6 Homework: Scattering kinematics

David Griffiths, Chapter 6, pp. 222-223, 6.4, 6.7, 6.8, 6.9, 6.10, 6.13, 6.14, 6.15

6.1 Classical cross section (seminar)

6.4. A nonrelativistic particle of mass m and (kinetic) energy E scatters from a fixed repulsive potential, $V(r) = k/r^2$, where k is a constant.

(a) Find the scattering angle, θ, as a function of the impact parameter, b. 0.2 POINTS

(b) Determine the differential cross section $d\sigma/d\Omega$, as a function of θ (not b). 0.2 POINTS

(c) Find the total cross section. 0.2 POINTS

6.2 Kinematics for cross sections

6.7. (a) Derive eq.(6.41),

$$\sqrt{(p_1 . p_2)^2 - (m_1 m_2)^2} = (E_1 + E_2)|\vec{p}_1| \tag{6.41}$$

for scattering of particles 1 and 2 in the CM frame. 0.2 POINTS

(b) Obtain the corresponding formula for the Lab frame (particle 2 at rest). 0.2 POINTS

6.8. Consider elastic scattering, $a + b \to a + b$, in the Lab frame (b initially at rest), assuming the target is so heavy ($m_b \gg E_a$) that its recoil is negligible. Determine the differential scattering cross section. [*Hint*: In this limit the Lab frame and the CM frame are the same.] 0.2 POINTS

6.9. Consider the collision, $1 + 2 \to 3 + 4$, in the Lab frame (2 at rest), with particles 3 and 4 massless. Obtain the formula for the differential cross section. 0.2 POINTS

6.10. (a) Analyze the problem of elastic scattering, ($1 + 2 \to 3 + 4$, with $m_1 = m_3$, $m_2 = m_4$) in the Lab frame (particle 2 at rest). Derive the formula for the differential cross section. 0.2 POINTS

(b) If the incident particle is massless ($m_1 = 0$), show that the result in part (a) simplifies. 0.2 POINTS

6.3 Matrix elements of the ABC theory

6.13. Calculate $d\sigma/d\Omega$ for $A + A \to B + B$ in the CM frame, assuming $m_B = m_C = 0$. Find the total cross section, σ. 0.2 POINTS

6.14. Find $d\sigma/d\Omega$ and σ for $A + A \to B + B$ in the Lab frame. (Let E be the energy and $\vec{p}$ the momentum of the incident A. Assume $m_B = m_C = 0$.) Determine the nonrelativistic and the ultrarelativistic limit of your formula. 0.2 POINTS

6.4 Full problem in the ABC theory

6.15. (a) Determine the lowest order amplitude for $A + B \to A + B$. (There are two diagrams.) 0.4 POINTS

(b) Find the differential cross section for this process in the CM frame, assuming $m_A = m_B = m$ and $m_C = 0$. Express your answer in terms of the incident energy (of A), E, and the scattering angle (for particle A), θ. 1.2 POINTS

(c) Find $d\sigma/d\Omega$ for this process in the Lab frame, assuming B is much heavier than A and remains stationary. A is incident with energy E. [*Hint*: See Problem (6.8). Assume $m_B \gg m_A, m_C$, and E.] 1.2 POINTS

(d) In case (c), find the total cross section, σ. 1.2 POINTS

Reading assignment: David Griffiths, Chapter 7; for more interested students: also P. B. Pal, *Dirac, Majorana and Weyl fermions*, arXiv:1006.1718 [hep-ph].

7 Spinors and photons; amplitudes and cross sections

David Griffiths, Chapter 7, pp. 268-273, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7, 7.8, 7.9, 7.11, 7.12, 7.13, 7.14, 7.15, 7.16, 7.17, 7.18, 7.19, 7.20, 7.21, 7.22, 7.23, 7.24, 7.25, 7.26, 7.27, 7.28, 7.29, 7.30, 7.31, 7.32, 7.33, 7.34, 7.35, 7.36, 7.37, 7.38, 7.39, 7.40, 7.41, 7.42, 7.44, 7.46, 7.47, 7.51,

7.1 Dirac spinors

7.3. Derive Equation 7.45, using Equations 7.43, 7.46, and 7.47, i.e.: Derive the normalization N (and show that it is the same) for the spinors

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{7.46}$$

$$v^{(1)} = u^{(4)}(-E,-p) = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad v^{(2)} = -u^{(3)}(-E,-p) = -N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \tag{7.47}$$

using the normalization condition

$$u^{(i)\dagger}u^{(i)} = v^{(i)\dagger}v^{(i)} = 2E \ . \tag{7.43}$$

0.1 POINTS

7.4. Show that $u^{(1)}$ and $u^{(2)}$ are *orthogonal*, in the sense that $u^{(1)\dagger}u^{(2)} = 0$. Likewise, show that $v^{(1)}$ and $v^{(2)}$ are orthogonal. Are $u^{(1)}$ and $v^{(1)}$ orthogonal? 0.1 POINTS

7.5. Show that for $u^{(1)}$ and $u^{(2)}$ (Equation 7.46) the lower components (u_B) are smaller than the upper ones (u_A), in the nonrelativistic limit, by a factor v/c. [This simplifies matters, when we are doing nonrelativistic approximations; we think of u_A as the 'big' components and u_B as the '1ittle' components. (For $v^{(1)}$ and $v^{(2)}$ the roles are reversed.) In the relativistic limit, by contrast, u_A and u_B are comparable in size.]

0.1 POINTS

7.6. If the z-axis (= 3-axis) points along the direction of motion, calculate the spinors $u^{(1)}$, $u^{(2)}$, $v^{(1)}$ and $v^{(2)}$. Confirm that these are eigenspinors of $S_z = \frac{\hbar}{2}\gamma^0\gamma^3\gamma^5$, and find their eigenvalues. 0.1 POINTS

7.7. Construct the normalized spinors $u^{(+)}$ and $u^{(-)}$ representing an electron of momentum $\vec{p}$ with helicity ± 1. That is, find the u's that satisfy Equation 7.49

$$(\not{p} - m)u = (\gamma^\mu p_\mu - m)u = 0 \tag{7.49}$$

and are eigenspinors of the helicity operator $(\hat{p} \cdot \vec{\Sigma})$ with eigenvalues ± 1. 0.1 POINTS

7.8. *The purpose of this problem is to demonstrate that particles described by the Dirac equation carry 'intrinsic' angular momentum ($\vec{S}$) in addition to their orbital angular momentum ($\vec{L}$), neither of which seperately conserved, although their sum is. It should be attempted only if you are reasonably familiar with quantum mechanics.*

(a) Construct the Hamiltonian, H for the Dirac equation. [*Hint*: Solve eq. 7.19 for p^0. Solution: $H = \gamma^0(\vec{\gamma} \cdot \vec{p} + m)$, where $\vec{p} = (\hbar/i)\vec{\nabla}$ is the momentum operator.]

0.2 POINTS

(b) Find the commutator of H with the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$. [*Solution*: $[H, \vec{L}] = -i\hbar\gamma^0(\vec{\gamma} \times \vec{p})$.] Since $[H, \vec{L}]$ is non zero, $\vec{L}$ by itself is not conserved. Evidently there is some *other* form of angular momentum lurking here. Introduce the 'spin angular momentum', $\vec{S}$, defined by eq. 7.51.

$$\vec{S} = \frac{\hbar}{2}\vec{\Sigma} \ , \quad \text{with } \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{7.51}$$

0.2 POINTS

(c) Find the commutator of H with the spin angular momentum, $\vec{S} = (\hbar/2)\vec{\Sigma}$. [*Solution*: $[H, \vec{S}] = i\hbar\gamma^0(\vec{\gamma} \times \vec{p})$.] It follows that the *total* angular momentum, $\vec{J} = \vec{L} + \vec{S}$, *is* conserved. 0.2 POINTS

(d) Show that every bispinor is an eigenstate of $\vec{S}^2$, with eigenvalue $\hbar^2 s(s+1)$, and find s. What, then, is the spin of a particle described by the Dirac equation?

