## Translation and Rotation Operators

- The momentum operator $\vec{P} = -i\frac{\partial}{\partial\vec{x}} = -i\vec{\partial}$ generates translations:
  - in index notation: $P_k = -i\frac{\partial}{\partial x^k} = -i\partial_k$

$$e^{ia^kP_k}f(x) = e^{a^k\partial_k}f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}(a^k\partial_k)^n f(x) \tag{1}$$
$$= f(x) + a^k\partial_k f(x) + \frac{1}{2}a^j a^k\partial_j\partial_k f(x) + \ldots$$

  - the Taylorseries of $f(x+a)$ is

$$f(x+a) = f(x) + a^k\partial_k f(x) + \frac{1}{2}a^j a^k\partial_j\partial_k f(x) + \cdots = e^{i\vec{a}\vec{P}}f(x) \tag{2}$$

$\Rightarrow$ the operator $e^{i\vec{a}\vec{P}}$ moves the function $f$ by the amount $\vec{a}$

- The angular momentum operator $\vec{L} = \vec{X} \times \vec{P}$ generates rotations
  - in index notation: $L_j = \epsilon_{jk\ell}\, x^k P_\ell = -i\epsilon_{jk\ell}\, x^k\partial_\ell$ (3)
  - or $L_x = i(z\partial_y - y\partial_z)\,,\;\; L_y = i(x\partial_z - z\partial_x)\,,\;\; L_z = i(y\partial_x - x\partial_y)$ (4)

## Translation and Rotation Operators

- The components of $\vec{L}$ do not commute:
  - if you rotate around the $\hat{x}$-axis and then around the $\hat{y}$-axis, you get a different result than rotating first around $\hat{y}$ and then $\hat{x}$.
  - mathematically:

$$\begin{aligned}
[L_y, L_x] &= i^2[(x\partial_z - z\partial_x)(z\partial_y - y\partial_z) - (z\partial_y - y\partial_z)(x\partial_z - z\partial_x)] \\
&= i^2[(x\partial_y + xz\partial_z\partial_y - xy\partial_z^2 - z^2\partial_x\partial_y + zy\partial_x\partial_z) \\
&\quad -(zx\partial_y\partial_z - z^2\partial_y\partial_x - yx\partial_z^2 + y\partial_x + yz\partial_z\partial_x)] \\
&= i^2[x\partial_y - y\partial_x] = -iL_z
\end{aligned} \tag{5}$$

  - or in index notation: $[L_j, L_k] = i\epsilon_{jk\ell}L_\ell \quad \Rightarrow \quad$ Rotationgroup (6)
- but the square $L^2 = \vec{L}\cdot\vec{L} = L_k L_k$ does commute:

$$\begin{aligned}
[L^2, L_j] &= L_k[L_k, L_j] + [L_k, L_j]L_k = L_k i\epsilon_{kj\ell}L_\ell + i\epsilon_{kj\ell}L_\ell L_k \\
&= L_h i\epsilon_{hjm}L_m + i\epsilon_{mjh}L_h L_m = i(\epsilon_{hjm} + \epsilon_{mjh})L_h L_m = 0
\end{aligned} \tag{7}$$

$\Rightarrow$ use $L^2$ and $L_z$ to describe quantum mechanical states (particles)

## Eigenstates of the Rotationgroup

- We write an eigenstate of the operators $L^2$ and $L_z$ as $|\lambda, m\rangle$

$$L^2|\lambda, m\rangle = \lambda|\lambda, m\rangle \qquad \text{and} \qquad L_z|\lambda, m\rangle = m|\lambda, m\rangle \tag{8}$$

  - $|f\rangle$ is called a ket and used to denote a quantum mechanical state.

