# 8. General Relativity — Gravitational Waves

## Historical ramblings ... ( arXiv:1609.09400 [physics.hist-ph] )

1864 Maxwell predicts electromagnetic waves

1867 Hertz discovers electromagnetic waves

1915 Einstein publishes GR

1916 – 1937 pro and contra gravtitational waves

1939 – 1945 second world war

1957 Meeting at Chapel Hill on General Relativity

- ∗ with Gravitational Waves (GW) as one topic
- ∗ Feynman gives the "sticky bead" argument, convincing that GWs are detectable

$\Rightarrow$ start of GW detection experiments

1983 initial LIGO project (Weiss, Thorne, Drever)

1988 NSF funds LIGO

1993 starting of VIRGO

2003 VIRGO completed

2010 beginning of aLIGO

2015 aLIGO: engineering test and detection of GW

# 8. General Relativity — Gravitational Waves

## What is actually a ”wave” ?

- "Why the Notion of Radiation is Non-Trivial" ( arXiv:2201.11634 [gr-qc] )
  - for EM we know, how a source $J^\mu(t',\vec{x}')$ generates the field $A^\mu(t,\vec{x})$:

$$A^\mu(t,\vec{x}) = \frac{1}{4\pi}\int_V d^3x' \int_T dt' \frac{J^\mu(t',\vec{x}')}{||\vec{x}-\vec{x}'||}\delta(t'+||\vec{x}-\vec{x}'||-t) \quad (1)$$

    - ∗ but not all changing fields are radiation    example: eddy current break

- from a ”wave” we usually require
  - it is periodic with a frequency $\omega$
    - ∗ and has the corresponding wavelength $\lambda = \frac{2\pi c}{\omega}$
  - it travels also far away from the source, which has a finite size $d$
    - ∗ so we can write $r = ||\vec{x}|| \gg d$
  - the field of the wave drops like $\frac{1}{r}$ with the distance

- for the radiation zone we assume    $d \ll \lambda \ll r$    (2)

# 8. General Relativity — Gravitational Waves

## Rigorous treatment ( arXiv:2201.11634 [gr-qc] )

- construct a conformal completion of spacetime
  - start with outgoing Eddington-Finkelstein coordinates $(u = t - r^*, r, \theta, \phi)$
    - built from the tortoise coordinate $r^* = r + r_S \ln |r/r_S - 1|$ with $r_S = 2GM$ (3)
  - conformally rescale the line element with $\rho = \frac{1}{r}$ ( to move $r = \infty$ to $\rho = 0$ )

$$d\tilde{s}^2 = \rho^2(du^2 + 2dudr - r^2 d^2\Omega) = du^2 - 2dud\rho - d^2\Omega \qquad (4)$$

$\Rightarrow$ the "boundary" at infinity is discussed as $\rho \to 0$

- use the Newman-Penrose Null Tetrad Formalism — compare c04Riga-gr2.pdf
  - to formulate the limit $\rho \to 0$ in mathematical exact terms
    - and to replace tensors by complex functions, i.e. Newman-Penrose scalars
  - apply the peeling theorem to identify the radiation

- for Penrose diagrams use additionally the rescaling $\tan(t \pm r) = \hat{t} \pm \hat{r}$ (5)
  - to draw the infinities on a finite sheet of paper . . .

## Arguments for linearized gravity

- we experience gravity as weak (in terms of the GR description)
  $\Rightarrow$ detection of GWs allows the weak-field description

- GWs travel to us through nearly empty space
  $\Rightarrow$ propagation of GWs allows the weak-field description

- we will see: some production of GWs can be described in weak-field limit

- gravity with a perturbation series of the metric compare (3) of c04Riga-gr3.pdf

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1 \quad \text{and} \quad \eta_{\mu\nu} \text{ Minkovsky or FRW} \tag{6}$$

  $\Rightarrow$ we need to linearize $\Gamma^\lambda_{\mu\nu}$, $R^\lambda_{\mu\nu\rho}$, $R_{\mu\nu}$, $R$, and $G_{\mu\nu}$ in terms of $h_{\mu\nu}$

- gravity is a gauge theory: $$x^\mu \to x^{\mu\prime} = x^\mu + \xi^\mu(x) \tag{7}$$
  - $x^{\mu\prime}$ has to give the same physics as the original $x^\mu$

  $\Rightarrow$ restrictions for the parametrisation of $h_{\mu\nu}$

## 8. General Relativity — gauging linearized gravity

### parametrizing metric and gauge transformations

we distinguish between time (index 0) and space (indices $i, j, k, \ldots$)

