7. General Relativity — CMB

Measurements of the Cosmic Microwave Background Radiation

- first (involuntary) measurements by Penzias and Wilson in 1965
  ⇒ Nobel Prize in 1978

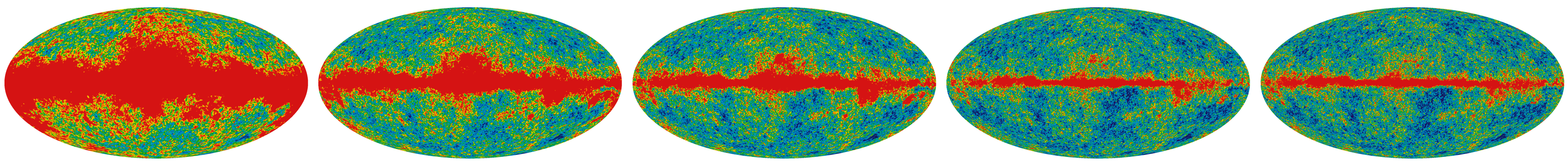
- COBE (Cosmic Background Explorer)
  is launched in 1989, takes data until 1991
  - FIRAS (Far Infrared Absolute Spectrophotometer)
    measures the frequency distribution in 1990
    ⇒ the CMB is a thermal blackbody radiation with $T \sim 2.725\,K$
  - DMR (Differential Microwave Radiometer)
    discovers the primary temperature anisotropy in 1992

  ⇒ Nobel Prize in 2006

- BOOMERanG and MAXIMA measure the acoustic oscillations
  in the angular power spectrum of the CMB anisotropy in 1999

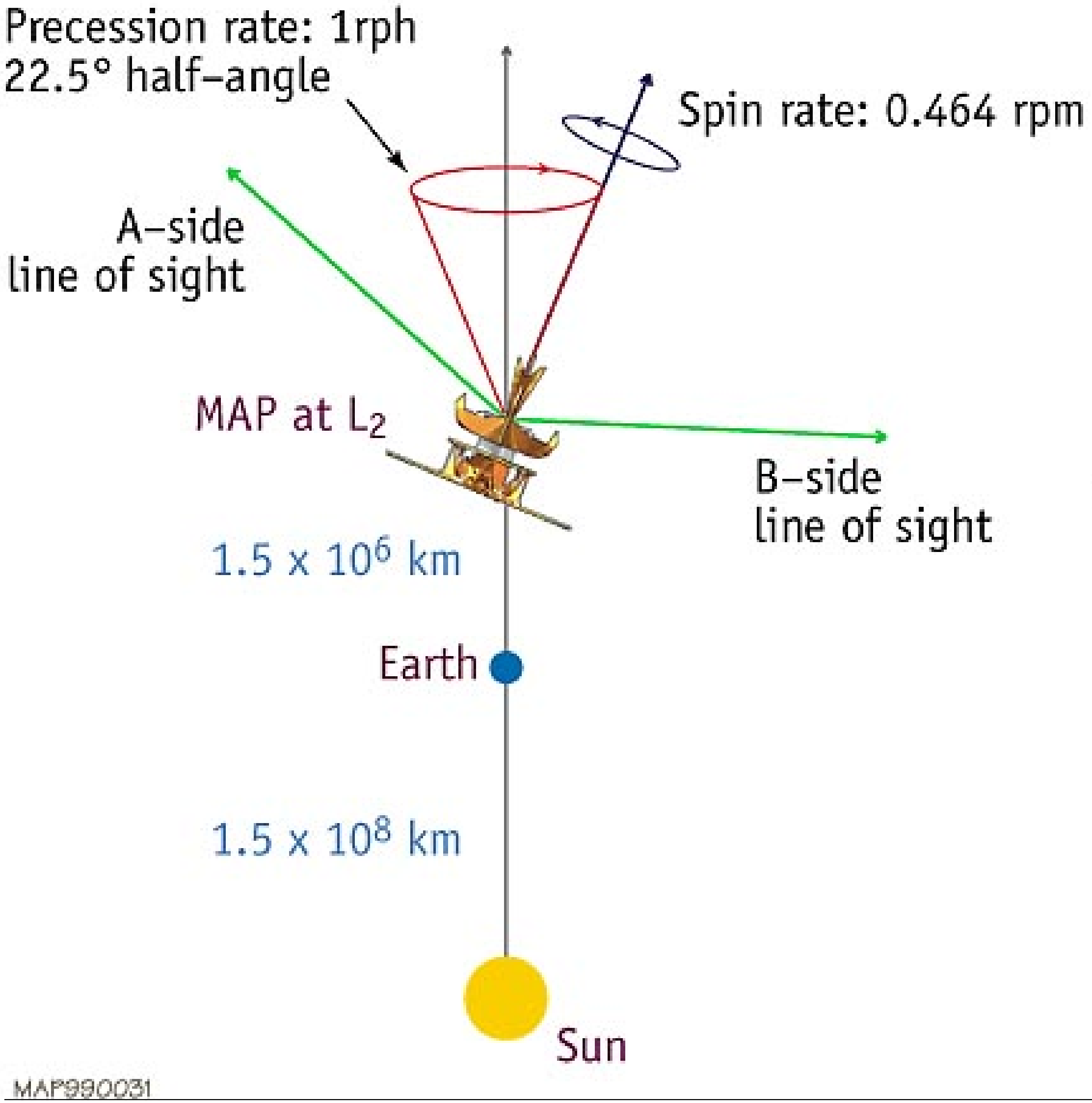
How is the Cosmic Microwave Background Radiation measured ?