- We define the ladder operators $L_\pm = L_x \mp iL_y$ with

$$[L^2, L_\pm] = [L^2, L_x] \mp i[L^2, L_y] = 0 \qquad \text{and} \tag{9}$$

$$[L_z, L_\pm] = [L_z, L_x] \mp i[L_z, L_y] = iL_y \mp i(-iL_x) = \pm(L_x \mp iL_y) = \pm L_\pm \tag{10}$$

$\Rightarrow$ $L_\pm|\lambda, m\rangle$ is also an eigenstate of $L^2$ and $L_z$ :

$$\begin{aligned} L^2(L_\pm|\lambda, m\rangle) &= ([L^2, L_\pm] + L_\pm L^2)|\lambda, m\rangle = 0 + L_\pm L^2|\lambda, m\rangle \\ &= L_\pm\lambda|\lambda, m\rangle = \lambda(L_\pm|\lambda, m\rangle) \end{aligned} \tag{11}$$

and

$$\begin{aligned} L_z(L_\pm|\lambda, m\rangle) &= ([L_z, L_\pm] + L_\pm L_z)|\lambda, m\rangle = (\pm L_\pm + L_\pm L_z)|\lambda, m\rangle \\ &= (\pm L_\pm + L_\pm m)|\lambda, m\rangle = (m \pm 1)(L_\pm|\lambda, m\rangle) \end{aligned} \tag{12}$$

## Eigenstates of the Rotationgroup

- $L_\pm$ does not change the eigenvalue $\lambda$ of the state $|\lambda, m\rangle$
- $L_\pm$ changes the eigenvalue $m$ of the state $|\lambda, m\rangle$

$\Rightarrow$ the states $|\lambda, m+n\rangle$ with $n \in Z$ are related

$\Rightarrow$ for each $\lambda$ there would be $\infty$ many states unless there is

* $a = m_{\text{min}}$ with $L_-|\lambda, a\rangle = 0$ and
* $b = m_{\text{max}}$ with $L_+|\lambda, b\rangle = 0$

- using

$$\begin{aligned} L_\pm L_\mp &= (L_x \mp iL_y)(L_x \pm iL_y) = L_x^2 \pm iL_xL_y \mp iL_yL_x + L_y^2 \\ &= (L_x^2 + L_y^2 + L_z^2) - L_z^2 \pm i[L_x, L_y] = L^2 - L_z^2 \pm i(iL_z) \\ &= L^2 - L_z(L_z \pm 1) \end{aligned} \tag{13}$$

we can relate $a$ and $b$.

## Eigenstates of the Rotationgroup

- relating $a$ and $b$:
  - $$0 = L_+L_-|\lambda, a\rangle = (\lambda - (a^2 + a))|\lambda, a\rangle \quad \Rightarrow \quad \lambda = a^2 + a \tag{14}$$
  - $$0 = L_-L_+|\lambda, b\rangle = (\lambda - (b^2 - b))|\lambda, b\rangle \quad \Rightarrow \quad \lambda = b^2 - b \tag{15}$$

$$a(a+1) = b(b-1) \quad \text{or} \quad a = -b \tag{16}$$

- Applying $(L_+)$ $n$ times on the state $|\lambda, a\rangle$ gives $|\lambda, a+n\rangle$
- for some $n$ we have to reach $|\lambda, b\rangle \quad \Rightarrow \quad a + n = b$
- with $a = -b$ we get $-b + n = b$ or $m_{\text{max}} = b = \frac{n}{2}$
- The rotationgroup allows for half integer eigenstates

$$\Rightarrow \quad \text{Spinors}$$

... we will encounter spinors again ...

## 3. Special Relativity (SR) — Algebra of the Poincaré group

### Groups — what is a group?

- a set $G$ together with a "multiplication $\circ$" with the properties:
  - for $a, b \in G \Rightarrow c = a \circ b \in G$
  - $(a \circ b) \circ c = a \circ (b \circ c)$
  - $\forall a \in G : \exists e \in G$ with $a \circ e = e \circ a = a$
  - $\forall a \in G : \exists a^{-1} \in G$ with $a \circ a^{-1} = a^{-1} \circ a = e$
- if $a \circ b = b \circ a \ \forall a, b \in G$ : abelian group, otherwise non-abelian
  - abelian: $\{\mathcal{R}, +\}$ or $\{\mathcal{R}^+, \times\}$
  - non-abelian: regular square matrices with the matrix multiplication
- continuous groups: the elements depend on a continuous parameter
  - example: rotations around an axis $R[\theta]$ with $\theta \in [0, 2\pi)$
- Lie group: a continuous group with an analytic multiplication
  - $g[\vec{x}] \circ g[\vec{y}] = g[f(\vec{x}, \vec{y})]$ with $f(\vec{x}, \vec{y})$ analytic in $\vec{x}$ and $\vec{y}$
  - the unit element is $e = g[\vec{0}]$