0.2 POINTS

7.9. The charge conjugation operator (C) takes a Dirac spinor ψ into the 'charge-conjugate' spinor ψ_c, given by $\psi_c = i\gamma^2\psi^*$, where γ^2 is the third Dirac gamma matrix. [See Halzen and Martin, Sect. 5.4.] Find the charge-conjugates of $u^{(1)}$ and $u^{(2)}$, and compare them with $v^{(1)}$ and $v^{(2)}$. 0.1 POINTS

7.11. Confirm the transformation rule for spinors, equation (7.52),

$$\psi \to \psi' = \mathbf{S}\psi \ , \tag{7.52}$$

with eqs. (7.53) and (7.54), given below. [*Hint*: we want it to carry solutions to the Dirac equation in the original frame to solutions in the primed frame:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0 \quad \Leftrightarrow \quad i\hbar\gamma^\mu\partial'_\mu\psi' - mc\psi' = 0 \ . \tag{1}$$

where $\psi' = \mathbf{S}\psi$ and

$$\partial'_\mu = \frac{\partial}{\partial x^{\mu\prime}} = \frac{\partial x^\nu}{\partial x^{\mu\prime}}\frac{\partial}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x^{\mu\prime}}\partial_\nu \ . \tag{2}$$

It follows that

$$(\mathbf{S}^{-1}\gamma^\mu\mathbf{S})\frac{\partial x^\nu}{\partial x^{\mu\prime}} = \gamma^\nu \ . \tag{3}$$

The (inverse) Lorentz transformations tell us $\frac{\partial x^\nu}{\partial x^{\mu\prime}}$. Take it from there.] 0.1 POINTS

7.12. Derive the transformation rule for parity, eq. (7.61),

$$\psi \to \psi' = \gamma^0\psi \ , \tag{7.61}$$

using the method in Problem 7.11. 0.1 POINTS

7.13. (a) Starting with eq. (7.53),

$$\mathbf{S}(\Lambda) = a_+ + a_-\gamma^0\gamma^1 \ , \tag{7.53}$$

with eq. (7.54),

$$a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)} \ , \tag{7.54}$$

and Λ being a Lorentz transformation only along the 1-axis, calculate $\mathbf{S}^\dagger\mathbf{S}$, and confirm that it is not the unity matrix.

0.1 POINTS

(b) Show that $\mathbf{S}^\dagger\gamma^0\mathbf{S} = \gamma^0$. 0.1 POINTS

(*) Show that $\bar{\mathbf{S}}(\Lambda) := \gamma^0\mathbf{S}^\dagger(\Lambda)\gamma^0 = \mathbf{S}^{-1}(\Lambda)$ 0.1 POINTS

7.14. Show that $\bar\psi\gamma^5\psi$ is invariant under $\psi \to \psi' = \mathbf{S}\psi$ with $\mathbf{S}$ from Problem 7.11.
0.1 POINTS

7.15. Show that the adjoint spinors $\bar u^{(1,2)}$ and $\bar v^{(1,2)}$ satisfy the equations

$$\bar u(\gamma^\mu p_\mu - m) = 0 \qquad \text{and} \qquad \bar v(\gamma^\mu p_\mu + m) = 0 \ . \tag{4}$$

0.1 POINTS

7.16. Show that the normalisation condition from Problem 7.3., expressed with the adjoint spinors, becomes

$$\bar u u = -\bar v v = 2mc \ . \tag{5}$$

0.1 POINTS

7.17. Show that $V^\mu = \bar\psi\gamma^\mu\psi$ is a four-vector, by confirming that its components transform according to the Lorentztransformation $V'^\mu = \Lambda^\mu_\nu V^\nu$ by transforming the spinors. Check that it transforms as a (polar) vector under parity (i.e., the 'time' component is invariant, whereas the 'spatial' components switch sign). 0.2 POINTS

7.18. Show that the spinor representing an electron at rest, eq. (7.30),

$$\begin{aligned} \psi^{(1)} &= e^{-i(m/\hbar)t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \;, \quad \psi^{(2)} = e^{-i(m/\hbar)t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ \psi^{(3)} &= e^{+i(m/\hbar)t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \;, \quad \psi^{(4)} = e^{+i(m/\hbar)t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \end{aligned} \tag{7.30}$$

is an eigenstate of the parity operator P. What is its intrinsic parity? How about the positron? What if you changed the sign convention in eq. (7.61)? Notice that whereas the *absolute* parity of a spin-$\frac{1}{2}$ particle is in a sense arbitrary, the fact that particles and antiparticles carry *opposite* parity is *not* arbitrary. 0.1 POINTS

7.24. Using $u^{(1)}$, $u^{(2)}$, eq. (7.46), and $v^{(1)}$, $v^{(2)}$, eq. (7.47), prove the completeness relations for spinors, eq. (7.99):

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + m) \quad \text{and} \quad \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = (\gamma^\mu p_\mu - m) \;. \tag{7.99}$$

0.1 POINTS

7.2 Electrodynamics

7.20. (a) Derive Maxwell's equations, eq. (7.70)

$$\left\{ \begin{array}{llll} \text{(i)} & \vec{\nabla}\cdot\vec{E} = 4\pi\rho & \text{(iii)} & \vec{\nabla}\cdot\vec{B} = 0 \\ \text{(ii)} & \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 & \text{(iv)} & \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \end{array} \right\} \;, \tag{7.70}$$

from eq. (7.73)

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \;. \tag{7.73}$$

0.2 POINTS

(b) Prove eq. (7.74),

$$\partial_\mu J^\mu = 0 \;, \tag{7.74}$$

from eq. (7.73) 0.2 POINTS

7.21. Show that the continuity equation, eq. (7.74), enforces conservation of charge. [If you don't see how to do this, look in any electrodynamics textbook.] 0.6 POINTS

7.22. Show that we are always free to pick $A^0 = 0$, in free space. That is, given a potential A^μ which does *not* satisfy this constraint, find a gauge function λ, consistent with eq. (7.85),

$$\Box\lambda = 0 \;, \tag{7.85}$$

such that A_0' in eq. (7.81) is zero.

$$A_\mu' = A_\mu + \partial_\mu\lambda \;, \tag{7.81}$$

0.1 POINTS

7.23. Suppose we apply a gauge transformation, eq. (7.81), to the plane wave potential, eq. (7.89),

$$A^\mu(x) = a e^{-(i/\hbar)p.x} \epsilon^\mu(p) \ , \tag{7.89}$$

using as a gauge function

$$\lambda = i\hbar\kappa a e^{-(i/\hbar)p.x} \ , \tag{6}$$

where κ is an arbitrary constant and p is the photon four-momentum

(a) Show that this λ satisfies eq. (7.85). 0.1 POINTS

(b) Show that this gauge transformation has the effect of modifying ϵ^μ: $\epsilon^\mu \to \epsilon^\mu + \kappa p^\mu$. (In particular, if we choose $\kappa = -\epsilon^0/p^0$ we obtain the Coulomb gauge polarization vector, Equation 7.92)

$$\epsilon^0 = 0 \ , \quad \text{so} \quad \vec{\epsilon} \cdot \vec{p} = 0 \ . \tag{7.92}$$