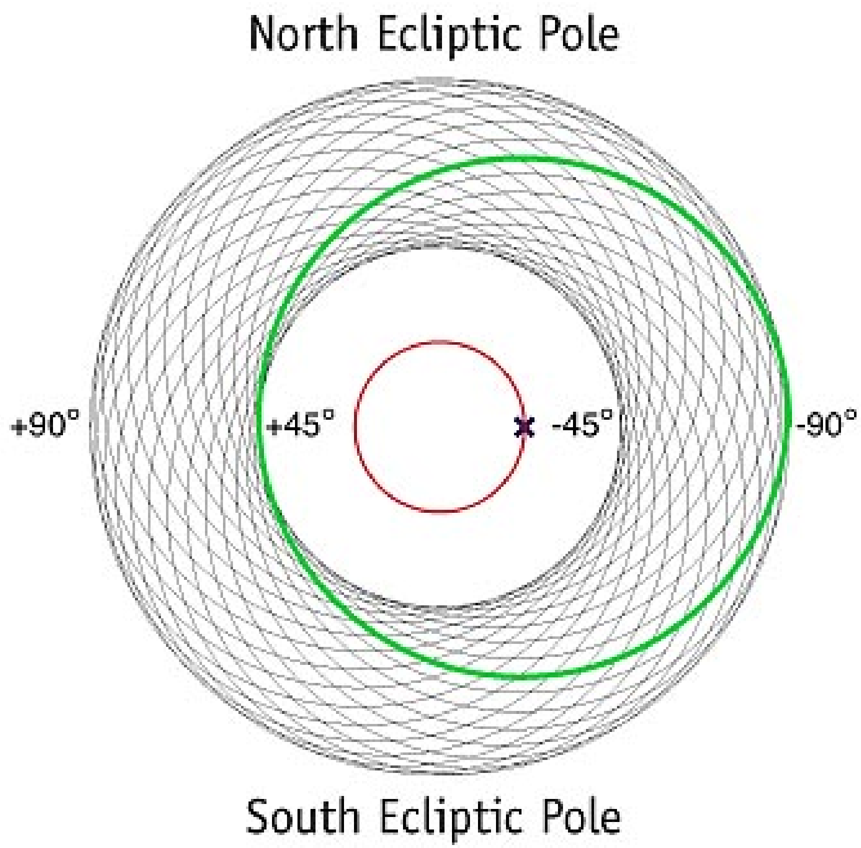
- it is mainly a microwave radiation $\Rightarrow$ radio antenna
  - $\Rightarrow$ directional measurements possible
- for higher accuracy in temperature differences
  - $\Rightarrow$ differential measurement
    - ∗ comparing the radiation coming from two different directions
- WMAP (Wilkinson Microwave Anisotropy Probe) measured
  - – in 5 radio bands (23, 33, 41, 61, and 94 GHz with $\sim$ 22% bandwidth)
  - – 393,216 sky pixels with a solid angle of (0.77, 0.44, 0.26, 0.12, and 0.05) degree
    - ∗ each sky pixel is measured 1000 to 5000 times per year