- for simplicity we assume Cartesian space coordinates $(x, y, z)$
  - with the space metric $\delta_{jk}$, allowing simple raising and lowering
- then we write the metric as the line element: $d^2s = g_{\mu\nu}dx^\mu dx^\nu$

  we keep $a^2(t)$ for generality: we can always set $a(t) = a_0$ for Minkovsky space.

$$d^2s = a^2(t)[(1 + 2A)dt^2 - 2B_i dt dx^i - (\delta_{jk} + h_{jk})dx^j dx^k] \tag{8}$$

with $A$, $B_i = \partial_i B + \hat{B}_i$ and $h_{jk}$ as perturbations

- that can be decomposed into S(calar)-V(ector)-T(ensor) parts

  . . . which stay among themselves in first order perturbation theory

$$\tfrac{1}{2}h_{jk} = C\delta_{jk} + (\partial_j\partial_k - \tfrac{1}{3}\delta_{jk}\vec{\partial}^2)E + \tfrac{1}{2}(\partial_j\hat{E}_k + \partial_k\hat{E}_j) + \hat{E}_{jk} \tag{9}$$

- vectors and tensors are transverse: $\partial^j\hat{B}_j = \partial^j\hat{E}_j = \partial^j\hat{E}_{jk} = \partial^k\hat{E}_{jk} = 0$ (10)
- and the tensor is traceless: $\delta^{jk}\hat{E}_{jk} = 0$ (11)

## 8. General Relativity — gauging linearized gravity

### parametrizing metric and gauge transformations

- the perturbations $A$, $B$, $C$, $E$, $\hat{B}_i$, $\hat{E}_i$, and $\hat{E}_{jk}$ change under the transformation eq.(7) $x^\mu \to x^{\mu\prime} = x^\mu + \xi^\mu(x)$

- but the Bardeen variables do not:

$$\begin{aligned} \Psi &= A + \frac{\dot a}{a}(B - \frac{d}{dt}E) + \frac{d}{dt}(B - \frac{d}{dt}E) & & \hat{E}_{jk} \\ -\Phi &= C + \frac{\dot a}{a}(B - \frac{d}{dt}E) - \frac{1}{3}\vec\partial^2 E) & \hat\Phi_i &= -(\hat B_i - \frac{d}{dt}\hat E_i) \end{aligned} \tag{12}$$

  – they are gauge invariant

    * and have 2 scalar , 2 vector , and 2 tensor degrees of freedom (d.o.f.)
    * the four gauge transformations eq.(7) remove 4 d.o.f.s from the 10 of the symmetric $h_{\mu\nu}$

- one can always choose a gauge that the local metric is given by

$$d^2s = a^2(t)[(1+2\Psi)dt^2 - 2\hat\Phi_i dt dx^i - (1-2\Phi)d\vec x^2 - 2\hat E_{jk}dx^j dx^k] \tag{13}$$

  – so $$h_{00} = 2\Psi\,,\ h_{0i} = h_{i0} = -2\hat\Phi_i\,,\ h_{jk} = 2\Phi\delta_{jk} - 2\hat E_{jk} \tag{14}$$

    * and

$$\partial^\mu h_{\mu 0} = \partial^0 h_{00} + \partial^i h_{i0} = 2\partial^0\Psi - 2\partial^i\hat\Phi_i \;=\; 2\partial_0\Psi \tag{15}$$

$$\partial^\mu h_{\mu k} = \partial^0 h_{0k} + \partial^i h_{ik} = -2\partial^0\hat\Phi_k + 2\partial^i\Phi\delta_{ik} - 2\partial^i\hat E_{ik} \;=\; -2\partial_0\hat\Phi_k + 2\partial_k\Phi \tag{16}$$

# 8. General Relativity — linearised Einstein equations

linearised Christoffel symbols $$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \quad (17)$$

- up to first order in the perturbations $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{[0]\lambda}_{\mu\nu} + \hat{\Gamma}^{\lambda}_{\mu\nu}$

$$\Gamma^{[0]\lambda}_{\mu\nu} = \frac{1}{2} a^{-2} \eta^{\lambda\rho}(\partial_\mu a^2 \eta_{\nu\rho} + \partial_\nu a^2 \eta_{\mu\rho} - \partial_\rho a^2 \eta_{\mu\nu}) \quad (18)$$