## Lie groups and Lie algebras

- The $n \times n$ (complex) matrices form representations of Lie groups
- group multiplication is analytic $\Rightarrow$ expansion around unit element
  - unit element $e = \mathbf{1}_{n\times n}$
  - representation $T(g[\alpha]) = \exp[i\alpha_i X_i] \quad \Rightarrow \quad X_k = -i\frac{\partial T(g[\alpha])}{\partial \alpha_k}|_{\vec{\alpha}=0}$ (17)
  - generators $\{X_k\}$ span the representation of the Lie group
- the generators $\{X_k\}$ fulfill the Lie algebra $\quad [X_j, X_k] = C_{jk}{}^{\ell} X_\ell$ (18)
  - with the antisymmetric structure constants $C_{jk}{}^{\ell} = -C_{kj}{}^{\ell}$
  - rank of the group: number of commuting generators
  - a Casimir operator commutes with all generators $\quad \Rightarrow \quad \propto e$
- the indices $i, j, k, \ell$ need not indicate single numbers!
  - for the generators we will have $X_i = X_{[mn]} = -X_{[nm]}$

## Representations of the Lie group

- using the Jacobi identity

$$
\begin{aligned}
0 &= [A,[B,C]] + [B,[C,A]] + [C,[A,B]] &(19)\\
&= ABC - ACB - BCA + CBA \;+\; BCA - BAC - CAB + ACB \;+\; CAB - CBA - ABC + BAC
\end{aligned}
$$

we get for the structure constants

$$
\begin{aligned}
0 &= C_{bc}{}^{d}[A,D] + C_{ca}{}^{d}[B,D] + C_{ab}{}^{d}[C,D] &(20)\\
&= C_{bc}{}^{d}C_{ad}{}^{e} + C_{ca}{}^{d}C_{bd}{}^{e} + C_{ab}{}^{d}C_{cd}{}^{e} = -(C_{ca}{}^{d}C_{db}{}^{e} - C_{cb}{}^{d}C_{da}{}^{e}) + C_{ab}{}^{d}C_{cd}{}^{e}
\end{aligned}
$$

- writing the structure constants as matrices $(M_k)_j{}^{\ell} = C_{jk}{}^{\ell}$ we have

$$
0 = -[(M_a)_c{}^{d}(M_b)_d{}^{e} - (M_b)_c{}^{d}(M_a)_d{}^{e}] + C_{ab}{}^{d}(M_d)_c{}^{e} \tag{21}
$$

or

$$
[M_a, M_b] = C_{ab}{}^{d} M_d \tag{22}
$$

$\Rightarrow$ the structure constants form the adjoint representation of the Lie group

## Lie Algebra of the rotation group

- a rotation around the $\hat{z}$-axis by the angle $\theta$ is done by the matrix

$$R[\theta] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{i\theta L_z} \tag{23}$$

  – in index notation: $R[\theta]^j{}_k = \cos\theta(\delta^j_1\delta^1_k + \delta^j_2\delta^2_k) - \sin\theta(\delta^j_1\delta^2_k - \delta^j_2\delta^1_k) + \delta^j_3\delta^3_k$ (24)

- so the generator of the rotations, $iL_z$, is

$$iL_z = \left.\frac{\partial R[\theta]}{\partial\theta}\right|_{\theta=0} = \left.\begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\right|_{\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{25}$$

  – and similar

$$iL_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad iL_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{26}$$

- these rotations (incl. $L_x$ and $L_y$) act on 3d column vectors $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

Lie Algebra of the rotation group

- with simple matrix multiplication we can see:

$$[iL_x, iL_y] = -iL_z \qquad [iL_y, iL_z] = -iL_x \qquad [iL_z, iL_x] = -iL_y \tag{27}$$

  - or in index notation with $x = 1$, $y = 2$, and $z = 3$: $[L_j, L_k] = i\epsilon_{jk\ell}L_\ell$ (28)

- but there is a smaller dimensional realisation of the rotation group!
  - using the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{29}$$