This observation leads to a beautifully simple test for the gauge invariance of QED results: the answer must be unchanged if you replace ϵ^μ by $\epsilon^\mu + \kappa p^\mu$.
0.1 POINTS

7.25. Using $\epsilon^{(1)}$ and $\epsilon^{(2)}$, eq. (7.93), confirm the completeness relations for photons, eq. (7.105):

$$\sum_{s=1,2} \epsilon_i^{(s)} \epsilon_j^{(s)*} = \delta_{ij} - \hat{p}_i \hat{p}_j \ . \tag{7.105}$$

0.1 POINTS

7.3 γ-matrices (Clifford algebra)

7.2. Show that eq. (7.17)

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \ , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \ , \tag{7.17}$$

satisfies eq. (7.15)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ , \tag{7.15}$$

0.1 POINTS

7.19. (a) Express $\gamma^\mu\gamma^\nu$ as a linear combination of 1, γ^5, γ^μ, $\gamma^\mu\gamma^5$, and $\sigma^{\mu\nu}$. 0.1 POINTS

(b) Construct the matrices σ^{12}, σ^{13}, and σ^{23}, eq. (7.69),

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \ , \tag{7.69}$$

and relate them to Σ^1, Σ^2, and Σ^3, eq. (7.51). 0.1 POINTS

7.29. (a) Show that $\gamma^0\gamma^{\nu\dagger}\gamma^0 = \gamma^\nu$, for $\nu = 0$, 1, 2, or 3. 0.1 POINTS

(b) If Γ is any product of γ-matrices ($\Gamma = \gamma^a\gamma^b \cdots \gamma^c$) show that

$$\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0 \ , \tag{7.119}$$

is the same product in reverse order, $\bar{\Gamma} = \gamma^c \cdots \gamma^b\gamma^a$. 0.1 POINTS

7.31. (a) Prove the trace theorem 1, 2, and 3, in Section 7.7.

$$1. \quad \mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) \tag{1.}$$
$$2. \quad \mathrm{Tr}(\alpha A) = \alpha \mathrm{Tr}(A) \tag{2.}$$
$$3. \quad \mathrm{Tr}(AB) = \mathrm{Tr}(BA)\ . \tag{3.}$$

0.1 POINTS

(b) Prove Equation 4

$$4. \quad g_{\mu\nu} g^{\mu\nu} = 4\ . \tag{4.}$$

0.1 POINTS

(c) Using the anticommutation relation 5, prove 5′

$$5. \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \qquad 5'. \quad \not{a}\not{b} + \not{b}\not{a} = 2(a.b)\ . \tag{5.}$$

0.1 POINTS

7.32. (a) Use the anticommutation relation 5 to prove the contraction theorems 6, 7, 8, and 9.

$$6. \quad \gamma_\mu\gamma^\mu = 4 \tag{6.}$$
$$7. \quad \gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu \qquad 7'. \quad \gamma_\mu\not{a}\gamma^\mu = -2\not{a} \tag{7.}$$
$$8. \quad \gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda} \qquad 8'. \quad \gamma_\mu\not{a}\not{b}\gamma^\mu = 4(a.b) \tag{8.}$$
$$9. \quad \gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu \qquad 9'. \quad \gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a} \tag{9.}$$

0.1 POINTS

(b) From 7 prove 7′, from 8 prove 8′, from 9 prove 9′. 0.1 POINTS

7.33. (a) Confirm the trace theorems 10, 11, 12, and 13.

10. The trace of the product of an *odd* number of gamma matrices is *zero*

$$11. \quad \mathrm{Tr}[1] = 4 \tag{11.}$$
$$12. \quad \mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \qquad 12'. \quad \mathrm{Tr}[\not{a}\not{b}] = 4(a.b) \tag{12.}$$
$$13. \quad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}) \tag{13.}$$
$$13'. \quad \mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4[(a.b)(c.d) - (a.c)(b.d) + (a.d)(b.c)] \tag{13′.}$$

0.1 POINTS

(b) From 12 prove 12′, from 13 prove 13′. 0.1 POINTS

7.34. (a) Proof theorems 14, 15, and 16.

$$14. \quad \mathrm{Tr}[\gamma^5] = 0 \tag{14.}$$
$$15. \quad \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0 \qquad 15'. \quad \mathrm{Tr}[\gamma^5\not{a}\not{b}] = 0 \tag{15.}$$
$$16. \quad \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] = 4i\epsilon^{\mu\nu\lambda\sigma} \tag{16.}$$
$$16'. \quad \mathrm{Tr}[\gamma^5\not{a}\not{b}\not{c}\not{d}] = 4i\epsilon^{\mu\nu\lambda\sigma}a_\mu b_\nu c_\lambda d_\sigma = 4i\epsilon^{abcd} \tag{16′.}$$

0.1 POINTS

(b) From 15 prove 15′, from 16 prove 16′. 0.1 POINTS

7.35. (a) Show that $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\tau} = -6\delta^{\sigma}_{\tau}$. (Summation over μ, ν, λ implied) 0.1 POINTS

(b) Show that $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\theta\tau} = -2(\delta^{\lambda}_{\theta}\delta^{\sigma}_{\tau} - \delta^{\lambda}_{\tau}\delta^{\sigma}_{\theta})$. 0.1 POINTS

(c) Find the analogous formula for $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\phi\theta\tau}$. 0.1 POINTS

(d) Find the analogous formula for $\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\omega\phi\theta\tau}$. 0.1 POINTS

7.36. Evaluate the following traces:

(a) $\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}(1-\gamma^5)\gamma^{\lambda}(1+\gamma^5)\gamma_{\lambda}]$ 0.1 POINTS

(b) $\mathrm{Tr}[(\not{p}+m)(\not{q}+M)(\not{p}+m)(\not{q}+M)]$, where p is the four momentum of a (real) particle of mass m and q is the four momentum of a (real) particle of mass M. Express your answer in terms of m, M, and $(p.q)$. 0.1 POINTS

7.4 QED matrix elements

7.26. Evaluate the amplitude for electron-muon scattering, eq. (7.106),

$$\mathcal{M} = -\frac{g_e^2}{(p_1-p_3)^2}[\bar{u}^{(s_3)}(p_3)\gamma^{\mu}u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_{\mu}u^{(s_2)}(p_2)] \quad . \tag{7.106}$$

in the CM system, assuming the e and μ approach one another along the $\hat{z}$-axis, repel, and return back along the $\hat{z}$-axis. Assume the initial and final particles all have helicity +1. [*Answer*: $\mathcal{M} = -2g_e^2$] 0.4 POINTS

7.27. Derive the amplitudes, eq. (7.133) and (7.134),

$$\mathcal{M}_1 = \frac{g_e^2}{(p_1-p_3)^2-m^2}\bar{v}(2)\not{\epsilon}_4(\not{p}_1-\not{p}_3+m)\not{\epsilon}_3u(1) \tag{7.133}$$

$$\mathcal{M}_2 = \frac{g_e^2}{(p_1-p_4)^2-m^2}\bar{v}(2)\not{\epsilon}_3(\not{p}_1-\not{p}_4+m)\not{\epsilon}_4u(1) \tag{7.134}$$

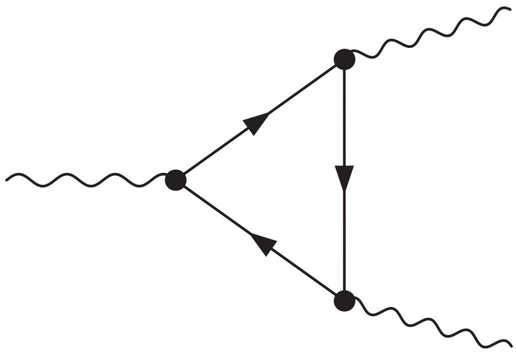
for pair annihilation $e^+ + e^- \to \gamma + \gamma$. 0.1 POINTS

7.42. Derive Equation 7.176 for the loop diagram on p.262

$$\mathcal{M} = \frac{-ig_e^4}{q^4}[\bar{u}(3)\gamma^{\mu}u(1)]\left\{\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}[\gamma_{\mu}(\not{k}+m)\gamma_{\nu}(\not{k}-\not{q}+m)]}{(k^2-m^2)[(k-q)^2-m^2]}\right\}[\bar{u}(4)\gamma^{\nu}u(2)] \tag{7.176}$$

You'll need one last Feynman rule: for a closed fermion loop include a factor −1 and take the trace. 0.8 POINTS

7.46. Why can't the photon 'decay', by the process $\gamma \to \gamma + \gamma$, Fig. (7.12)? Calculate the amplitude for this diagram. [This is an example of *Furry's theorem*, which says that any diagram containing a closed electron loop with an odd number of vertices has an amplitude of zero.]



