Supersymmetry (SUSY) — Theory

General features of Supersymmetry

- connects bosons and fermions
- provides an extension to the Poincaré algebra
- unbroken global Supersymmetry
  - bosons and fermions of the same superfield have the same mass
  - the energy of the vacuum is always positive
  - loop corrections involving superfields vanish

  $\Rightarrow$ there is no renormalisation of couplings or masses
- broken global Supersymmetry
  - bosonic and fermionic loop corrections nearly cancel each other

  $\Rightarrow$ solves the hierarchy problem of the Standardmodel
  - better unification of couplings: Grand unified Theories (GUTs)
- local Supersymmetry
  - includes Gravity $\Rightarrow$ Supergravity (SUGRA)
- needed for consistent 10 dimensional Stringtheory $\Rightarrow$ Superstrings

Supersymmetry as an extension of the Poincaré algebra

- the Poincaré algebra describes space-time transformations of particles
- due to statistics bosons and fermions transform separately
- so all normal symmetries are bosonic
- according to the Coleman-Mandula theorem
  - all symmetry generators have to commute with the generators of the Poincaré algebra
  - no other symmetries are allowed for a meaningful QFT
- Supersymmetry evades this restriction by using anticommutators
  - Supersymmetry generators are fermionic
  - they extend the Poincaré algebra to the Super-Poincaré algebra

The Super-Poincaré algebra

- the Poincaré algebra with generators $M^{\alpha\beta}$ and P^μ

$$[M^{\kappa\lambda}, M^{\rho\sigma}] = i(-g^{\kappa\rho}M^{\lambda\sigma} + g^{\lambda\rho}M^{\kappa\sigma} + g^{\kappa\sigma}M^{\lambda\rho} - g^{\lambda\sigma}M^{\kappa\rho}) \;, \qquad (1)$$

$$[P^\mu, P^\nu] = 0 \qquad \text{and} \qquad [M^{\kappa\lambda}, P^\mu] = i(-g^{\kappa\mu}P^\lambda + g^{\lambda\mu}P^\kappa) \qquad (2)$$

- is extended by the Supersymmetry generators Q_α and $\bar{Q}_{\dot{\alpha}}$ to

$$[M^{\mu\nu}, Q_\alpha] = \tfrac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \;, \qquad [M^{\mu\nu}, \bar{Q}_{\dot{\alpha}}] = -\tfrac{1}{2}(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\bar{Q}_{\dot{\beta}} \;, \qquad (3)$$

$\Rightarrow$ Q_α and $\bar{Q}_{\dot{\alpha}}$ transform as spinors under Lorentz transformations

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu \;, \qquad \text{and} \qquad [Q_\alpha, P^\mu] = [\bar{Q}_{\dot{\alpha}}, P^\mu] = 0 \qquad (4)$$

$\Rightarrow$ the algebra closes

- mass dimensions of the generators:
  - P^μ has 1
  - $M^{\kappa\lambda}$ has 0
  - Q, $\bar{Q}$ have half the dimension of P^μ, so $[Q] = [\bar{Q}] = \frac{1}{2}$

Superspace

- the normal coordinates x^μ can be extended to $(x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$
  - θ^α, $\bar{\theta}_{\dot{\alpha}}$ with $\alpha, \dot{\alpha} = 1, 2$ are Grassman valued parameters
- The SUSY generators can be represented by differential operators:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \qquad \text{and} \qquad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{5}$$

  - similar to $P_\mu = -i\frac{\partial}{\partial x^\mu}$
- it is convenient to introduce differential operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \qquad \text{and} \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{6}$$

  - they anticommute with Q_α and $\bar{Q}_{\dot{\alpha}}$ and have a similar algebra:

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \tag{7}$$

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \qquad \text{and} \qquad \{D_\alpha, \bar{D}_{\dot{\beta}}\} = 2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \tag{8}$$