  – with the only nonvanishing $\Gamma^{[0]0}_{00} = \frac{\dot{a}}{a}$ and $\Gamma^{[0]0}_{jk} = \frac{\dot{a}}{a}\delta_{jk}$ (19)

    * which are zero for Minkovsky space background

- at first order we get more non-vanishing pieces ...

$$\hat{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2a^2} h^{\lambda\rho}(\partial_\mu a^2 \eta_{\nu\rho} + \partial_\nu a^2 \eta_{\mu\rho} - \partial_\rho a^2 \eta_{\mu\nu}) + \frac{1}{2a^2} \eta^{\lambda\rho}(\partial_\mu a^2 h_{\nu\rho} + \partial_\nu a^2 h_{\mu\rho} - \partial_\rho a^2 h_{\mu\nu}) \quad (20)$$

  – for a result we have to specify the decomposition of the perturbation $h_{\mu\nu}$

    * and split the sums over $\mu$ into 0 and $i$

- for detection and local propagation of GWs, we can set $a$ constant

  – then $$\Gamma^{\lambda}_{\mu\nu} = \hat{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2} \eta^{\lambda\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}) \quad (21)$$

    * $h_{\mu\nu}$ here is the general one from eq.(6)
    * the linearised approach allows a general solution, that is simpler to write with this general $h_{\mu\nu}$

## 8. General Relativity — linearised Einstein equations

linearised Ricci tensor $$R_{\mu\nu} = R^{[0]}{}_{\mu\nu} + \hat{R}_{\mu\nu} = R^{[0]\lambda}{}_{\mu\lambda\nu} + \hat{R}^{\lambda}{}_{\mu\lambda\nu} \qquad (22)$$

- gives at zero order the background curvature

$$\begin{aligned} R^{\lambda}{}_{\mu\lambda\nu} &= \partial_\lambda \Gamma^{[0]\lambda}_{\nu\mu} - \partial_\nu \Gamma^{[0]\lambda}_{\lambda\mu} - \Gamma^{[0]\lambda}_{\nu\kappa}\Gamma^{[0]\kappa}_{\lambda\mu} + \Gamma^{[0]\lambda}_{\lambda\kappa}\Gamma^{[0]\kappa}_{\nu\mu} \\ &= \partial_0 \Gamma^{[0]0}_{\nu\mu} - \partial_\nu \Gamma^{[0]0}_{0\mu} - \Gamma^{[0]0}_{\nu 0}\Gamma^{[0]0}_{0\mu} + \Gamma^{[0]0}_{00}\Gamma^{[0]0}_{\nu\mu} \\ &= \delta_{\mu(\nu\neq 0)}[\tfrac{d}{dt}\tfrac{\dot{a}}{a} + (\tfrac{\dot{a}}{a})^2] \qquad (23) \end{aligned}$$

- and to first order

$$\hat{R}^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \hat{\Gamma}^{\lambda}_{\nu\mu} - \partial_\nu \hat{\Gamma}^{\lambda}_{\lambda\mu} - \Gamma^{[0]\lambda}_{\nu\kappa}\hat{\Gamma}^{\kappa}_{\lambda\mu} - \hat{\Gamma}^{\lambda}_{\nu\kappa}\Gamma^{[0]\kappa}_{\lambda\mu} + \Gamma^{[0]\lambda}_{\lambda\kappa}\hat{\Gamma}^{\kappa}_{\nu\mu} + \hat{\Gamma}^{\lambda}_{\lambda\kappa}\Gamma^{[0]\kappa}_{\nu\mu} \qquad (24)$$

- for $a = a_0$ we get

$$\begin{aligned} R_{\mu\nu} &= \hat{R}_{\mu\nu} = \partial_\lambda \hat{\Gamma}^{\lambda}_{\nu\mu} - \partial_\nu \hat{\Gamma}^{\lambda}_{\lambda\mu} \qquad (25) \\ &= \tfrac{1}{2}(\partial_\mu \partial^\rho h_{\nu\rho} + \partial_\nu \partial^\rho h_{\mu\rho} - \partial_\rho \partial^\rho h_{\mu\nu} - \partial_\nu \partial_\mu h) \end{aligned}$$

  – where $h = \eta^{\lambda\rho} h_{\lambda\rho}$ is the "trace" of $h_{\mu\nu}$

- the Ricci scalar $$R = \eta^{\mu\nu} R_{\mu\nu} = (\partial^\lambda \partial^\rho h_{\lambda\rho} - \partial_\lambda \partial^\lambda h) \qquad (26)$$