(Fig. 7.12)

Fig. 7.12: Decay of the photon: $\gamma \to \gamma + \gamma$
– a process forbidden by Furry's theorem (Problem 7.46). 0.4 POINTS

7.5 Casimir's trick

7.28. Work out the analog to Casimir's trick, eq. (7.125),

$$\sum_{\text{all spins}} [\bar{u}(a)\Gamma_1 u(b)]\,[\bar{u}(a)\Gamma_2 u(b)]^* = \mathrm{Tr}[\Gamma_1(\not{p}_b + m_b)\bar{\Gamma}_2(\not{p}_a + m_a)] \ , \tag{7.125}$$

for *antiparticles*

$$\sum_{\text{all spins}} [\bar{v}(a)\Gamma_1 v(b)]\,[\bar{v}(a)\Gamma_2 v(b)]^* \tag{7}$$

and for 'mixed' cases

$$\sum_{\text{all spins}} [\bar{u}(a)\Gamma_1 v(b)]\,[\bar{u}(a)\Gamma_2 v(b)]^* \quad \text{and} \quad \sum_{\text{all spins}} [\bar{v}(a)\Gamma_1 u(b)]\,[\bar{v}(a)\Gamma_2 u(b)]^* \ . \tag{8}$$

0.1 POINTS

7.30. Use Casimir's trick, eq. (7.125), to obtain an expression analogous to eq. (7.126)

$$|\mathcal{M}|^2 = \frac{g_e^4}{4(p_1 - p_3)^4}\mathrm{Tr}[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_3 + m)] \times \mathrm{Tr}[\gamma_\mu(\not{p}_2 + M)\gamma_\nu(\not{p}_4 + M)] \ . \tag{7.126}$$

for Compton scattering. Note that there are four terms here:

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + \mathcal{M}_1\mathcal{M}_2^* + \mathcal{M}_1^*\mathcal{M}_2 \ . \tag{9}$$

0.8 POINTS

7.37. Starting with Equation 7.107,

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)] + \frac{g_e^2}{(p_1 - p_4)^2}[\bar{u}(4)\gamma^\mu u(1)][\bar{u}(3)\gamma_\mu u(2)] \tag{7.107}$$

determine the spin-averaged amplitude, analogous to eq. (7.129),

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle =& \frac{4g_e^4}{(p_1 - p_3)^4}\left\{p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}[m^2 - (p_1.p_3)]\right\} \\ & \times \left\{p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}[M^2 - (p_2.p_4)]\right\} \\ =& \frac{8g_e^4}{(p_1 - p_3)^4}[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) \\ & - (p_1.p_3)M^2 - (p_1.p_3)m^2 + 2(mM)^2]\end{aligned} \tag{7.129}$$

for elastic electron-electron scattering. Assume we're working at high energies, so that the mass of the electron an be ignored (i.e., set $m = 0$). [*Hint*: You can read $(|\mathcal{M}_1|^2)$ and $(|\mathcal{M}_2|^2)$ from Equation 7.129. For $(\mathcal{M}_1\mathcal{M}_2^*)$ use the same strategy as Casimir's trick to get

$$(\mathcal{M}_1\mathcal{M}_2^*) = \frac{-g_e^4}{4(p_1 - p_3)^2(p_1 - p_4)^2}\mathrm{Tr}[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_4\gamma_\mu\not{p}_2\gamma_\nu\not{p}_3] \tag{10}$$

Then exploit the contraction theorems to evaluate the trace. Notice, that for *massless* particles the conservation of momentum $(p_1 + p_2 = p_3 + p_4)$ implies, that $(p_1.p_2) = (p_3.p_4)$, $(p_1.p_3) = (p_2.p_4)$, and $(p_1.p_4) = (p_2.p_3)$.] 1.0 POINTS

7.38. (a) Starting with Equation 7.129, find the spin averaged amplitude for electron-muon scattering in the CM frame, in the high energy regime ($m, M \to 0$).
0.2 POINTS

(b) Find the CM differential cross section for electron-muon scattering at high energy. Let E be the electron energy and θ the scattering angle. 0.2 POINTS

7.39. (a) Using the result of Problem 7.37, determine the spin-averaged amplitude for electron-electron scattering in the CM in the high-energy regime ($m, \to 0$).
0.2 POINTS

(b) Find the CM differential cross section for electron-electron scattering at high energy. 0.2 POINTS

7.40. Starting with Equation 7.158,

$$\mathcal{M}_{\text{singlet}} = -2\sqrt{2} i g_e^2 (\vec{\epsilon}_3 \times \vec{\epsilon}_4)_z \tag{7.158}$$

calculate $|\mathcal{M}|^2$, and use eq. (7.105) to sum over photon polarizations. Check that the answer is consistent with eq. (7.163),

$$\mathcal{M}_{\text{singlet}} = -4g_e^2 \tag{7.163}$$

and explain why this method gives the correct answer (note that we are now summing over *all* photon polarizations, whereas in *fact* the photons must be in the singlet configuration). 0.2 POINTS

7.41. Starting with Equation 7.149,

$$\mathcal{M} = \frac{g_e^2}{m} \bar{v}(2)[(\vec{\epsilon}_3 \cdot \vec{\epsilon}_4)\gamma^0 + i(\vec{\epsilon}_3 \times \vec{\epsilon}_4) \cdot \vec{\Sigma}\gamma^3]u(1) \tag{7.149}$$

calculate $|\mathcal{M}|^2$ for $e^+ + e^- \to \gamma + \gamma$, and use it to get the differential cross section for pair annihilation. Compare eq. (7.167) (see footnote before example 7.8).

$$\frac{d\sigma}{d\Omega} = \frac{1}{v}\left(\frac{\hbar\alpha}{m}\right)^2 \tag{7.167}$$

0.8 POINTS

7.6 Historical interest

7.44. Derive Equation 7.187

$$q^2 = -4|\vec{p}|^2 \sin^2 \frac{\theta}{2} \tag{7.187}$$

0.1 POINTS

7.45. Evaluate the correction term in eq. (7.192)

$$\alpha(q^2) = \alpha(0)\left\{1 + \frac{\alpha(0)}{3\pi} f\left(\frac{-q^2}{m^2}\right)\right\} \ , \tag{7.192}$$

where

$$f(x) = 6\int_0^1 z(1-z)\ln[1+xz(1-z)]dz$$
$$= -\frac{5}{3} + \frac{4}{x} + \frac{2(x-2)}{x}\sqrt{\frac{x+4}{x}}\tanh^{-1}\sqrt{\frac{x}{x+4}} \ , \qquad (7.184)$$

for the case of a head-on collision in the CM; assume the electron is traveling at $\frac{1}{10}c$. In the experiment [9], the beam energies were 57.8 GeV; what should the measured fine structure 'constant' have been? Look up the actual result, and compare it with your prediction. 0.2 POINTS