Superfields in Superspace

- a Superfield S is a function of $(x^\mu, \theta^\alpha, \bar\theta_{\dot\alpha})$
- it can be expanded in component fields
  - the expansion terminates since $(\theta^\alpha)^2 = (\bar\theta_{\dot\alpha})^2 = 0$
  - the highest term is the coefficient of $\theta\theta = \theta^\alpha\theta_\alpha$ or $\bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$

$$S = s + \theta\chi + \bar\theta\bar\chi + \theta\theta\, m + \bar\theta\bar\theta\, n + \theta\sigma^\mu\bar\theta v_\mu + \theta\theta\,\bar\theta\bar\lambda + \bar\theta\bar\theta\,\theta\lambda + \theta\theta\,\bar\theta\bar\theta\, d \qquad (9)$$

- The SUSY transformation with spinorial parameter η is $\delta_\eta = \eta Q + \bar\eta\bar Q$

$$\begin{aligned} \delta_\eta S &= \delta_\eta s + \theta\delta_\eta\chi + \bar\theta\delta_\eta\bar\chi + \theta\theta\,\delta_\eta m + \bar\theta\bar\theta\,\delta_\eta n + \theta\sigma^\mu\bar\theta\delta_\eta v_\mu && (10) \\ &\quad + \theta\theta\,\bar\theta\delta_\eta\bar\lambda + \bar\theta\bar\theta\,\theta\delta_\eta\lambda + \theta\theta\,\bar\theta\bar\theta\,\delta_\eta d \\ &= (\eta Q + \bar\eta\bar Q)S = (\eta\tfrac{\partial}{\partial\theta} + i(\eta\sigma^\mu\bar\theta)\partial_\mu + \bar\eta\tfrac{\partial}{\partial\bar\theta} + i(\theta\sigma^\mu\bar\eta)\partial_\mu)S && (11) \end{aligned}$$

  gives the transformation property of the component fields.

- since Qs and Ds anticommute
  - constraints written with Ds are unaffected!

Supersymmetry (SUSY) — Theory

Superfields containing bosons and fermions

- Superfields describe multiplets of component fields
- but they have too many (unnecessary) degrees of freedom (dof)

$\Rightarrow$ constraints:

- chiral Superfields are defined as $\bar{D}_{\dot{\alpha}}\Phi = D_\alpha \Phi^\dagger = 0$ (12)
  - solved by $\Phi = \Phi(y = x - i\theta\sigma^\mu\bar{\theta}, \theta, 0)$, so $\Phi = \phi + 2\theta\psi - \theta\theta\, F$ (13)
  - $\delta_\eta\phi = 2\eta^\alpha\psi_\alpha\,,\;\; \delta_\eta\psi_\alpha = -\eta_\alpha F - i(\sigma^\mu\bar{\eta})_\alpha\partial_\mu\phi\,,\;\; \delta_\eta F = -2i(\partial_\mu\psi)\sigma^\mu\bar{\eta}$ (14)
  - F has mass dimension 2 $\Rightarrow$ auxiliar field
  - complex scalar ϕ has 2 dof, Weyl spinor ψ has 2 dof
- Vector Superfields are defined as $V = V^\dagger$ (15)
  - in Wess Zumino gauge: $V = \theta\sigma^\mu\bar{\theta}v_\mu + \theta\theta\,\bar{\theta}\bar{\lambda} + \bar{\theta}\bar{\theta}\,\theta\lambda + \theta\theta\,\bar{\theta}\bar{\theta}\,D$ (16)
    - $*$ then $V^2 = -\frac{1}{2}\theta\theta\,\bar{\theta}\bar{\theta}v^\mu v_\mu$ and $V^3 = 0$ (17)

Supersymmetry (SUSY) — Theory

The minimal supersymmetric Standardmodel (MSSM)