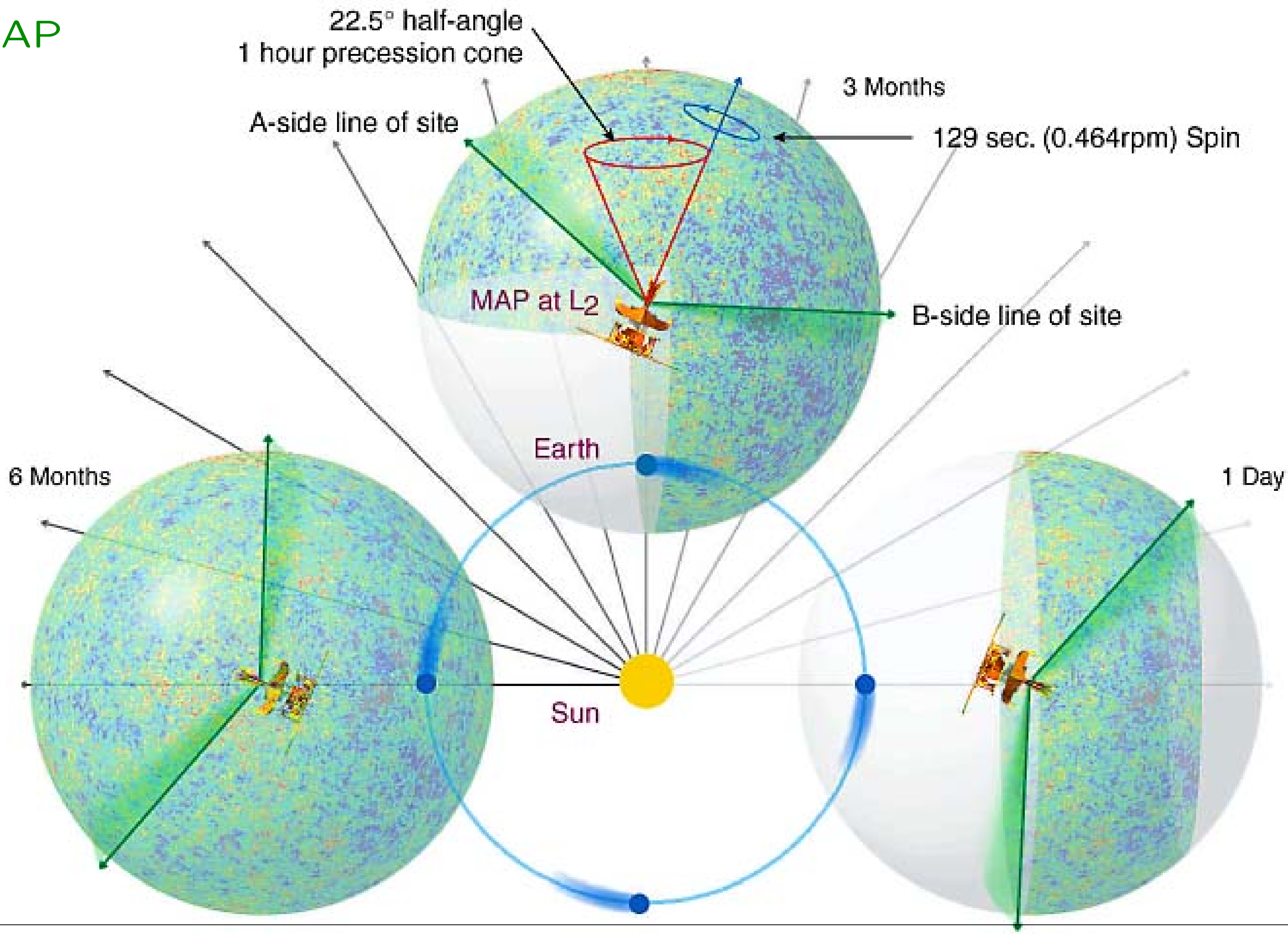
23 GHz 33 GHz 41 GHz 61 GHz 94 GHz

WMAP


Precession rate: 1rph
22.5° half-angle
Spin rate: 0.464 rpm
A-side
line of sight
MAP at L_2
B-side
line of sight
1.5 x 10^6 km
Earth
1.5 x 10^8 km
Sun
MAP990031


North Ecliptic Pole
+90°
+45°
-45°
-90°
South Ecliptic Pole

WMAP


22.5° half-angle
1 hour precession cone
A-side line of site
3 Months
129 sec. (0.464rpm) Spin
MAP at L_2
B-side line of site
Earth
6 Months
1 Day
Sun

WMAP gives the temperature $\mathsf{T}(\theta,\phi)$ of the CMB radiation

- the average temperature is

$$\langle \mathsf{T} \rangle = \frac{1}{4\pi} \int \mathsf{T}(\theta,\phi) \sin\theta \, d\theta \, d\phi = 2.725\,\mathsf{K} \tag{1}$$

  - the temperature fluctuation

$$\frac{\delta \mathsf{T}}{\mathsf{T}}(\theta,\phi) = \frac{\mathsf{T}(\theta,\phi) - \langle \mathsf{T} \rangle}{\langle \mathsf{T} \rangle} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\theta,\phi) \tag{2}$$

    can be described by spherical harmonics

  - the spherical harmonics are orthonormal basis functions

$$Y_\ell^m(\theta,\phi) = N e^{im\phi} P_\ell^m(\cos\theta) \quad \text{with} \quad \int Y_\ell^{*m} Y_{\ell'}^{m'} \sin\theta \, d\theta \, d\phi = \delta_{\ell\ell'} \delta^{mm'} \tag{3}$$

    that satisfy an addition theorem

$$\sum_{m=-\ell}^{\ell} Y_\ell^{*m}(\hat{n}_1) Y_\ell^m(\hat{n}_2) = \frac{2\ell+1}{4\pi} P_\ell(\cos\theta_{12}) \tag{4}$$

    with Legendre polynomial P_ℓ and the angle $\cos\theta_{12} = \hat{n}_1 \cdot \hat{n}_2$

multipoles of the CMB radiation

- the multipoles are given by

$$a_{\ell m} = \int Y_\ell^{m*}(\theta,\phi)\frac{\delta \mathrm{T}}{\mathrm{T}}(\theta,\phi)\,\sin\theta\,d\theta\,d\phi \tag{5}$$