7.47. Starting with your answer to Problem 7.30, derive the *Klein-Nishina formula* for Compton scattering (in the rest frame of the target):

$$\frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right] \ , \qquad (11)$$

where ω and ω' are the frequencies of the incident and scattered photon (Problem 3.27). 0.2 POINTS

7.7 Full problem

7.51. Spin-$\frac{1}{2}$ particles that are electrically neutral could conceivably be their own antiparticles (if so, they are called "Majorana" fermions – in the Standard Model the only possible candidates are the neutrinos)

(a) According to Problem 7.9, the charge conjugate spinor is $\psi_c = i\gamma^2\psi^*$. Evidently, if a particle is the same as its antiparticle, then $\psi = \psi_c$. Show that this condition is Lorentz invariant (if true in one inertial frame, it is true in any inertial frame). [*Hint*: Use eqs. (7.52) and (7.53).] 1.0 POINTS

(b) Show that if $\psi = \psi_c$, the "lower" two elements of ψ are related to the "upper" two by $\psi_B = -i\sigma_y\psi_A^*$. For Majorana particles, then, we only need a two-component spinor $\chi \equiv \psi_A$. This makes sense: A Dirac spinor takes four elements to represent the two spin states (each) of the particle and the antiparticle, but in this case the latter two are redundant. Show that the Dine equation for a Majorana particle can be written in 2-component form as

$$i\hbar[\partial_0\chi + i(\vec{\sigma}\cdot\vec{\nabla})\sigma_y\chi^*] - m\chi = 0 \ , \qquad (12)$$

Check that the equation you get for the "lower" elements is consistent with this. 1.0 POINTS

(c) Construct spinors χ representing plane wave Majorana states. [*Hint*: Form the general linear combination $\psi = a_1\psi^{(1)} + a_2\psi^{(2)} + a_3\psi^{(3)} + a_4\psi^{(4)}$, eqs. (7.46) and (7.47), impose the constraint in part (b), and solve for a_3 and a_4 (in terms of a_1 and a_2); then pick (say) $a_1 = 1$, $a_2 = 0$ for $\chi^{(1)}$, and $a_1 = 0$, $a_2 = 1$ for $\chi^{(2)}$.] 1.0 POINTS

8 quarks and gluons; amplitudes and cross sections

David Griffiths, Chapter 8, pp. 303-306, 8.4, 8.5, 8.7, 8.8, 8.9, 8.10, 8.11, 8.12, 8.13, 8.14, 8.15, 8.17, 8.19, 8.20, 8.21, 8.23, 8.24, 8.25, 8.26, 8.27, 8.28,

8.1 Quark scattering

8.4. Prove Equation 8.16

$$q_\mu K^{\mu\nu} = 0 \ . \tag{8.16}$$

[*Hint*: First show that $q_\mu L^{\mu\nu} = 0$. Then argue that we *may as well* take $K^{\mu\nu}$ such that $q_\mu K^{\mu\nu} = O$, in the sense that any term in $K^{\mu\nu}$ that does not obey $q_\mu K^{\mu\nu} = O$ will contribute nothing to $L^{\mu\nu} K_{\mu\nu}$.] *Comment*: Equation 8.16 actually follows more simply and generally from charge conservation at the proton vertex. but I have not developed the formalism here to make this argument (see Halzen and Martin [2], Sections 8.2 and 8.3). [One way to proceed is as follows. Take $q^\mu = (0, 0, 0, q)$; then $q_\mu L^{\mu\nu} = 0 \Rightarrow L^{\mu 3} = L^{3\mu} = 0$. So $L^{\mu\nu} K_{\mu\nu} = L^{jk} K_{jk} + L^{j3} K_{j3} + L^{3k} K_{3k} + L^{33} K_{33} = L^{jk} K_{jk}$ with $j, k = 0, 1, 2$, and then K_{j3} and K_{3k} might as well be zero.] 0.2 POINTS

8.5. Prove eq. (8.17)

$$K_4 = \frac{M^2}{q^2} K_1 + \frac{1}{4} K_2 \quad \text{and} \quad K_5 = \frac{1}{2} K_2 \tag{8.17}$$

from eq. (8.16) [*Hint*: First contract $K^{\mu\nu}$ with q_μ, then with p_ν.] 0.2 POINTS

8.7. Derive eq. (8.19)

$$\langle |\mathcal{M}|^2 \rangle = \left(\frac{2 g_e^2}{q^2} \right)^2 \left\{ K_1 [(p_1.p_3) - 2m^2] + K_2 \left[\frac{(p_1.p)(p_3.p)}{M^2} + \frac{q^2}{4} \right] \right\} \ . \tag{8.19}$$

0.2 POINTS

8.8. Derive eq. (8.20)

$$\langle |\mathcal{M}|^2 \rangle = \frac{g_e^4}{4 E E' \sin^4(\theta/2)} \left(2 K_1 \sin^2 \frac{\theta}{2} + K_2 \cos^2 \frac{\theta}{2} \right) \ . \tag{8.20}$$

0.2 POINTS

8.9. Derive eq. (8.21)

$$E' = \frac{E}{1 + (2E/M) \sin^2(\theta/2)} \ . \tag{8.21}$$

0.2 POINTS

8.10. Check that the Rosenbluth formula, eq. (8.23),

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha \hbar}{4 M E \sin^2(\theta/2)} \right)^2 \frac{E'}{E} [2 K_1 \sin^2 \frac{\theta}{2} + K_2 \cos^2 \frac{\theta}{2}] \ , \tag{8.23}$$

agrees with the Mott formula, eq. (7.131),

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha \hbar}{2 |\vec{p}\,|^2 \sin^2(\theta/2)} \right)^2 [m^2 + |\vec{p}\,|^2 \cos^2(\theta/2)] \ , \tag{7.131}$$

in the intermediate energy regime ($m \ll E \ll M$). Use the expressions for K_1 and K_2 appropriate to a 'Dirac' proton (Problem 8.6). 0.2 POINTS

8.2 Color

8.11. Why can't the 'ninth gluon' be the photon? (For reading and thinking about it ...)
0.2 POINTS

8.12. Color $SU(3)$ transformations relable 'red', 'blue', and 'green' according to the transformation rule

$$c \to c' = Uc \ , \tag{13}$$

where U is any unitary ($UU^\dagger = 1$) 3×3-matrix of determinant 1 and c is a three-element column vector. For example

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{14}$$

would take $r \to g$, $g \to b$, $b \to r$. The ninth gluon, $|9\rangle$, is obviously invariant under U, but the octet gluons are *not*. Show that $|3\rangle$ and $|8\rangle$ go into linear combinations of one another:

$$|3'\rangle = \alpha|3\rangle + \beta|8\rangle \ , \quad |8'\rangle = \gamma|3\rangle + \delta|8\rangle \ . \tag{15}$$

Find the numbers α, β, γ, and δ. 0.1 POINTS

8.13. Show that $\mathrm{Tr}[\lambda^a\lambda^b] = 2\delta^{ab}$. (Notice that all λ matrices are traceless.) 0.1 POINTS

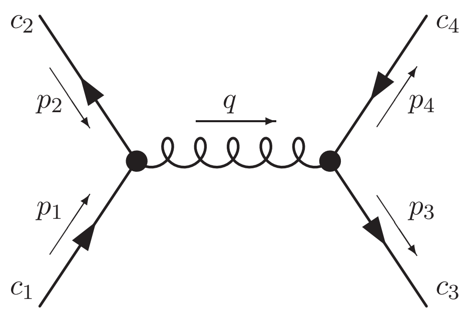
8.14. What are the structure constants for $SU(2)$? That is, what are the numbers f^{ijk} in $[\sigma^i, \sigma^j] = 2if^{ijk}\sigma^k$. 0.1 POINTS

8.15. (a) Given that $f^{\alpha\beta\gamma}$ is completely antisymmetric (so that $f^{112} = 0$ automatically, and having calculated f^{123} we don't need to bother with f^{213}, f^{231}, etc.) how many *distinct* nontrivial structure constants remain? 0.1 POINTS
(Of these, it turns out that only nine are nonzero (those listed in eq. (8.36)), and among these there are only three different *numbers*.)