- gives a superpartner to each particle in the Standarmodel:
  - fermions get the scalar sfermions
  - vector bosons (gauge bosons) get the fermionic gauginos
  - the 2 doublets of Higgs bosons get the fermionic higgsinos

the Lagrangian consists of

- supersymmetric parts:
  - vector superfields in the SM gauge group $SU(3)_{\text{color}} \times SU(2)_L \times U(1)_Y$
  - chiral superfields give the matter fields and the higgses
  - the interaction between chiral and vector superfields
- and soft breaking terms:
  - they are not supersymmetric invariant
  - the couplings have a mass dimension ≥ 1
  - they also break spontaneously $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{em}}$

$\Rightarrow$ no SUSY breaking $\Rightarrow$ no masses in the MSSM

Supersymmetry (SUSY) — Theory

MSSM and Renormalisation

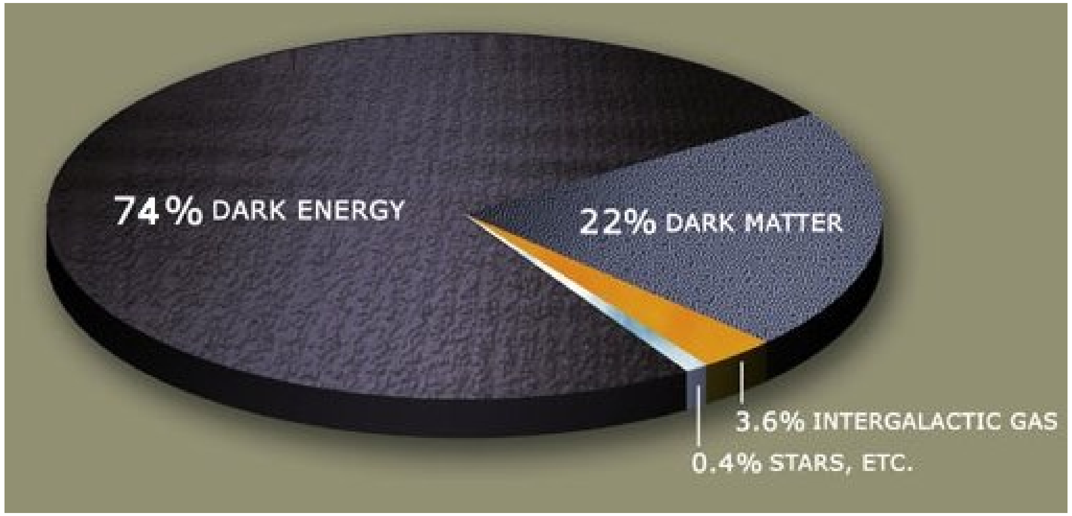
- the soft breaking terms (masses and trilinear couplings) break SUSY
  - they generate a mass splitting between superpartners:

$$\tilde{m}^2 = m^2_{\text{fermion}} - m^2_{\text{boson}} \sim \mathcal{O}(1)\ \text{TeV} \tag{18}$$

- the natural cut-off scale for the SM is the Plank scale: $m_P \sim 10^{19}$GeV
  - the loop corrections to the Higgs mass should be $\Delta m_H^2 \sim \mathcal{O}(\alpha m_P^2)$
  - in the MSSM the loop corrections are $\Delta m_H^2 \sim \mathcal{O}(\alpha \tilde{m}^2)$
  - $\Rightarrow$ stabilizes the Higgspotential
    - $*$ the Higgs self coupling is also a gauge coupling
- the running of the couplings are changed
  - the three gauge couplings meet at a single scale $m_{\text{GUT}} \sim 10^{18}$GeV
    - $*$ a grand unified theory (GUT) describes the three forces as different representations of one single force
  - $\Rightarrow$ predictions for proton decay are consistent with experiment
    - $*$ non-SUSY GUTs give a too fast proton decay

Supersymmetry (SUSY) — Theory

MSSM and Cosmology



- Astronomers measure the content of the universe:
  - 74% Dark Energy
  - 22% Dark Matter
  - 3.6% Intergalactic gas
  - 0.4% Stars, etc ...
- the MSSM has a discrete symmetry: R-parity
  - supersymmetric particles can only be produced in pairs
  - a SUSY particle can only decay into a SUSY particle
  - ⇒ the lightest supersymmetric particle (LSP) is stable
    - ∗ this is usually the neutralino $\tilde{\chi}_1^0$ with $m_{\tilde{\chi}_1^0} > 50\text{GeV}$