- the two point correlation is

$$\begin{aligned} C(\theta_{12}) &= \langle \tfrac{\delta \mathrm{T}}{\mathrm{T}}(\hat{n}_1)\tfrac{\delta \mathrm{T}}{\mathrm{T}}(\hat{n}_2)\rangle \\ &= \sum_{\ell_1,\ell_2,m_1,m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} \int Y_{\ell_1}^{m_1}(\hat{n}_1) Y_{\ell_2}^{m_2}(\hat{n}_2)\,\sin\theta\,d\theta\,d\phi \end{aligned} \tag{6}$$

  - use Clebsch-Gordan coefficients to express the product of two Ys

$$Y_{\ell_1}^{m_1}(\hat{n}_1) Y_{\ell_2}^{m_2}(\hat{n}_2) = |\ell_1 m_1\rangle \otimes |\ell_2 m_2\rangle = |\ell_1 m_1 \ell_2 m_2\rangle \tag{7}$$

    as a sum over single Ys

$$|(\ell_1\ell_2)\ell_3 m_3\rangle = \textstyle\sum_{m_1,m_2} |\ell_1 m_1 \ell_2 m_2\rangle\langle \ell_1 m_1 \ell_2 m_2|(\ell_1\ell_2)\ell_3 m_3\rangle \tag{8}$$

  - since we integrate over the angles $\Rightarrow$ $\ell_3 = m_3 = 0$

    $\Rightarrow$ m_1 and m_2 have to sum up to zero and $\ell_1 = \ell_2$

$\Rightarrow$ only the "diagonal" terms contribute: $C(\theta_{12}) = \langle (\frac{\delta \mathrm{T}}{\mathrm{T}})^2 \rangle$ (9)

multipoles of the CMB radiation

- using the addition theorem (and $C_\ell = \sum_m |a_{\ell m}|^2$) we get

$$C(\theta_{12}) = \sum_{\ell,m} a_{\ell m} a^*_{\ell m} \int Y_\ell^{m*}(\hat{n}_1) Y_\ell^m(\hat{n}_2) \sin\theta \, d\theta \, d\phi$$
$$= \sum_\ell C_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos\theta_{12}) \tag{10}$$

- we are looking for the autocorrelation of density fluctuations
  - the angle between a direction and the same direction is zero

    ⇒ $\cos\theta_{12} = 1$ and $P_\ell(1) = 1$

- when dealing with a large sum, one can estimate it with an integral
  - in this case it is convenient to display the logarithm of ℓ

$$C = \sum_{\ell=0}^{\infty} C_\ell \frac{2\ell+1}{4\pi} P_\ell(1) \sim \int C_\ell \frac{\ell(2\ell+1)}{4\pi} d(\ln\ell) \tag{11}$$

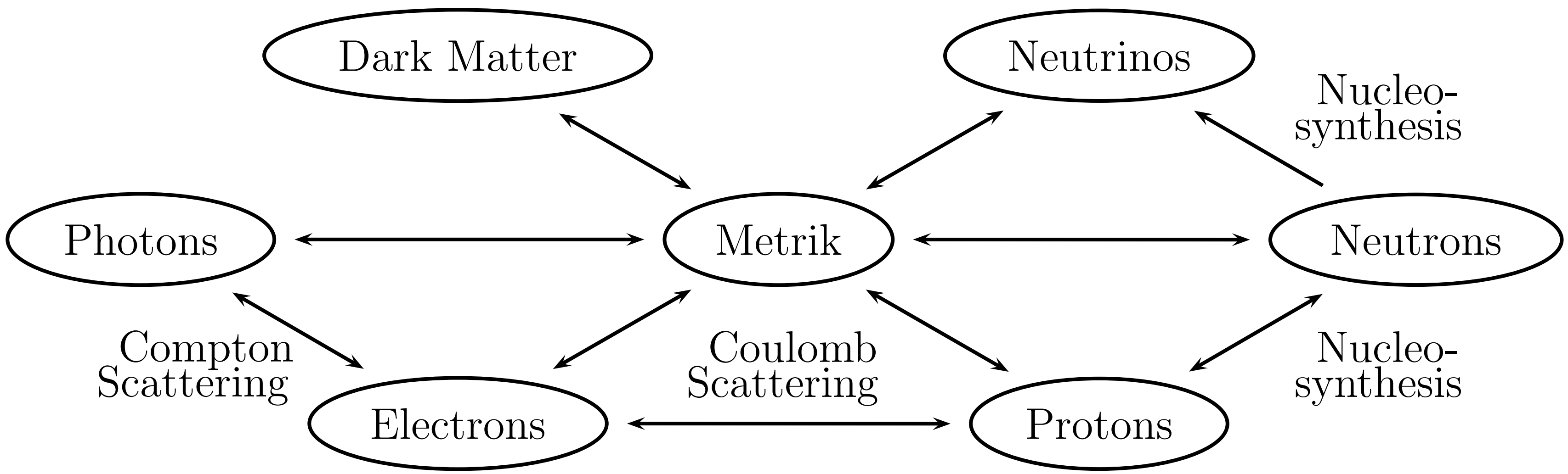
  - the interesting quantity is the integrand, more exactly C_ℓ

7. General Relativity — Analysing the CMB

How can we predict the multipoles of the CMB radiation ?