  - one can define the Spin matrices $S_k = \frac{1}{2}\sigma_k$ , which give

$$[S_j, S_k] = i\epsilon_{jk\ell}S_\ell \tag{30}$$

- these Spin matrices act on 2d complex column vectors $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$ $\Rightarrow$ Spinors

$\Rightarrow$ fundamental representation of the rotation group $SU(2)$

## Rotations of Spinors

- with simple matrix multiplication we can see for the Pauli matrices:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{2\times 2} \tag{31}$$

- So the finite rotation of a spinor around the $\hat{y}$-axis is

$$\begin{aligned} R[\theta] &= e^{i\theta S_y} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta \tfrac{1}{2}\sigma_y)^n = \sum_{n=2m} \frac{1}{n!}(i\tfrac{\theta}{2})^n \sigma_y^n + \sum_{n=2m+1} \frac{1}{n!}(i\tfrac{\theta}{2})^n \sigma_y^n \\ &= \sum_{n}^{\infty} \frac{(-1)^n (\frac{\theta}{2})^{2n}}{(2n)!}(\sigma_y^2)^n + i\sum_{n}^{\infty} \frac{(-1)^n (\frac{\theta}{2})^{2n+1}}{(2n+1)!}(\sigma_y^2)^n \sigma_y \\ &= \cos\tfrac{\theta}{2} * 1_{2\times 2} + i\sin\tfrac{\theta}{2}\sigma_y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \end{aligned} \tag{32}$$

  - acting on the spinor $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

⇒ spinors rotate only with half of the rotation angle $\theta$

## 3. Special Relativity (SR) — Algebra of the Poincaré group

Lorentz transformations ( like Galilean transformations )
consist of Boosts and Rotations

- a boost in $\hat{x}$ was done by

$$\Lambda(\eta)^{\mu}{}_{\nu} = \cosh\eta(\delta^{\mu}_{0}\delta^{0}_{\nu} + \delta^{\mu}_{1}\delta^{1}_{\nu}) - \sinh\eta(\delta^{\mu}_{0}\delta^{1}_{\nu} + \delta^{\mu}_{1}\delta^{0}_{\nu}) + \delta^{\mu}_{2}\delta^{2}_{\nu} + \delta^{\mu}_{3}\delta^{3}_{\nu} \quad (33)$$

- a rotation between $\hat{y}$ and $\hat{z}$ can be done by

$$\Lambda(\theta)^{\mu}{}_{\nu} = \delta^{\mu}_{0}\delta^{0}_{\nu} + \delta^{\mu}_{1}\delta^{1}_{\nu} + \cos\theta(\delta^{\mu}_{2}\delta^{2}_{\nu} + \delta^{\mu}_{3}\delta^{3}_{\nu}) - \sin\theta(\delta^{\mu}_{2}\delta^{3}_{\nu} - \delta^{\mu}_{3}\delta^{2}_{\nu}) \quad (34)$$

- we obtain the generators for boosts with $-i\frac{\partial\Lambda(\eta)^{\mu}{}_{\nu}}{\partial\eta}|_{\eta=0} =$

$$-i\sinh\eta(\delta^{\mu}_{0}\delta^{0}_{\nu} + \delta^{\mu}_{1}\delta^{1}_{\nu}) + i\cosh\eta(\delta^{\mu}_{0}\delta^{1}_{\nu} + \delta^{\mu}_{1}\delta^{0}_{\nu})|_{\eta=0} = i(\delta^{\mu}_{0}\delta^{1}_{\nu} + \delta^{\mu}_{1}\delta^{0}_{\nu}) \quad (35)$$

- we obtain the generators for rotations with $-i\frac{\partial\Lambda(\theta)^{\mu}{}_{\nu}}{\partial\theta}|_{\theta=0} =$

$$+i\sin\theta(\delta^{\mu}_{2}\delta^{2}_{\nu} + \delta^{\mu}_{3}\delta^{3}_{\nu}) + i\cos\theta(\delta^{\mu}_{2}\delta^{3}_{\nu} - \delta^{\mu}_{3}\delta^{2}_{\nu})|_{\theta=0} = i(\delta^{\mu}_{2}\delta^{3}_{\nu} - \delta^{\mu}_{3}\delta^{2}_{\nu}) \quad (36)$$