- and the Einstein tensor $$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} R$$

$$= \tfrac{1}{2}(\partial_\mu \partial^\rho h_{\nu\rho} + \partial_\nu \partial^\rho h_{\mu\rho} - \partial_\rho \partial^\rho h_{\mu\nu} - \eta_{\mu\nu}\partial^\lambda \partial^\rho h_{\lambda\rho} - \partial_\nu \partial_\mu h + \eta_{\mu\nu}\partial_\lambda \partial^\lambda h) \qquad (27)$$

## 8. General Relativity — linearised Einstein equations

### simplifying the Einstein tensor

- using a trace-reversed perturbation $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ (28)

  $\Rightarrow$ the trace term in $G_{\mu\nu}$ disappears

- choosing additionally the *Lorentz gauge* $\partial^\mu \bar{h}_{\mu\nu} = 0$ (29)

  $\Rightarrow$ most terms in $G_{\mu\nu}$ vanish: $G_{\mu\nu} = -\frac{1}{2}\partial_\rho\partial^\rho \bar{h}_{\mu\nu} =: -\frac{1}{2}\Box\bar{h}_{\mu\nu}$ (30)

$\Rightarrow$ Einstein equations become simple wave equations in vacuum:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \Rightarrow \quad \Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \overset{\text{in vacuum}}{=} 0 \tag{31}$$

- with the simple solution $\bar{h}_{\mu\nu} = \int d^3k C_{\mu\nu}(\vec{k}) e^{i(\omega t - \vec{k}\cdot\vec{x})}$ (32)

  - the Lorentz gauge leads to transversity: $k^\mu C_{\mu\nu} = 0$ (33)

- comparing to the Bardeen variables, eqs.(12) and (13):

  - only the transverse-traceless $\hat{E}_{ik}$ survives the Lorentz condition

    $\Rightarrow$ transverse-traceless gauge $h_{jk}^{\mathsf{TT}} = 2\hat{E}_{jk}$ (34)

## The transverse-traceless gauge

The transverse-traceless gauge: $k^\mu C_{\mu\nu} = 0 \quad C^\mu{}_\mu = 0 \quad C_{\mu 0} = C_{0\mu} = 0$

$\partial^\mu h^{\mathrm{TT}}_{\mu\nu} = 0 \quad \eta^{\mu\nu} h^{\mathrm{TT}}_{\mu\nu} = 0 \quad h^{\mathrm{TT}}_{\mu 0} = h^{\mathrm{TT}}_{0\mu} = 0$

For a wave propagating in the $z$ direction: $k^\mu = (\omega, 0, 0, \omega)$

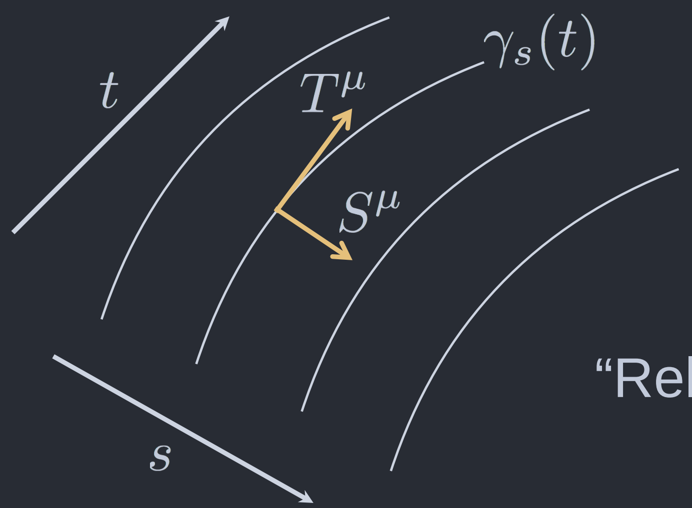
$$h^{\mathrm{TT}}_{\mu\nu}(z,t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\omega(z-t)}$$

Consider a particle initially at rest: $U^\mu = (1, 0, 0, 0)$

$$\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\nu\lambda} U^\nu U^\lambda = 0 \implies \text{initially, } \left.\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau}\right|_{\tau=0} = -\Gamma^\mu_{00} = 0 \implies \text{zero acceleration}$$

TT gauge coordinates stay attached to particles

## Geodesic deviation




Tangent vector: $T^\mu = \dfrac{\partial x^\mu}{\partial t}$

Deviation vector: $S^\mu = \dfrac{\partial x^\mu}{\partial s}$

"Relative velocity of geodesics": $V^\mu = (\nabla_T S)^\mu = T^\nu \nabla_\nu S^\mu$