- first we have to realize how the CMB is produced
  - the radiation left over from the hot big bang
- how exactly?
  - studying the distribution of photons
    - coming from the pair annihilations and scatterings
    - of the available particles during the expansion

$\Rightarrow$ coupled Einstein-Boltzmann equations



the Boltzman transport equation

$$\frac{d}{dt} f_i(\vec{r}, \vec{p}, t) = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}} + \vec{F} \cdot \nabla_{\vec{p}} \right) f_i(\vec{r}, \vec{p}, t) = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} \tag{12}$$

- describes the change in the phase space density of particle i:
  - the flow $\vec{v}$ of particles changes their number in a region of space
  - the force $\vec{F}$ acting on the particles changes their momentum
  - the collisions (and decays) can change number and momentum
- how can we understand this equation in a covariant way?
  - we have $3 + 1$ and not only 3 dimensions ...

⇒ field equations (equations of motion) constrain p^μ:
  - in flat space:

$$m^2 = p^2 \quad \Rightarrow \quad p^0 = E = \sqrt{m^2 + \vec{p}^{\,2}} \tag{13}$$

7. General Relativity — fluctuation of densities

in curved space

- field equations (equations of motion)
  - couple to the Einstein equations
- taking the Robertson-Walker metric as a background:

$$g_{\mu\nu} \approx \begin{pmatrix} 1 & & & \\ & -a^2 & & \\ & & -a^2 & \\ & & & -a^2 \end{pmatrix} \tag{14}$$

  - with scalar metric perturbations

$$g_{00} = 1 - 2\Phi \qquad \text{and} \qquad g_{jk} = -\delta_{jk} a^2 (1 - 2\Psi) \tag{15}$$

    * Φ corresponds to the Newtonian potential
    * Ψ is the curvature perturbation
  - one gets the constraint on the momentum

$$m^2 = g_{\mu\nu} p^\mu p^\nu = E^2 (1 - 2\Phi) - \vec{p}^{\,2} a^2 (1 - 2\Psi) \tag{16}$$

    * that contains already a dependence on the curvature ...

7. General Relativity — fluctuation of densities

curvature is still given by Einsteins equations

- but now linearlized
  - the scale factor a is determined without perturbations
  - Φ and Ψ are determined by the first order in the perturbations
- the stress energy tensor is given by the particles
  - weighted by their densities:

$$T^{\mu\nu} = \sum_{i=\text{all particles}} n_i T_i^{\mu\nu} \tag{17}$$

  - the density is given by the phase space density

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{r}, \vec{p}, t) \tag{18}$$

    * with g_i the number of degrees of freedom per particle
    * which is also the zero-order "moment" of the phase space density
  - the velocity $\vec{v}_i$ is then the first-order moment

$$\vec{v}_i = \frac{g_i}{n_i} \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}_i}{E_i} f_i(\vec{r}, \vec{p}, t) = \left\langle \frac{\vec{p}_i}{E_i} \right\rangle \tag{19}$$

7. General Relativity — fluctuation of densities

the integrated collision term $C[f]$ is given by

$$C[f] = \int_{p_1} \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \int_{p_1}\int_{p_2}\int_{p_3}\int_{p_4} (2\pi)^4 \delta^4(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu)|\mathcal{M}|^2$$
$$\times \{f_3 f_4 [1 \pm f_1][1 \pm f_2] - f_1 f_2 [1 \pm f_3][1 \pm f_4]\} \qquad (20)$$

- with $$\int_{p_i} := \int \frac{d^3 p_i}{(2\pi)^3 2E_i} \qquad (21)$$
  - $\mathcal{M}$ describes the matrix element for the process $1 + 2 \rightleftharpoons 3 + 4$
    - ∗ to be calculated in Quantum Field Theory (next semester)
  - $\pm$ describes Bose enhancement / Pauli blocking $(+/-)$ for bosons / fermions
  - for high temperatures these factors become less important

  ⇒ the distributions $f = [e^{\frac{E-\mu}{kT}} \mp 1]^{-1}$ approach the Boltzmann distribution $e^{-\frac{E-\mu}{kT}}$
    - ∗ with the chemical potential μ, which is related to the density

$$\frac{n_i}{g_i} = \int \frac{d^3 p}{(2\pi)^3} f_i(\vec{r}, \vec{p}, t) = e^{\mu/kT} \int \frac{d^3 p}{(2\pi)^3} e^{-E/kT} \approx \begin{cases} \left(\frac{m_i kT}{2\pi}\right)^{3/2} e^{-m_i/kT} & m_i \gg kT \\ \frac{(kT)^3}{\pi^2} & m_i \ll kT \end{cases} \qquad (22)$$