- The other boosts go in $\hat{y}$ or $\hat{z}$ direction: $i(\delta^\mu_0 \delta^i_\nu + \delta^\mu_i \delta^0_\nu)$,
  or with the indices $0i$ down: $(M_{0i})^\mu{}_\nu = i(\delta^\mu_0(-g_{i\nu}) + \delta^\mu_i g_{0\nu})$ . (37)
- The other rotations go in $\hat{x}\hat{y}$ or $\hat{x}\hat{z}$ direction: $i(\delta^\mu_j \delta^k_\nu - \delta^\mu_k \delta^j_\nu)$,
  or with the indices $jk$ up: $(M_{jk})^\mu{}_\nu = i(\delta^\mu_j(-g_{k\nu}) - \delta^\mu_k(-g_{j\nu}))$ . (38)
- both generators have now the same form: $(M_{\alpha\beta})^\mu{}_\nu = -i(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu})$
- with $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ we get

$$\Lambda(\omega)^\mu{}_\nu = \exp[i(M_{\alpha\beta}\omega^{\alpha\beta})^\mu{}_\nu] = \exp[(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu})\omega^{\alpha\beta}] \tag{39}$$

- these generators fulfill the Lie algebra of the Lorentz group:

$$[M_{\alpha\beta}, M_{\gamma\delta}]^\mu{}_\nu = i(g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\delta}M_{\beta\gamma} + g_{\beta\delta}M_{\alpha\gamma})^\mu{}_\nu \tag{40}$$

- unifying time and spatial translations $P_\mu = (H, P_i)$ . (41)
- we get the rest of the Poincaré algebra:

$$[P_\mu, P_\nu] = 0 \qquad \text{and} \qquad [M_{\alpha\beta}, P_\mu] = i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha) \tag{42}$$

## Invariants of the Poincaré group

- obviously $[ab,c] = a[b,c] + [a,c]b = abc - acb + acb - cab = abc - cab$

$$[P_\mu, P^2] = [P_\mu, P_\nu]P^\nu + P^\nu[P_\mu, P_\nu] = 0 \tag{43}$$

and

$$\begin{aligned}[M_{\alpha\beta}, P^2] &= g^{\mu\nu}[M_{\alpha\beta}, P_\mu]P_\nu + g^{\mu\nu}P_\mu[M_{\alpha\beta}, P_\nu] \\ &= g^{\mu\nu}i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)P_\nu + g^{\mu\nu}P_\mu i(g_{\alpha\nu}P_\beta - g_{\beta\nu}P_\alpha) \\ &= -2i[P_\alpha, P_\beta] = 0 \ .\end{aligned} \tag{44}$$

$\Rightarrow$ $P^2 = m^2$ invariant is a consequence of the Poincaré algebra!

- Another invariant is $W^2$
  - with the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}M_{\nu\rho}P_\lambda$ (45)
- since $P_\mu$ and $W^\mu$ commute: $[P_\kappa, W^\mu] = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}[P_\kappa, M_{\nu\rho}P_\lambda]$

$$= \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}([P_\kappa, M_{\nu\rho}]P_\lambda + M_{\nu\rho}[P_\kappa, P_\lambda]) = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda} i(g_{\rho\kappa}P_\nu - g_{\nu\kappa}P_\rho)P_\lambda = 0 \tag{46}$$

$\Rightarrow$ Particles can be characterised by the eigenvalues of $P^2$ and $W^2$

## Invariants of the Poincaré group

- the spin vector $W^\mu$ is orthogonal to $P_\mu$ :
$$(P.W) = P^\mu \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda} M^{\nu\rho} P^\lambda = 0 \tag{47}$$
- a particle at rest: $P_\mu = (m, 0)$ and $W_\mu = \frac{1}{2} m\, \epsilon_{\mu\nu\rho 0} M^{\nu\rho} = m(0, \vec{J}\,)$ (48)
  - then $$W^2 = -m^2 \vec{J}^2 = -m^2 s(s+1) \tag{49}$$
    - ⇒ the eigenvalues of $P^2$ is $m^2$ and of $W^2$ is $m^2 s(s+1)$
- a massless particle has $P_\mu = (\eta, \eta, 0, 0)$ and $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda} M_{\nu\rho} P_\lambda$ (50)
  - and we get $$P^2 = (P.W) = W^2 = 0 \tag{51}$$
    - ⇒ so the eigenvalues of $P^2$ and $W^2$ are $0$
  - noticing $$0 = \lambda^2 P^2 - 2\lambda(P.W) + W^2 = (\lambda P - W)^2 \tag{52}$$
  - we can require $W^\mu = \lambda P^\mu$ with the helicity $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$
    - ∗ $\lambda$ depends on the representation (i.e. the spin) of the particle

⇒ Particles are characterised by mass and spin !