"Relative acceleration of geodesics": $A^\mu = (\nabla_T V)^\mu$

Geodesic deviation equation:

$$A^\mu = \frac{D^2}{\mathrm{d}t^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

$$\frac{D}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \nabla_\mu$$

## Effect of gravitational waves on test particles

we use a single vector field for velocities of both particles

For nearby particles we have $\frac{D^2}{\mathrm{d}t^2} S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma$

particles are moving slowly, so we use "universal" time

separation between the particles

In the zeroth order in the perturbation, $U^\mu = (1, 0, 0, 0)$

Hence, $\frac{D^2}{\mathrm{d}t^2} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\nabla_\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}t}\nabla_\nu = \nabla_0\nabla_0 \quad = \partial_0\partial_0$ since, in TT gauge, $\Gamma^\mu_{0\nu} = 0$

The only required components are $R^\mu{}_{00\sigma} = \frac{1}{2}\partial_0\partial_0 h^{\mathrm{TT}\mu}{}_\sigma$

$$\frac{\partial^2}{\partial t^2} S^\mu = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^{\mathrm{TT}\mu}{}_\sigma$$

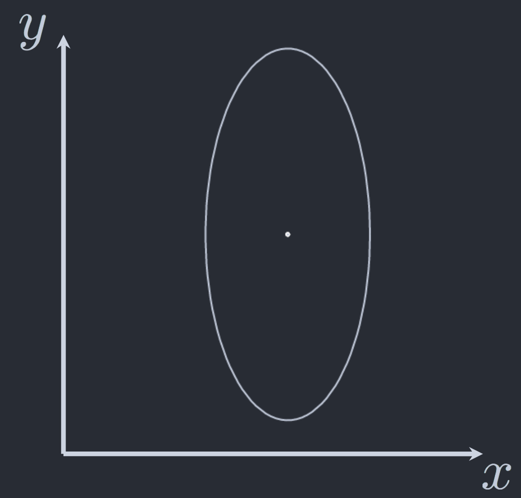
## Polarisation of gravitational waves

$$\frac{\partial^2}{\partial t^2} S^\mu = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^{\mathrm{TT}\mu}{}_\sigma$$

For a wave propagating in the $z$ direction: $k^\mu = (\omega, 0, 0, \omega)$

$$h^{\mathrm{TT}}_{\mu\nu}(z,t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\omega(z-t)}$$

"Plus" polarisation:




Let $h_\times = 0$:

$$\frac{\partial^2}{\partial t^2} S^x = \frac{1}{2} S^x h_+ \omega^2 \mathrm{e}^{\mathrm{i}\omega(z-t)} \qquad S^x = S^x(0)\left(1 + \frac{1}{2} h_+ \mathrm{e}^{\mathrm{i}\omega(z-t)} + \mathcal{O}(h_+^2)\right)$$

$$\Longrightarrow$$

$$\frac{\partial^2}{\partial t^2} S^y = -\frac{1}{2} S^y h_+ \omega^2 \mathrm{e}^{\mathrm{i}\omega(z-t)} \qquad S^y = S^y(0)\left(1 - \frac{1}{2} h_+ \mathrm{e}^{\mathrm{i}\omega(z-t)} + \mathcal{O}(h_+^2)\right)$$

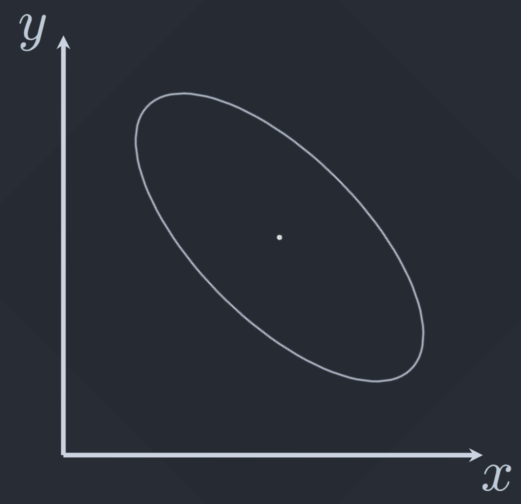
## Polarisation of gravitational waves

$$\frac{\partial^2}{\partial t^2} S^\mu = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^{\mathrm{TT}\mu}{}_\sigma$$