- in equilibrium $C[f] = 0 = (e^{(\mu_1+\mu_2)/kT} - e^{(\mu_3+\mu_4)/kT}) \int |\mathcal{M}|^2$

  ⇒ the chemical potentials have to be equal: $\mu_1 + \mu_2 = \mu_3 + \mu_4$

7. General Relativity — fluctuation of densities

introducing the equilibrium density $n_i^{(0)} = g_i \int \frac{d^3p}{(2\pi)^3} e^{-E/kT}$ (23)

- one can rewrite $e^{\mu/kT} = n_i / n_i^{(0)}$
- and defined the thermally averaged cross section

$$\langle v\sigma \rangle := \frac{(2\pi)^4}{n_1^{(0)} n_2^{(0)}} \int_{p_1} \int_{p_2} \int_{p_3} \int_{p_4} \delta^4(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu) |\mathcal{M}|^2 e^{-(E_1+E_2)/kT} \quad (24)$$

- then the Boltzmann equation for the number density becomes

$$\int_{p_1} \frac{d}{dt} f_1(\vec{r}, \vec{p}, t) = \frac{d(a^3 n_1)}{a^3\, dt} = n_1^{(0)} n_2^{(0)} \langle v\sigma \rangle \left\{ \frac{n_3}{n_3^{(0)}} \frac{n_4}{n_4^{(0)}} - \frac{n_1}{n_1^{(0)}} \frac{n_2}{n_2^{(0)}} \right\} \quad (25)$$

- now $\frac{d(a^3 n_1)}{a^3\, dt} \sim H n_1$ if $H \ll n_2 \langle v\sigma \rangle$

  $\Rightarrow$ the bracket has to become zero: $\frac{n_3}{n_3^{(0)}} \frac{n_4}{n_4^{(0)}} = \frac{n_1}{n_1^{(0)}} \frac{n_2}{n_2^{(0)}}$ (26)

  * chemical equilibrium ... for heavy relics of the early universe
  * nuclear statistical equilibrium ... for Big Bang nucleosynthesis
  * Saha equation ... for recombination and ionization balance

7. General Relativity — fluctuation of densities

applying this ansatz to

- dark matter particles and SM particles

  $\Rightarrow$ dark matter abundance

  * needs non-equilibrium solution for freeze-out

- protons, neutrons and nuclei

  $\Rightarrow$ Big Bang nucleosynthesis

  * needs non-equilibrium solution for neutron capture and decay

- electrons, positrons, photons, and neutrinos

  $\Rightarrow$ CMB temperature versus neutrino temperature

- electrons, nuclei, and photons

  $\Rightarrow$ recombination, CMB photon spectrum

  * still have to calculate the density fluctuations

$\Rightarrow$ we have to solve the linearized Einstein-Boltzmann equations

7. General Relativity — fluctuation of densities

linearized Einstein-Boltzmann equations

- introducing the metric perturbations Φ and Ψ
- introducing linearized density fluctuations for all particles:
  - for the photons $\Theta(\vec{x},\hat{p},t)$, indepentent of $|\vec{p}|$:

$$f_\gamma(\vec{x},\vec{p},t) = \left[\exp\left\{\frac{|\vec{p}|}{T(t)[1+\Theta(\vec{x},\hat{p},t)]}\right\} - 1\right]^{-1} \tag{27}$$

  - for the other particles as a density contrast $\delta_i(\vec{x},t)$ and a velocity $\vec{v}_i(\vec{x},t)$

$$n_i(\vec{x},t) = n_i^{(0)}(\vec{x},t)\,[1+\delta_i(\vec{x},t)] \quad \text{and} \quad \vec{v}_i(\vec{x},t) = \left\langle \frac{\vec{p}_i}{E_i} \right\rangle \tag{28}$$

    - both, $\delta_i(\vec{x},t)$ and $\vec{v}_i(\vec{x},t)$, are considered first order
    - the zero-order velocity is in equilibrium $\Rightarrow$ $\vec{v}_i^{(0)} = 0$
  - minimal amount of relevant particles:
    * photon * baryon (includes e^- !) * neutrino * dark matter

$\Rightarrow$ 10 coupled partial differential equations

  - Fourier transform $\Theta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}.\vec{x}} \Theta(\vec{k})$ decouples the Fourier modes of Θ_γ

$\Rightarrow$ CMB power spectrum in the multipoles $$C_\ell = \frac{1}{(-i)^\ell}\int_{-1}^{1} \frac{d(\hat{p}.\hat{k})}{2} P_\ell(\hat{p}.\hat{k})\Theta(\vec{k}) \tag{29}$$