(b) Work out $[\lambda^1, \lambda^2]$, and confirm that $f^{12\gamma} = 0$ for all γ except 3, while $f^{123} = 1$.
0.1 POINTS

(c) Similarly, compute $[\lambda^1, \lambda^3]$, and $[\lambda^4, \lambda^5]$, and determine the resulting structure constants. 0.1 POINTS

8.17. Find the amplitude $\mathcal{M}$ for diagram



What is the color factor, analogous to eq. (8.47),

$$f = \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_2^\dagger \lambda^\alpha c_4) \ , \tag{8.47}$$

in this case? Evaluate f in the color singlet configuration. Can you explain this result? 0.2 POINTS

8.19. Color factors always involve expressions of the form $\lambda^\alpha_{ij}\lambda^\alpha_{k\ell}$ (summed over α). There is a simple formula for this quantity, which shortens the arithmetic:

$$\lambda^\alpha_{ij}\lambda^\alpha_{k\ell} = 2\delta_{i\ell}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{k\ell} \ , \tag{16}$$

(see Kane [4]). Check this theorem for

(a) $i = j = k = \ell = 1$, see eq. (8.61) 0.1 POINTS

(b) $i = j = 1$, $k = \ell = 2$, see eq. (8.49) 0.1 POINTS

(c) $i = \ell = 1$, $j = k = 2$, see eq. (8.62) 0.1 POINTS

(d) Use it to confirm eq. (8.52) 0.1 POINTS

8.20. Derive eq. (8.70),

$$\begin{aligned}\mathcal{M}_3 = \ & i\frac{g_s^2}{4}\frac{1}{(p_3.p_4)}\bar{v}(2)[(\epsilon_3.\epsilon_4)(\not{p}_4 - \not{p}_3) + 2(p_3.\epsilon_4)\not{\epsilon}_3 - 2(p_4.\epsilon_3)\not{\epsilon}_4]u(1) \\ & \times f^{\alpha\beta\gamma} a_3^\alpha a_4^\beta (c_2^\dagger \lambda^\gamma c_1) \ ,\end{aligned} \tag{8.70}$$

starting from eq. (8.69)

$$\begin{aligned}\mathcal{M}_3 = \ & i\bar{v}(2)c_2^\dagger \left[-i\frac{g_s}{2}\lambda^\delta\gamma_\sigma\right] u(1)c_1 \left[-i\frac{g^{\sigma\lambda}\delta^{\delta\gamma}}{q^2}\right] \times \\ & \{-g_s f^{\alpha\beta\gamma}[g_{\mu\nu}(-p_3+p_4)_\lambda + g_{\nu\lambda}(-p_4-q)_\mu + g_{\lambda\mu}(q+p_3)_\nu]\}[\epsilon_3^\mu a_3^\alpha][\epsilon_4^\nu a_4^\beta] \ .\end{aligned} \tag{8.69}$$

0.6 POINTS

8.21. There is a simple test for the gauge invariance of an amplitude ($\mathcal{M}$) in QCD (or QED): Replace any gluon (or photon) polarization vector by its momentum ($\epsilon_3 \to p_3$, say), and you must get zero (see Problem 7.23). Show using this criterion that $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 =$ eq. (8.65) + eq. (8.68) + eq. (8.70), (remember, as in the book p. 295, no * on polarization vectors)

$$\mathcal{M}_1 = i\bar{v}(2)c_2^\dagger \left[-i\frac{g_s}{2}\lambda^\beta\gamma^\nu\right][\epsilon_{4\nu}a_4^\beta]\left[\frac{i(\not{q}+m)}{q^2-m^2}\right] \times \left[-i\frac{g_s}{2}\lambda^\alpha\gamma^\mu\right][\epsilon_{3\mu}a_3^\alpha]u(1)c_1 \ , \tag{8.65}$$

$$\mathcal{M}_2 = \frac{-g_s^2}{8}\frac{1}{(p_1.p_4)}\bar{v}(2)[\not{\epsilon}_3(\not{p}_1 - \not{p}_4 + m)\not{\epsilon}_4]u(1)a_3^\alpha a_4^\beta(c_2^\dagger\lambda^\alpha\lambda^\beta c_1) \ , \tag{8.68}$$

is gauge invariant, but $\mathcal{M}_1 + \mathcal{M}_2$ alone is *not*. [Thus the three gluon vertex is essential in QCD to preserve gauge invariance. Notice, by contrast, that $\mathcal{M}_1 + \mathcal{M}_2$ alone is gauge invariant in QED (Example 7.8). The fact that λ matrices do not commute makes the difference.] 1.5 POINTS

8.3 combining quarks and color

8.23. Determine the branching ratio $\Gamma(\eta_c \to 2g)/\Gamma(\eta_c \to 2\gamma)$. [*Hint*: Use eq. (8.90)

$$\Gamma(\eta_c \to 2g) = \frac{8\pi}{3}\left(\frac{\hbar\alpha_s}{m}\right)^2 |\psi(0)|^2 \ , \tag{8.90}$$

for the numerator, and a suitable modification of eqs. (7.168) and (7.171)

$$\sigma = \frac{4\pi}{v}\left(\frac{\hbar\alpha}{m}\right)^2 \ , \tag{7.168}$$

$$\Gamma = v\sigma|\psi(0)|^2 = 4\pi\left(\frac{\hbar\alpha}{m}\right)^2 |\psi(0)|^2 \ , \tag{7.171}$$

for the numerator. There are two modifications: (i) the quark charge is $\frac{2}{3}e$ and (ii) there is a color factor of 3, for quarks in the singlet state, eq. (8.30),

$$|9\rangle = (r\bar{r} + b\bar{b} + g\bar{g})/\sqrt{3} \ . \tag{8.30}$$

0.2 POINTS

8.4 Running coupling

8.24. (a) Calculate the energy ($\sqrt{|q^2|}$) at which the QED coupling constant, eq. (8.92),

$$\alpha(|q^2|) = \frac{\alpha(0)}{1 - [\alpha(0)/3\pi]\ln[|q^2|/m^2]} \qquad (|q^2| \gg m^2) \ . \tag{8.92}$$

blows up. (Remember, $\alpha(0) = 1/137$, the finestructure constant.) 0.1 POINTS

(b) At what energy do we get a 1% departure from $\alpha(0)$? Is this an accessible energy? 0.1 POINTS

8.25. Prove that the value of μ in eq. (8.93):

$$\alpha_s(|q^2|) = \frac{\alpha_s(\mu^2)}{1 + [\alpha_s(\mu^2)/12\pi](11n - 2f)\ln[|q^2|/\mu^2]} \qquad (|q^2| \gg \mu^2) \tag{8.93}$$

is arbitrary. [That is, suppose physicist A uses the value μ_a and physicist B uses a different value, μ_b, *Assume* A's version of eq. (8.93) is correct, and *prove* that B's version is also correct.] 0.1 POINTS