For a wave propagating in the $z$ direction: $k^\mu = (\omega, 0, 0, \omega)$

$$h_{\mu\nu}^{\mathrm{TT}}(z,t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}\omega(z-t)}$$

"Cross" polarisation:




Let $h_+ = 0$:

$$S^x = S^x(0) + \frac{1}{2} h_\times \mathrm{e}^{\mathrm{i}\omega(z-t)} S^y(0) + \mathcal{O}(h_\times^2)$$

$$S^y = S^y(0) + \frac{1}{2} h_\times \mathrm{e}^{\mathrm{i}\omega(z-t)} S^x(0) + \mathcal{O}(h_\times^2)$$

Wave is invariant under rotation by 180° around the direction of propagation

$\Longrightarrow$ gravitons should have spin 2

(360°/180° = 2)

## Energy flux

To discuss energy, we have to include the second order:

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots$$

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{\text{lin}}_{\mu\nu}(h^{(1)}_{\mu\nu}) + R^{\text{lin}}_{\mu\nu}(h^{(2)}_{\mu\nu}) + R^{\text{quad}}_{\mu\nu}(h^{(1)}_{\mu\nu}) + \dots$$

The Einstein equation in vacuum $R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = 0$ or $R_{\mu\nu} = 0$ gives:

zeroth order: $R^{(0)}_{\mu\nu} = 0$

first order: $R^{\text{lin}}_{\mu\nu}(h^{(1)}_{\mu\nu}) = 0$ (allows us to determine $h^{(1)}_{\mu\nu}$)

second order: $R^{\text{lin}}_{\mu\nu}(h^{(2)}_{\mu\nu}) - \frac{1}{2} R^{\text{lin}}(h^{(2)}_{\mu\nu})\eta_{\mu\nu} + \underbrace{R^{\text{quad}}_{\mu\nu}(h^{(1)}_{\mu\nu}) - \frac{1}{2} R^{\text{quad}}(h^{(1)}_{\mu\nu})\eta_{\mu\nu}}_{\equiv -8\pi G t_{\mu\nu}} = 0$

Calculation of the energy–momentum tensor yields

$$t_{\mu\nu} \equiv \frac{1}{32\pi G} \left\langle (\partial_\mu h^{\text{TT}}_{ij})(\partial_\nu h^{ij}_{\text{TT}}) \right\rangle$$

$\langle \dots \rangle$ — averaging over many wavelengths

and the energy flux is $f = t_{00}$

(in conventional units, $f = c t_{00}$)

## Quadrupole moment tensor

General solution of the wave equation $\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$:

$$\bar{h}_{ij}(t, \boldsymbol{x}) = 4G \int \frac{T_{ij}(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}\boldsymbol{x}'$$

In the limit $r \gg \delta r, \quad \lambda \gg \delta r$ we have $\bar{h}_{ij}(t, \boldsymbol{x}) = \frac{4G}{r} \int T_{ij}(t - r, \boldsymbol{x}') \, \mathrm{d}\boldsymbol{x}'$

distance to source wavelength source dimensions

Using $\partial_\mu T^{\mu\nu} = 0$, one may show that $\int T_{ij} \, \mathrm{d}\boldsymbol{x}' = \frac{1}{2} \frac{\partial^2}{\partial t^2} \int T_{00} x_i x_j \, \mathrm{d}\boldsymbol{x}'$

Employing the Newtonian approximation $T_{00}(t - r, \boldsymbol{x}') = \rho(t - r, \boldsymbol{x}')$
we define the quadrupole moment

$$I_{ij}(t - r) = \int \rho(t - r, \boldsymbol{x}') x_i x_j \, \mathrm{d}\boldsymbol{x}'$$

The perturbation is then

$$\bar{h}_{ij}(t, \boldsymbol{x}) = \frac{2G}{r} \frac{\mathrm{d}^2}{\mathrm{d}t^2} I_{ij}(t - r)$$

## Energy loss due to radiation

$$f = \frac{1}{32\pi G}\left\langle(\partial_0 h_{ij}^{\mathrm{TT}})(\partial_0 h^{ij}_{\mathrm{TT}})\right\rangle$$

We are interested in the radiation power $\dfrac{\mathrm{d}E}{\mathrm{d}t} = \displaystyle\int_{S^2_\infty} f r^2 \mathrm{d}\Omega$

We have $\bar{h}_{ij}(t,\boldsymbol{x}) = \dfrac{2G}{r}\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} I_{ij}(t-r)$