  - $\vec{v}_i$ is already the "dipole" of n_i

7. General Relativity — Big Bang nucleosynthesis (following Dodelson)

cosmic plasma at a temperature of 1 MeV (universe a few minutes old)

- in equilibrium: electrons, positrons, photons ... relativistic
- decoupled neutrinos ... still relativistic
  - for $e\nu \rightleftharpoons e\nu$ the coupling $\langle v\sigma \rangle < H$
- coupled baryons: protons and neutrons ... non-relativistic
  - antibaryons have annihilated with the baryons
  - the remaining baryons can come from Baryogenesis (introduced by Sakharov)
    - the baryon asymmetry is estimated as $(n_b - n_{\bar{b}})/s \approx 10^{-10}$
    - compatible with the ratio of baryons to photons today: $n_b/n_\gamma \approx 5.5 \cdot 10^{-10}$
  - these baryons can form nuclei ... step by step
  - ? coupled equations for all the elements until iron
- simplifications:
  - with $\sim 10^{10}$ photons per baryon and a temperature $kT \sim 0.1$ MeV
    - a binding energy of deuterium (^{2}H) of $E_b = 2.2$ MeV
    - there are $10^{10} \times e^{-2.2\text{MeV}/0.1\text{MeV}} \sim 2.2$ photons with $E_\gamma > E_b$ per nucleon
    - $\Rightarrow$ nearly all nuclei are disintegrated by high energy photons
  - Li, Be, and B are less bound than He
  - $\Rightarrow$ ^{n}H and ^{n}He are relevant, ^{n}Li only marginal

7. General Relativity — Big Bang nucleosynthesis (following Dodelson)

cosmic plasma at a temperature of 0.1 MeV (universe a few minutes old)

- for the reaction $p + n \rightleftharpoons D + \gamma$ we have the equilibrium

$$\frac{n_D}{n_p n_n} = \frac{n_D^{(0)}}{n_p^{(0)} n_n^{(0)}} = \frac{g_D \int \frac{d^3p}{(2\pi)^3} \exp\{-E_D/kT\}}{g_p \int \frac{d^3p}{(2\pi)^3} \exp\{-E_p/kT\} g_n \int \frac{d^3p}{(2\pi)^3} \exp\{-E_n/kT\}} \quad (30)$$

$$\approx \frac{3 \left[\frac{m_D kT}{(2\pi)}\right]^{3/2} e^{-\frac{m_D}{kT}}}{2 \left[\frac{m_p kT}{(2\pi)}\right]^{3/2} e^{-\frac{m_p}{kT}} \, 2 \left[\frac{m_n kT}{(2\pi)}\right]^{3/2} e^{-\frac{m_n}{kT}}} = \frac{3}{4} \left[\frac{2\pi m_D}{m_p m_n kT}\right]^{3/2} \exp\left\{\frac{m_p + m_n - m_D}{kT}\right\}$$

  - $m_p \simeq m_n \simeq m_D/2 \simeq 1$ GeV, but $m_p + m_n - m_D = E_b \sim 2.2$ MeV
  - at $kT \sim 1$ MeV the densities for p and n are similar to n_b

$$\frac{n_D}{n_b} = \frac{n_D}{n_p n_n} n_n \approx \frac{3}{4} \left[\frac{2\pi m_D}{m_p m_n kT}\right]^{3/2} e^{\frac{E_b}{kT}} \frac{n_b}{n_\gamma} n_\gamma \approx \frac{3}{4} \left[\frac{4\pi}{m_p kT}\right]^{3/2} e^{\frac{E_b}{kT}} \, 5.5 \cdot 10^{-10} \frac{(kT)^3}{\pi^2}$$

$$\approx 1.86 \cdot 10^{-9} \left[\frac{kT}{m_p}\right]^{3/2} \exp\left\{\frac{E_b}{kT}\right\} \quad (31)$$

  - which becomes smaller than 1 for $kT > 63.6$ keV

$\Rightarrow$ n_D is at higher kT exponentially suppressed: $\left.\frac{n_D}{n_b}\right|_{1\,\text{MeV}} \approx 5.3 \cdot 10^{-13}$

7. General Relativity — Big Bang nucleosynthesis (following Dodelson)

cosmic plasma at a temperature of 0.1 MeV (universe a few minutes old)

- the neutron-proton ratio comes mainly from $p + e \rightleftharpoons n + \nu$
  - for equilibrium we have $n_p^{(0)}/n_n^{(0)} = e^{(m_n - m_p)/kT} = e^{\mathcal{Q}/kT}$
  - the electrons are still in thermal equilibrium: $n_e = n_e^{(0)}$
  - the neutrinos decouple with this reaction completely
  - starting with the Boltzman equation

$$\frac{d(a^3 n_n)}{a^3\, dt} = n_n^{(0)} n_\nu^{(0)} \langle v\sigma \rangle \left\{ \frac{n_p}{n_p^{(0)}} - \frac{n_n}{n_n^{(0)}} \right\} = n_\nu^{(0)} \langle v\sigma \rangle \left\{ n_p e^{-\mathcal{Q}/kT} - n_n \right\} \quad (32)$$

