta_{p2}) = \frac{n_2^{(e)}}{\lambda_2}, \tag{118}$$

$$n_2^{(e)} = \left(\cos^2(\theta_{p1}+\alpha)/n_{o2}^2 + \sin^2(\theta_{p1}+\alpha)/n_{e2}^2\right)^{-1/2} \tag{119}$$

Find θ_{p1}, then calculate $\theta_{p2} = \theta_{p1} + \alpha$. Finally, find θ_{p3}:

$$\theta_{p3} = \arccos\left(\frac{k_1(\theta_{p1})\cos(\theta_{p1}) + k_2(\theta_{p2})\cos(\theta_{p2})}{k_3}\right), \tag{120}$$

where $k_3 = n_p(\lambda_3)/\lambda_3$.

Type eoo

First, find θ_{p1} solving numerically the equation:

$$k_3^2 - \left[k_1^2(\theta_{p1}) + k_2^2 + 2k_1(\theta_{p1})k_2\right] = 0, \tag{121}$$

where $k_2 = n_p(\lambda_2)/\lambda_2$, $k_3 = n_p(\lambda_3)/\lambda_3$,

$$k_1(\theta_{p1}) = \frac{n_1^{(e)}}{\lambda_1}, \tag{122}$$

$$n_1^{(e)} = \left(\cos^2(\theta_{p1})/n_{o1}^2 + \sin^2(\theta_{p1})/n_{e1}^2\right)^{-1/2}, \tag{123}$$

Find θ_{p1}, then calculate $\theta_{p2} = \theta_{p1} + \alpha$. Finally, find θ_{p3}:

$$\theta_{p3} = \arccos\left(\frac{k_1(\theta_{p1})\cos(\theta_{p1}) + k_2\cos(\theta_{p2})}{k_3}\right). \tag{124}$$

Type oeo

First, find θ_{p1} solving numerically the equation:

$$k_3^2 - \left[k_1^2 + k_2^2(\theta_{p1} + \alpha) + 2k_1k_2(\theta_{p1} + \alpha)\right] = 0, \tag{125}$$

where $k_1 = n_p(\lambda_1)/\lambda_1$, $k_3 = n_p(\lambda_3)/\lambda_3$,

$$k_2(\theta_{p2}) = \frac{n_2^{(e)}}{\lambda_2}, \tag{126}$$

$$n_2^{(e)} = \left(\cos^2(\theta_{p2})/n_{o2}^2 + \sin^2(\theta_{p2})/n_{e2}^2\right)^{-1/2}, \tag{127}$$

Find θ_{p1}, then calculate $\theta_{p2} = \theta_{p1} + \alpha$. Finally, find θ_{p3}:

$$\theta_{p3} = \arccos\left(\frac{k_1\cos(\theta_{p1}) + k_2(\theta_{p2})\cos(\theta_{p2})}{k_3}\right). \tag{128}$$

4.3 PP crystals.

4.3.1 Equations

Main equations:

$$\frac{n_3(T)}{\lambda_3} - \frac{n_1(T)}{\lambda_1} - \frac{n_2(T)}{\lambda_2} = \frac{1}{\Lambda} \tag{129}$$

and

$$\frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}. \tag{130}$$

4.3.2 Notations. Down-conversion

- Indices 1,2,3 stand for signal, idler and pump waves, respectively.
- Λ is the grating period.
- $n_o(\lambda, T)$ and $n_e(\lambda, T)$ are the temperature-dependent principle refractive indices of the uniaxial crystal.
- $n_x(\lambda, T)$, $n_y(\lambda, T)$ and $n_z(\lambda, T)$ are the temperature-dependent principle refractive indices of the biaxial crystal.
- The meaning of $n_1(T)$, $n_2(T)$ and $n_3(T)$ in Eq. (129) depends on the interaction type.

4.3.3 Quasi-phasematching. Down-conversion

- Temperature T and pump wavelength λ_3 are given.
- From (129) and (130) equations one calculates either λ_1 when Λ is given or Λ when λ_1 is provided.
- From (129) and (130) equations one also calculates $\lambda_1(T)$ dependence when Λ is given.

4.3.4 Notations. Up-conversion

- Indices 1,2,3 stand for pump 1, pump 2 and sum frequency waves, respectively.
- Λ is the grating period.
- $n_o(\lambda, T)$ and $n_e(\lambda, T)$ are the temperature-dependent principle refractive indices of the uniaxial crystal.
- $n_x(\lambda, T)$, $n_y(\lambda, T)$ and $n_z(\lambda, T)$ are the temperature-dependent principle refractive indices of the biaxial crystal.
- The meaning of $n_1(T)$, $n_2(T)$ and $n_3(T)$ in Eq. (129) depends on the interaction type.

4.3.5 Quasi-phasematching. Up-conversion

- Temperature $T = 300$ K.
- Pump wavelengths λ_1 and λ_2 are given.
- From (129) and (130) equations one calculates Λ and λ_3.
- Fixing λ_1 and varying λ_2 one calculates Λ for each λ_3. As a result, $\lambda_3(\Lambda)$ dependence is obtained.