One may show that $h_{ij}^{\mathrm{TT}}(t,\boldsymbol{x}) = \bar{h}_{ij}^{\mathrm{TT}}(t,\boldsymbol{x}) = \dfrac{2G}{r}\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} I_{ij}^{\mathrm{TT}}(t-r) = \dfrac{2G}{r}\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} J_{ij}^{\mathrm{TT}}(t-r)$

with reduced quadruple moment $J_{ij} = I_{ij} - \dfrac{1}{3}\delta_{ij}\delta^{kl} I_{kl}$

To convert to the TT gauge, we use the formula $J_{ij}^{\mathrm{TT}} = \left(P_i{}^k P_j{}^l - \dfrac{1}{2} P_{ij} P^{kl}\right) J_{kl}$

with a projection tensor $P_{ij} = \delta_{ij} - n_i n_j$

unit vector along the direction of propagation

Finally, the integral may be performed to yield $\dfrac{\mathrm{d}E}{\mathrm{d}t} = -\dfrac{G}{5}\left\langle \dfrac{\mathrm{d}^3 J_{ij}}{\mathrm{d}t^3}\dfrac{\mathrm{d}^3 J^{ij}}{\mathrm{d}t^3}\right\rangle$

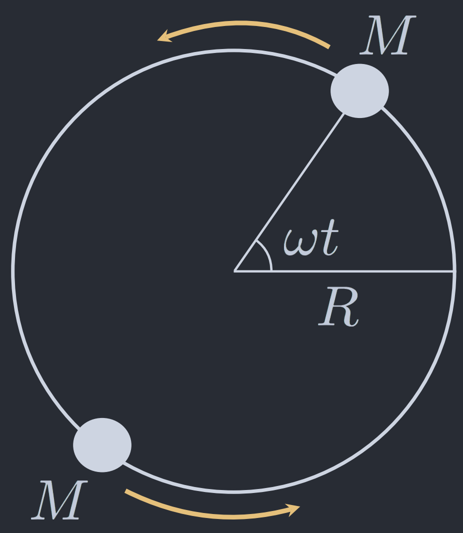
## Hulse–Taylor binary pulsar — energy loss

Mass density of the system:

$$\rho(t, \boldsymbol{x}') = M\delta(x_3)\big(\delta(x_1 - R\cos\omega t)\delta(x_2 - R\sin\omega t) + \delta(x_1 + R\sin\omega t)\delta(x_2 + R\cos\omega t)\big)$$

Straightforward calculation of $I_{ij}(t-r) = \int \rho(t-r, \boldsymbol{x}')x_i x_j \, \mathrm{d}\boldsymbol{x}'$

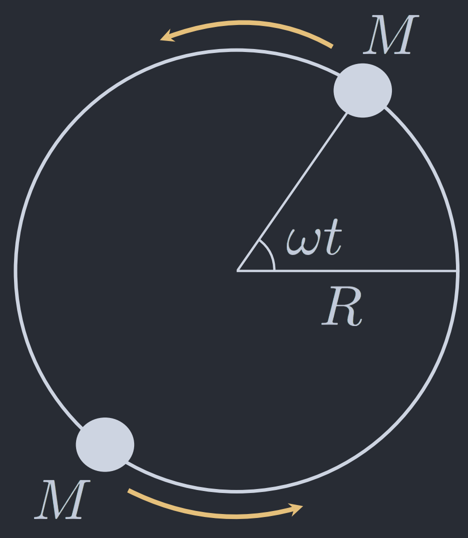
and then $J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}I_{kl}$




leads to the energy loss due to gravitational radiation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{5}\left\langle \frac{\mathrm{d}^3 J_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 J^{ij}}{\mathrm{d}t^3}\right\rangle = -\frac{128}{5}GM^2R^4\omega^6$$

## Hulse–Taylor binary pulsar — orbital decay




$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{5}\left\langle \frac{\mathrm{d}^3 J_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 J^{ij}}{\mathrm{d}t^3}\right\rangle = -\frac{128}{5}GM^2R^4\omega^6$$

Using the equation of motion $\frac{GM^2}{(2R)^2} = M\frac{v^2}{R}$

we get the energy as $E = -\frac{GM^2}{2R} + Mv^2 = -\frac{GM^2}{4R}$

Expressing $R$ in terms of the period $T$, $E = -M\left(\frac{G\pi M}{2}\right)^{2/3} T^{-2/3}$

Differentiating with respect to time and substituting the energy loss, we get the orbital decay

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{48\pi}{5}\left(\frac{4\pi GM}{T}\right)^{5/3}$$