  - $\lambda_{np} = n_\nu^{(0)} \langle v\sigma \rangle$ describes the rate for neutron-proton conversion
  - the neutron fraction $X_n = \frac{n_n}{n_p + n_n}$ "freezes" below $kT \sim 0.5$ MeV
  - nucleosynthesis starts at $kT = 70$ keV with $X_n \approx 0.11$

$\Rightarrow$ He mass ratio $\sim 4 \cdot X_n/2 \approx 22\%$

$*$ all neutrons are bound in He, since $E_b(^4\text{He}) \sim 28\,\text{MeV} \gg E_b(^2\text{D}) \sim 2.2\,\text{MeV}$

7. General Relativity — Big Bang nucleosynthesis (following Dodelson)

cosmic plasma at a temperature < 0.1 MeV (universe a few minutes old)

- only at $kT = 70$ keV nucleosynthesis really starts
  - nearly all D is processed further to ^{4}He
  - the left over D depends on the baryon density n_b

    $\Rightarrow$ measurement of the ratio D/H $\sim 3 \cdot 10^{-5}$ determines $\Omega_b \sim 0.04$
  - the produced ^{7}Li gives also tight bounds

cosmic plasma at a temperature < 14 eV

- Recombination goes by the process $p + e \rightleftharpoons H + \gamma$
  - for equilibrium we have $\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$
  - with the free electron fraction $X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}$ we get

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left(\frac{m_e kT}{2\pi} \right)^{3/2} e^{(m_H - m_p - m_e)/kT} \tag{33}$$

  - for $kT \sim \epsilon_0 = m_p + m_e - m_H$ all Hydrogen is ionized
  - recombination has to end in an excited state
    - ∗ a photon from ground state recombination has $E_\gamma \geq \epsilon_0$

    $\Rightarrow$ instant reionization
  - solving the equation for the electron fraction

    $\Rightarrow$ determines the decoupling temperature (or redshift ~ 1000)

$\Rightarrow$ CMB pattern: C_ℓ-distribution, CMB polarization

dark matter balance

- for simplicity we take a single particle X
  - with a (very) weak coupling $X + X \rightleftharpoons Y + Z$
- the Standard Model particles Y and Z are in thermal equilibrium

  $\Rightarrow$ $n_{Y,Z} = n_{Y,Z}^{(0)}$ and the number density equation becomes

$$\frac{d(a^3 n_X)}{a^3\, dt} = \langle v\sigma\rangle \left\{ (n_X^{(0)})^2 - (n_X)^2 \right\} \tag{34}$$

  - eventually we want to express the density in terms of the temperature kT
    - the temperature scales inverse to the scale factor: $T \sim a^{-1}$
    - in the radiation dominated time $H(a) = H(a_1)(a_1/a)^2$
    - with $x = m_X a$ we have $\frac{dx}{dt} = m\dot{a} = mHa = xH_m(x_m/x)^2 = H_m/x$
    - using $Y := a^3 n_X$ and $\frac{d}{dt} = \frac{dx}{dt}\frac{d}{dx} = \frac{H_m}{x}\frac{d}{dx}$ we get

$$\frac{H_m}{x}\frac{dY}{dx} = \frac{m_X^3}{x^3}\langle v\sigma\rangle \left(Y_{\text{EQ}}^2 - Y^2\right) \quad \text{or} \quad \frac{dY}{dx} = -\frac{\lambda}{x^2}\left(Y^2 - Y_{\text{EQ}}^2\right) \tag{35}$$

  - Riccati equation with $\lambda = m_X^3 \langle v\sigma\rangle / H_m$

dark matter balance

- for estimating $\lambda = m_X^3 \langle v\sigma \rangle / H_m$ we need
  - the mass m_X
  - the cross section $\sigma_{X+X \to Y+Z}$
  - the Hubble parameter H_m at the mass scale m_X
- for X from Supersymmetry (SUSY) we know the cross section σ
  - we get limits on the mass m_X
  - v and H_m are given by the Einstein-Boltzmann equations

$\Rightarrow$ for the lightest supersymmetric particle (LSP) $\neq$ gravitino

$$\Omega_X \sim 0.3 \left(\frac{x_f}{10}\right) \left(\frac{g_*(m_X)}{100}\right)^{1/2} \frac{10^{-39}\text{cm}^2}{\langle v\sigma \rangle} \tag{36}$$

- the gravitino couples as $\frac{E_X^2}{M_P^2} \sim \frac{(10^3\,\text{GeV})^2}{(10^{19}\,\text{GeV})^2} \sim 10^{-32} \ll \alpha_{\text{em}}$

  $\Rightarrow$ it is only at the Plank epoch in thermal equilibrium

  * all estimates for Ω_X have to be reassessed