⇒ the MSSM provides a Dark Matter candidate
  - if SUSY particles are found by LHC
    - ⇒ some properties of dark matter can be investigated

Supersymmetry (SUSY) — Theory

supersymmetric flat space

- as in GR it will be helpful to use forms
  - they are easily generalized to superspace
  - but they no longer vanish for $p > n$, as there are spinors, too
- as in GR we can choose any basis, not only $dz^M = (dz^m, d\theta^\mu, d\bar{\theta}_{\dot{\mu}})$
  - m denotes the space-time index, μ and $\dot{\mu}$ the spinor indices
  - a better basis is given by the supersymmetric covariant derivatives

$$D_a = \frac{\partial}{\partial x^a} \quad D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu} \quad \bar{D}^{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + i\theta_\alpha \sigma^\mu_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{19}$$

  - i.e. $e^A = (e^a, e^\alpha, e_{\dot{\alpha}})$ and $d = dz^M \partial_M = e^A D_A$ (20)
    - this relation also defines the vielbein: $e^A = dz^M {e_M}^A$ (21)
    - $d(dz^M) = 0$ but $de^A = dz^M dz^N \partial_N {e_M}^A = {e^M}_B e^B e^C D_C e^A \neq 0$ (22)
  - this defines supersymmetric flat space
- but even in flat space, the supersymmetric torsion does not vanish:

$$T_{\alpha\dot{\beta}}{}^c = T_{\dot{\beta}\alpha}{}^c = 2i\sigma^c_{\alpha\dot{\beta}} \tag{23}$$

Supersymmetry (SUSY) — Theory

local Supersymmetry

- SUSY always includes fermions, the basic quantities are
  - the local vielbein ${E_M}^A$
  - and the spin connection ${\omega_M}^A{}_B$
    - A and B are called Lorentz indices
    - M and N are Einstein indices (general coordinate transformations)
- the Lorentz group can be seen as the local symmetry group
  - with local Lorentz transformations (LLTs) ${\Lambda_B}^A$
  - the space time and spinorial parts are linked:

$$\sigma^a_{\alpha\dot\alpha}\sigma^b_{\beta\dot\beta}\Lambda_{ab} = -2\epsilon_{\alpha\beta}\Lambda_{\dot\alpha\dot\beta} + 2\epsilon_{\dot\alpha\dot\beta}\Lambda_{\alpha\beta} \tag{24}$$

- a general coordinate transformation in superspace $z'^M = z^M - \xi^M$

$$\delta_\xi V^A = -\xi^B {E_B}^M \partial_M V^A + V^B {\Lambda_B}^A = -\xi^B \nabla_B V^A + V^B \xi^C {\omega_{CB}}^A + V^B {\Lambda_B}^A \tag{25}$$

  - taking ${\Lambda_B}^A = -\xi^C {\omega_{CB}}^A$ gives supergauge transformations
    - or gauged supersymmetry transformations

Supersymmetry (SUSY) — Theory

Supergravity

- all of the introduced quantites are superfields
  - they can be expanded in the superspace coordinates
  - the metric is replaced by the vielbein as the dynamic quantity
  - constraints on the torsion: $T_{\alpha\dot{\beta}}{}^{c} = T_{\dot{\beta}\alpha}{}^{c} = 2i\sigma^{c}_{\alpha\dot{\beta}}$ (26)

    $\Rightarrow$ express the connection in terms of the vielbein
  - the vielbein contains graviton and gravitino

- gives a scenario for SUSY breaking
  - with mechanisms to motivate the soft breaking terms

- but this SUSY breaking happens at high energies
  - relevant for Cosmology
    - $\ast$ but only for the first nanoseconds

but it does not make GR a renormalisable QFT