Taking into account that the masses are different and that the orbit is not circular, we find $\frac{\mathrm{d}T}{\mathrm{d}t} = -(2.40247 \pm 0.00002) \times 10^{-12}\ \mathrm{s/s}$

Measured: $\frac{\mathrm{d}T}{\mathrm{d}t} = -(2.4086 \pm 0.0052) \times 10^{-12}\ \mathrm{s/s}$

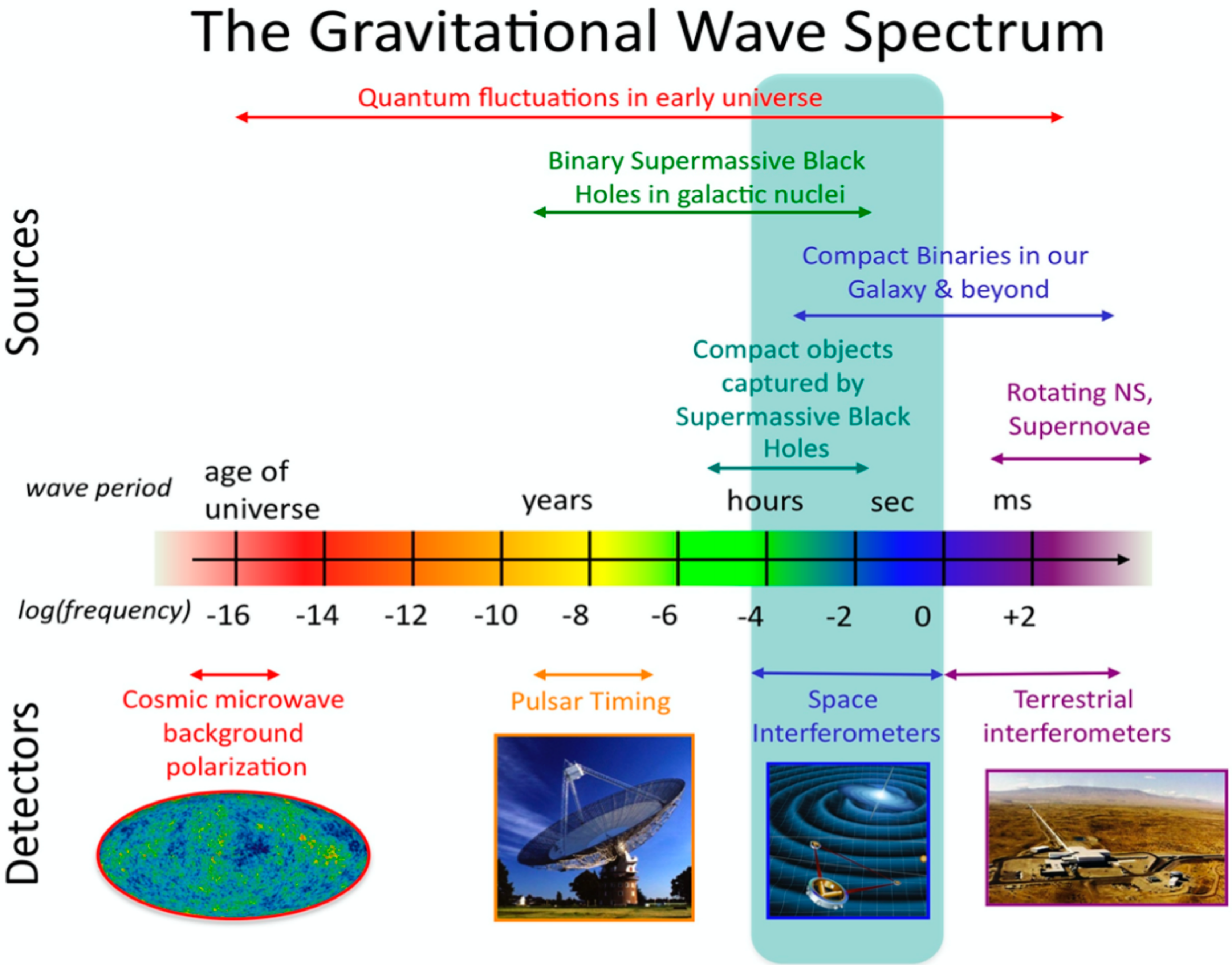



**Figure 7.** Gravitational wave spectrum showing wavelength and frequency along with some anticipated sources and the kind of detectors one might use. Figure credit: NASA Goddard Space Flight Center.