4.3.6 Interaction types

Uniaxial crystals

- **Type eee**. $n_1(T) = n_e(\lambda_1, T)$, $n_2(T) = n_e(\lambda_2, T)$, $n_3(T) = n_e(\lambda_3, T)$.
- **Type ooe**. $n_1(T) = n_o(\lambda_1, T)$, $n_2(T) = n_o(\lambda_2, T)$, $n_3(T) = n_e(\lambda_3, T)$.
- **Type oeo**. $n_1(T) = n_o(\lambda_1, T)$, $n_2(T) = n_e(\lambda_2, T)$, $n_3(T) = n_o(\lambda_3, T)$.
- **Type eoo**. $n_1(T) = n_e(\lambda_1, T)$, $n_2(T) = n_o(\lambda_2, T)$, $n_3(T) = n_o(\lambda_3, T)$.

Biaxial crystals

- **Type ZZZ**. $n_1(T) = n_z(\lambda_1, T)$, $n_2(T) = n_z(\lambda_2, T)$, $n_3(T) = n_z(\lambda_3, T)$.
- **Type YZY**. $n_1(T) = n_y(\lambda_1, T)$, $n_2(T) = n_z(\lambda_2, T)$, $n_3(T) = n_y(\lambda_3, T)$.
- **Type YYZ**. $n_1(T) = n_y(\lambda_1, T)$, $n_2(T) = n_y(\lambda_2, T)$, $n_3(T) = n_z(\lambda_3, T)$.
- **Type XZX**. $n_1(T) = n_x(\lambda_1, T)$, $n_2(T) = n_z(\lambda_2, T)$, $n_3(T) = n_x(\lambda_3, T)$.
- **Type XXZ**. $n_1(T) = n_x(\lambda_1, T)$, $n_2(T) = n_x(\lambda_2, T)$, $n_3(T) = n_z(\lambda_3, T)$.

4.4 Dispersion parameters

***c/v*: refractive index n**

- Sellmeier equations from [1] were utilised.
- The refractive index n formulas for any type of interaction are given in Section 4.1.2.

***c/u*: fraction of speed of light to the group velocity**

$$\frac{c}{u} = c\frac{dk}{d\omega}, \; k = \frac{2\pi n}{\lambda}, \; \lambda = \frac{2\pi c}{\omega}. \tag{131}$$

***GVD*: group velocity dispersion coefficient g**

$$g = \frac{d^2k}{d\omega^2}, \; k = \frac{2\pi n}{\lambda}, \; \lambda = \frac{2\pi c}{\omega}. \tag{132}$$

***walk-off*: the walk of angle β** (for *Bullk Crystals* only).
For extraordinary wave:

$$\beta = \arctan\left(\frac{\tan(\theta_p)(n_o^2 - n_e^2)}{n_e^2 + n_o^2\tan^2(\theta_p)}\right). \tag{133}$$

- uniaxial crystal: θ_p is the Euler angle θ.
- biaxial crystal: in XY plane θ_p is the Euler angle ϕ. In XZ and YZ planes θ_p is the Euler angle θ.

For ordinary wave:

$$\beta = 0. \tag{134}$$

d_{eff}: the effective nonlinear susceptibility (for *Bullk Crystals* only).
For each crystal, the formulas were taken from [1].

5 Edit crystals' database

- Click *Edit Database* in either *Bulk Crystals* or *PP Crystals* module, see Figs. 2 and 3, respectively.
- The menu window will be opened, Fig. 55
- Button *GO Back* returns to the previous window.
- Button *Reset DB* resets the database. All user-defined crystal are removed, only crystals from the main list remain. Crystals' main list is described in this tutorial. After clicking *Reset DB* button you will be asked one more time if you are sure to do this. It is recommended to restart the program after the reset of the database.

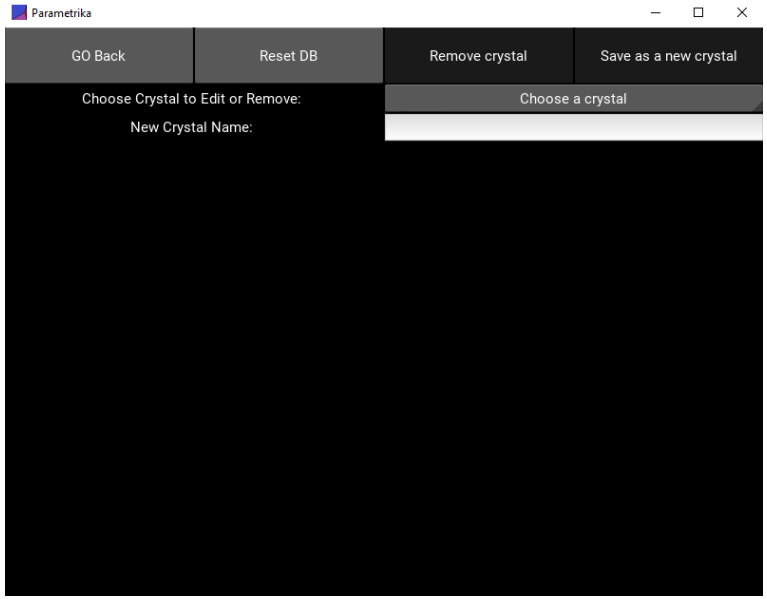


Figure 55: *Edit Database* menu window.