## Investigating the Lorentz group

- distinguishing again boosts and rotations

$$K_i = M_{0i} = -M^{0i} \qquad \text{and} \qquad J_i = \tfrac{1}{2}\epsilon_{ijk}M^{jk} \quad , \tag{53}$$

  the Lorentz algebra gives

$$[J_j, J_k] = i\epsilon_{jk\ell}J_\ell \quad , \quad [K_j, K_k] = -i\epsilon_{jk\ell}J_\ell \quad , \quad [J_j, K_k] = i\epsilon_{jk\ell}K_\ell \tag{54}$$

- defining

$$L_i = N_i = \tfrac{1}{2}(J_i + iK_i) \qquad \text{and} \qquad R_i = N_i^\dagger = \tfrac{1}{2}(J_i - iK_i) \tag{55}$$

  one gets

$$[L_j, R_k] = 0 \quad , \qquad [L_j, L_k] = i\epsilon_{jk\ell}L_\ell \quad , \qquad [R_j, R_k] = i\epsilon_{jk\ell}R_\ell \tag{56}$$

- the Lorentz algebra is similar to $SU(2)_L \otimes SU(2)_R$ !
- two invariants: $L_iL_i = n(n+1)$ and $R_iR_i = m(m+1)$
  - since $J_i = L_i + R_i \quad \Rightarrow \quad$ spin $j = n + m$ (57)

## classifying particles

- according to the eigenstates $(n, m)$ of $SU(2)_L \otimes SU(2)_R$
  - $(0,0)$ is a scalar
  - $(\frac{1}{2},0)$ is the $\chi_\alpha$ left-handed Weyl-spinor
  - $(0,\frac{1}{2})$ is the $\bar{\eta}^{\dot{\alpha}}$ right-handed Weyl-spinor
  - $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ is $\Psi = \binom{\chi_\alpha}{\bar{\eta}^{\dot{\alpha}}}$, the Dirac-spinor
  - $(\frac{1}{2},0) \otimes (0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$ is $(\chi\sigma^\mu\bar{\eta}) = \chi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}$, the spin-1 four-vector

$\Rightarrow$ in that sense is the spinor the square root of the vector

- under Parity: $J_i \xrightarrow{\mathsf{P}} J_i,\ K_i \xrightarrow{\mathsf{P}} -K_i, \quad \Rightarrow \quad L_i \xleftrightarrow{\mathsf{P}} R_i,\ (n,m) \xleftrightarrow{\mathsf{P}} (m,n)$
  - the scalar stays the same
  - $(\frac{1}{2},0) \xleftrightarrow{\mathsf{P}} (0,\frac{1}{2})$, therefore $\chi_\alpha \xleftrightarrow{\mathsf{P}} \bar{\eta}^{\dot{\alpha}}$
  - $(\frac{1}{2},0) \oplus (0,\frac{1}{2}) \xleftrightarrow{\mathsf{P}} (0,\frac{1}{2}) \oplus (\frac{1}{2},0) = (\frac{1}{2},0) \oplus (0,\frac{1}{2})$,

    $\Rightarrow$ so a Dirac-spinor stays a Dirac-spinor
  - the four-vector stays the same

## 3. Special Relativity (SR) — Algebra of the Poincaré group

classifying particles

- Charge conjugation also interchanges $SU(2)_L \Leftrightarrow SU(2)_R$
  - like Parity

$\Rightarrow$ the combined transformation CP leaves $SU(2)_L$ and $SU(2)_R$ invariant
  - but it still includes mathematically a complex conjugation

- Time reversal T is an antiunitary transformation
  - it includes a complex conjugation

any quantum field theory

- built from the representations of the Poincaré algebra
  - that means: scalars, spinors, vectors

$\Rightarrow$ has to be invariant under CPT