8.26. Derive eq. (8.95)

$$\alpha_s(|q^2|) = \frac{12\pi}{(11n - 2f)\ln[|q^2|/\Lambda^2]} \qquad (|q^2| \gg \Lambda^2) \tag{8.95}$$

from eqs. (8.93) and (8.94)

$$\ln\Lambda^2 = \ln\mu^2 - 12\pi/[(11n - 2f)\alpha_s(\mu^2)] \ . \tag{8.94}$$

0.1 POINTS

8.27. Calculate α_s at 10 and 100 GeV. Assume $\Lambda = 0.3\,\text{GeV}$. What if $\Lambda = 1\,\text{GeV}$? How about $\Lambda = 0.1\,\text{GeV}$? 0.1 POINTS

8.5 Full problem

8.28. (Gluon-gluon scattering)

(a) Draw the lowest order diagrams (there are four of them) representing the interaction of two gluons. 0.6 POINTS

(b) Write down the corresponding amplitudes. 1.2 POINTS

(c) Put the incoming gluons into the color singlet state; do the same for the outgoing gluons. Compute the resulting amplitudes. 0.9 POINTS

(d) Go to the CM frame, in which each incoming gluon has energy E; express all the kinematic factors in terms of E and the scattering angle θ. Add the amplitudes to get the total $\mathcal{M}$. 1.2 POINTS

(e) Find the differential scattering cross section. 1.2 POINTS

(f) Determine whether the force is attractive or repulsive (if it is the former, this may be a likely glueball configuration). 0.6 POINTS

9 Weak interactions: amplitudes and cross sections; chiral spinors

David Griffiths, Chapter 9, pp. 348-352, 9.1, 9.2, 9.3, 9.4, 9.6, 9.7, 9.8, 9.9, 9.10, 9.11, 9.12, 9.15, 9.18, 9.19, 9.21, 9.22, 9.23, 9.24, 9.25, 9.26, 9.31, 9.32, 9.34,

9.1. Derive the completeness relation for a massive particle of spin 1 (see Problem 9.27 for the massless analog). [*Hint*: Let the $\hat{z}$-axis point along p. First construct three mutually orthogonal polarization vectors ($\epsilon_\mu^{(1)}$, $\epsilon_\mu^{(2)}$, $\epsilon_\mu^{(3)}$) that satisfy $p^\mu \epsilon_\mu^{(i)} = 0$ and $\epsilon_\mu^{(k)} \epsilon^{(k)\mu} = -1$.] 0.1 POINTS

9.2. Calculate the trace $\mathrm{Tr}[\gamma^\mu(c_V - c_A\gamma^5)(\not{p}_1 + m_1)\gamma^\nu(c_V - c_A\gamma^5)(\not{p}_2 + m_2)]$ for arbitrary (real) numbers c_V and c_A. 0.1 POINTS

9.4. Show that eq. (9.30) is equivalent to eq. (9.29).

$$u_- < m_\mu - |\vec{p}_2| - |\vec{p}_4| < u_+ \ , \tag{9.29}$$

$$\left\{ \begin{array}{r} |\vec{p}_2| < \frac{1}{2}m_\mu \\ |\vec{p}_4| < \frac{1}{2}m_\mu \\ |\vec{p}_2| + |\vec{p}_4| > \frac{1}{2}m_\mu \end{array} \right\} \ . \tag{9.30}$$

0.1 POINTS

9.7. What is the average value of the electron energy in muon decay? 0.1 POINTS

9.8. Using the coupling $\gamma^\mu(1 + \epsilon\gamma^5)$ for $n \to p + W$, but $\gamma^\mu(1 - \gamma^5)$ for leptons, calculate the spin averaged amplitude for neutron beta decay. Show that your result reduces to eq. (9.41) when $\epsilon \to -1$.

$$\langle |\mathcal{M}|^2 \rangle = 2\left(\frac{g_w}{M_W}\right)^4 (p_1.p_2)(p_3.p_4) \ . \tag{9.41}$$

0.2 POINTS

9.9. (a) Derive eq. (9.52)

$$p_{\pm} = \frac{\frac{1}{2}(m_n^2 - m_p^2 + m_e^2) - m_n\sqrt{|\vec{p}_4|^2 + m_e^2}}{m_n - \sqrt{|\vec{p}_4|^2 + m_e^2} \mp |\vec{p}_4|} \ . \tag{9.52}$$

0.2 POINTS

(b) Derive eq. (9.58)

$$J \approx 4m_n^4 \eta\phi(\epsilon - \eta)^2 = 4E\sqrt{E^2 - m_e^2}[m_n - m_p - E]^2 \ . \tag{9.58}$$

0.2 POINTS

9.10. In the text, I said that electron energies in neutron decay range up to about $(m_n - m_p)$. This is not exact, since it ignores the *kinetic* energy of the proton and the neutrino. What kinematic configuration gives the maximum electron energy? Apply conservation of energy and momentum to determine the exact maximum electron energy. [Remember the EPP1 course.]
How far off is the approximate answer (give the percent error)? 0.1 POINTS

9.11. (a) Integrate eq. (9.59)

$$\frac{d\Gamma}{dE} = \frac{1}{\pi^3\hbar}\left(\frac{g_w}{2M_W}\right)^4 E\sqrt{E^2 - m_e^2}[m_n - m_p - E]^2 \ , \tag{9.59}$$

to get eq. (9.60)

$$\Gamma = \frac{1}{4\pi^3\hbar}\left(\frac{g_w}{2M_W}\right)^4 m_e^5\left[\frac{1}{15}(2a^4 - 9a^2 - 8)\sqrt{a^2 - 1} + a\ln[a + \sqrt{a^2 - 1}]\right] \ , \tag{9.60}$$

where $a = \frac{\Delta m}{m_e} = \frac{m_n - m_p}{m_e}$. 0.1 POINTS

(b) Approximate as suitable for $m_e \ll \Delta m = (m_n - m_p)$. Note that m_e now drops out. 0.1 POINTS

9.12. Obtain eq. (9.62)

$$\tau = \frac{1}{\Gamma} = 1318 \text{ s} \ . \tag{9.62}$$

0.1 POINTS

9.15. Show that if $m \ll E$

$$\gamma^5 u \approx \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} u \ , \tag{17}$$

where u is a particle spinor satisfying the Dirac equation. Combining eqs. (7.35) and (7.41):

$$u = \begin{pmatrix} u_A \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}u_A \end{pmatrix} \ . \tag{18}$$

Show therefore that the *projection matrix*

$$P_{\pm} = \frac{1}{2}(1 \pm \gamma^5) \tag{19}$$

picks out the helicity ± 1 component of u:

$$\vec{\Sigma} \cdot \hat{p}\,(P_{\pm}u) = \pm(P_{\pm}u) \ . \tag{20}$$

0.1 POINTS

9.18. (a) Show that as long as the CKM matrix is *unitary* ($V^{-1} = V^{\dagger}$), the GIM mechanism for eliminating $K^0 \to \mu^+\mu^-$ works for three (or any number of) generations. [*Note*: $u \to d + W^+$ carries a CKM factor V_{ud}; $d \to u + W^-$ carries a factor V_{ud}^*.] 0.1 POINTS

(b) How many independent real parameters are there in the general 3×3 unitary matrix? How about $n \times n$? [*Hint*: It helps to know that any unitary matrix (U) can be written in the form $U = e^{iH}$, where H is a hermitian matrix. So an equivalent question is, how many independent real parameters are there in the general *hermitian* matrix?]