- The user is allowed to put in a new crystal on the base of an existing crystal.
- Choose a crystal to edit or remove from the list by clicking *Choose crystal*, Fig. 56.

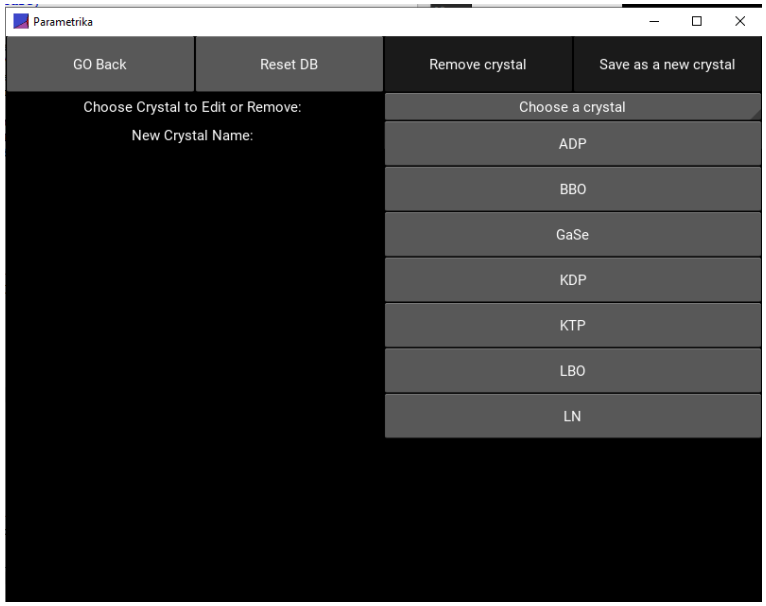


Figure 56: Crystals list drop-down menu. *Bulk crystals* module.

- Let's choose *BBO* crystal. Items to edit will appear, Fig, 57.
- One may choose a new crystal name. If one chooses an existing name, no changes will be applied.
- λ_{l1} and λ_{l2} are the limit wavelengths in the transparency range.
- Formulas for refractive index and other items are written in Python language. Use *np.sin(*)* and *np.cos(*)* for $\sin(*)$ and $\cos(*)$ functions. Use $a**n$ for power formula a^n.
- In the formulas, *theta* and *phi* denote the Euler angles θ, φ.
- In the module *PP Crystals*, refractive indices of uniaxial crystals depend both on wavelength λ and temperature T. For biaxial crystals, refractive index formula is given at $T = 300$ K and the formulas for the derivatives dn/dT should be provided.
- Press *Save as a new crystal* button to save the edited crystal. Note, that uniaxial crystal will remain uniaxial and biaxial crystal will be biaxial.
- Press button *Remove crystal* to remove a crystal. The crystals from the main list cannot be deleted.

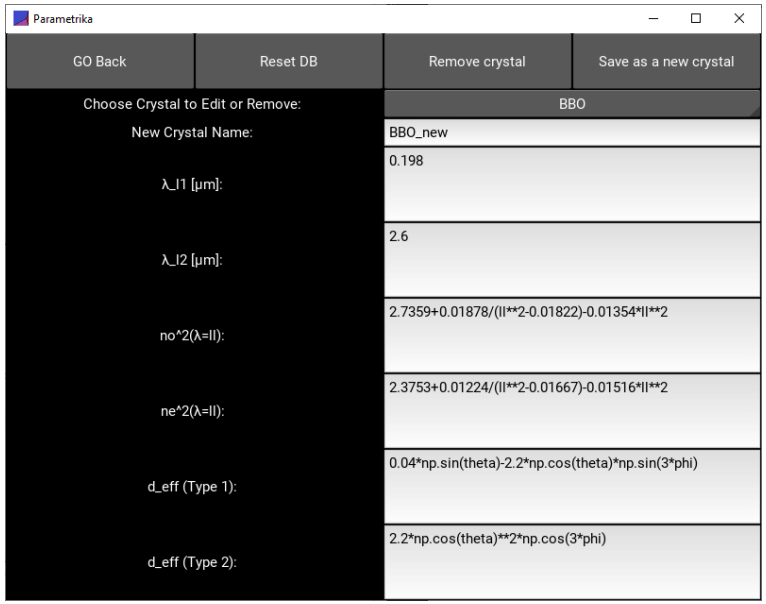


Figure 57: Crystal to edit: *BBO*. *Bulk crystals* module.

6 Financial Support

This work has received funding from European Regional Development Fund (Project No. 01.2.2-LMT-K-718-03-0004) under grant agreement with the Research Council of Lithuania (LMTLT).

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