We are free to change the phase of each quark wave function (normalization of u really only determines $|N|^2$; see Problem 7.3), so $2n$ of these parameters are arbitrary – or rather, $(2n - 1)$, since changing the phase of *all* quark wave functions by the same amount has no effect on V. *Question*: Can we thus reduce the CKM matrix to a *real* matrix (if it is real and unitary, then it is *orthogonal*: $V^{-1} = V^{\top}$). 0.1 POINTS

(c) How many independent real parameters are there in the general 3×3 (real) orthogonal matrix? How about $n \times n$? 0.1 POINTS

(d) So what is the asnwer? Can you reduce the CKM matrix to real form? How about for only two generations ($n = 2$)? 0.1 POINTS

9.19. Show that the CKM matrix, eq. (9.87)

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} , \tag{9.87}$$

where $c_{jk} = \cos\theta_{jk}$ and $s_{jk} = \sin\theta_{jk}$, is unitary for any (real) numbers θ_{12}, θ_{23}, θ_{13}, and δ. 0.1 POINTS

9.21. In Example 9.4 I used *muon* neutrinos, rather than *electron* neutrinos. As a matter of fact, ν_μ and $\bar{\nu}_\mu$ beams are easier to produce than ν_e and $\bar{\nu}_e$, but there is also a *theoretical* reason why $\nu_\mu + e^- \to \nu_\mu + e^-$ is simpler than $\nu_e + e^- \to \nu_e + e^-$ $\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$. Explain. 0.1 POINTS

9.22. (a) Calculate the differential and total cross section for $\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-$ in the GWS model. 0.2 POINTS

(b) Find the ratio $\sigma(\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-)/\sigma(\nu_\mu + e^- \to \nu_\mu + e^-)$. Assume the energy is high enough that you can set $m_e \to 0$. 0.2 POINTS

9.24. Estimate R (the total ratio of quark pair production to muon pair production in e^+e^- scattering), when the process is mediated by Z^0. For the sake of the argument, pretend the top quark is light enough so that eq. (9.109)

$$\sigma = \frac{1}{3\pi}\left(\frac{\hbar g_z^2 E}{4[(2E)^2 - M_Z^2]}\right)^2 [(c_V^f)^2 + (c_A^f)^2][(c_V^e)^2 + (c_A^e)^2] \tag{9.109}$$

can be used. Don't forget color. 0.2 POINTS

9.25. Graph the ratio in eq. (9.113)

$$\frac{\sigma(e^+e^- \to Z^0 \to \mu^+\mu^-)}{\sigma(e^+e^- \to \gamma \to \mu^+\mu^-)} = \frac{[\frac{1}{2} - 2\sin^2\theta_w + 4\sin^4\theta_w]^2}{(\sin\theta_w\cos\theta_w)^4} \times \frac{E^4}{[(2E)^2 - M_Z^2]^2 + (\hbar\Gamma_Z M_Z)^2} \tag{9.109}$$

as function of $x = 2E/M_Z$. Use $\Gamma_Z = 7.3(g_z^2/48\pi)(M_Z/\hbar)$ (Problem 9.23) 0.1 POINTS

9.26. (a) If $u(p)$ satisfies the (momentum space) Dirac equation, eq. (7.49), show that

$$u_L = P_L u = \tfrac{1}{2}(1-\gamma^5)u \quad \text{and} \quad u_R = P_R u = \tfrac{1}{2}(1+\gamma^5)u \tag{Table 9.2}$$

do *not* (unless $m = 0$). 0.1 POINTS

(b) Find the eigenvalues and eigenspinors of the matrices $P_{R,L} = P_\pm = \frac{1}{2}(1 \pm \gamma^5)$. 0.2 POINTS

(c) Can there exist spinors that are simultaneously eigenstates of P_+ (say) *and* of the Dirac operator $(\not{p} - m)$? 0.4 POINTS

9.31. Calculate the lifetime of the top quark. Note that because $m_t > m_b + m_W$, the top can decay into a *real* W ($t \to b + W^+$), whereas all other quarks must go via a *virtual* W. As a consequence, its lifetime is much shorter, and that's why it does not form bound states ('truthful' mesons and baryons). Take the b quark to be massless (compared to t and W). 0.2 POINTS

9.34. Find the threshold ν_μ energy for inverse muon decay (Example 9.1), assuming the target electron is at rest. Why is the answer so huge, when all we're doing is producing a muon? 0.2 POINTS

9.1 Full problems

9.3. (a) Calculate $\langle|\mathcal{M}|^2\rangle$ for $\nu_\mu + e^- \to \mu^- + \nu_e$ using the more general coupling $\gamma^\mu(1 + \epsilon\gamma^5)$. Check that your answer reduces to eq. (9.11) in the case $\epsilon = -1$. 0.2 POINTS

(b) Let $m_e = m_\mu = 0$, and calculate the CM differential cross section. Also find the *total* cross section. 0.2 POINTS

(c) If you had accurate experimental data on this reaction, how could you determine ϵ? 0.2 POINTS

9.6. Suppose the weak interaction were *pure vector*: no γ^5 in eq. (9.5),

$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5) \quad \text{weak vertecx factor} \ . \tag{9.5}$$

Would you still get the same shape for the graph in Figure 9.1 (Experimental spectrum of positron in $\mu^+ \to e^+ + \nu_e + \nu_\mu$)? 1.5 POINTS

9.23. (a) Calculate the decay rate for $Z^0 \to f + \bar{f}$, where f is any quark or lepton. Assume f is so light (compared to the Z^0) that its mass can be neglected. (You'll need the completeness relation for the Z^0 – see Problem 9.1) 1.5 POINTS

(b) Assuming these are the dominant decay modes. find the branching ratio for each species of quark and lepton (remember that the quarks come in three colors). Should you include the top quark among the allowed decays? 1.0 POINTS

(c) Calculate the lifetime, of the Z^0. Quantitatively, how would it change if there were a fourth generation (quarks and leptons)? (Notice that an accurate measurment of the Z^0 lifetime tells us how many quarks and leptons there can be with masses less than 45 GeV.) 1.0 POINTS

9.32. The radical new [your name] theory of weak interactions asserts that the W actually has spin 0 (not 1), and the coupling is 'scalar/pseudo-scalar', instead of 'vector/axial-vector'. Specifically, in your theory the W propagator is

$$\frac{-i}{q^2 - M_W^2} \approx \frac{i}{M_W^2} \tag{21}$$

replacing eq. (9.4) and the vertex factor is

$$\frac{-ig_w}{2\sqrt{2}}(1 - \gamma^5) \tag{22}$$

replacing eq. (9.5). Consider inverse muon decay ($\nu_\mu + e^- \to \mu^- + \nu_e$), in this theory:

(a) Draw the Feynman diagram and construct the amplitude $\mathcal{M}$. 0.6 POINTS

(b) Determine the spin-averaged quantity $\langle|\mathcal{M}|^2\rangle$. 1.5 POINTS

(c) Find the differential scattering cross section in the CM frame in terms of the electron energy E and the scattering angle θ. Assume $E \gg m_\mu \gg m_e$, so you can safely neglect the masses of both the electron and the muon (and of course the neutrinos) 1.5 POINTS

(d) Calculate the total cross section under the same conditions. 1.5 POINTS

(e) By comparing the orthodox predictions for this process, instruct the experimentalist how best to confirm your theory (and demolish the Standard Model). [*Note*: There is no reason to suppose that the weak coupling constant (g_w) in your theory has the same value as it does in the Standard Model, so a test that depends on this number is not going to be very persuasive.] 2.0 POINTS