6. General Relativity — Astro Particle Physics

description of the very early universe:

- curved space-time in the context of particle physics
- we need the particle physics description
  - formulated in Hamiltonian mechanics

    $\Rightarrow$ Quantum Mechanics
  - or formulated in Lagrangian mechanics

    $\Rightarrow$ using an action principle
    - Quantum Mechanics through the Pathintegral formulation
- Special Relativity is the local symmetry group

  $\Rightarrow$ Lagrangian mechanics as the unifying framework

? Can we formulate General Relativity in a Lagrangian picture?

  ? what is the dynamic degree of freedom?
  - the metric

  ? what are the consequences?

6. General Relativity — Lagrangian formulation of gravity

we can derive Einsteins equations also from an action:

- the starting point is the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g} \left(\mathcal{L}_m - \frac{R}{16\pi G} \right) \tag{1}$$

  - $R = R(g_{\mu\nu})$ is the Ricci scalar, G is Newtons gravitational constant
  - $g = \det(g_{\mu\nu})$ is the determinant of the metric

- why this $\sqrt{-g}$?
  - using the differential calculus the volume element should be written as a 4-form

$$d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \tfrac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{2}$$

  - that transforms under coordinate transformations $x \to x'$ with $\frac{\partial x^\mu}{\partial x'^\alpha} = \Lambda^\mu{}_\alpha$

$$d^4x = \tfrac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta dx'^\alpha \wedge dx'^\beta \wedge dx'^\gamma \wedge dx'^\delta = \det[\Lambda]\, d^4x' \tag{3}$$

  - the metric transforms under these coordinate transformations $x \to x'$

$$g'_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} \quad \Rightarrow \quad g' = \det[g'_{\alpha\beta}] = \det[\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu}] = \det[\Lambda]^2\, g \tag{4}$$

  - since $\det[\eta_{\mu\nu}] = \det[\mathrm{diag}(1,-1,-1,-1)] = -1 \quad \Rightarrow \quad g = \det[g_{\mu\nu}] < 0$ (5)

$\Rightarrow$ $(\sqrt{-g}\, d^4x)$ is invariant: $d^4x' \sqrt{-g'} = \det[\Lambda]^{-1} d^4x \sqrt{-g \det[\Lambda]^2} = d^4x \sqrt{-g}$ (6)

6. General Relativity — Lagrangian formulation of gravity

we can derive Einsteins equations also from an action:

- the variation of the action gives the Euler-Lagrange equations
- varying the Einstein-Hilbert action we get

$$\delta S = \int d^4x \left(\delta(\sqrt{-g}\mathcal{L}_m) - \delta(\sqrt{-g})\frac{g^{\alpha\beta}R_{\alpha\beta}}{16\pi G} - \sqrt{-g}\frac{(\delta g^{\alpha\beta})R_{\alpha\beta}}{16\pi G} - \sqrt{-g}\frac{g^{\alpha\beta}(\delta R_{\alpha\beta})}{16\pi G} \right) \quad (7)$$

  - the first term is the variation of the matter Lagrangian
  - the second term can be calculated from the identity

$$\mathrm{Tr}[\ln M] = \ln(\det[M]) \quad \Rightarrow \quad \mathrm{Tr}[M^{-1}\delta M] = \det[M]^{-1}\delta \det[M] \quad (8)$$

    * setting $M = g^{\mu\nu}$ we have

$$M^{-1} = g_{\mu\nu} \quad \text{and} \quad \det[M] = \det[g^{\mu\nu}] = \det[(g_{\mu\nu})^{-1}] = 1/\det[g_{\mu\nu}] = 1/g \quad (9)$$

    * so $\mathrm{Tr}[M^{-1}\delta M] = g_{\mu\nu}\delta g^{\mu\nu} = g\delta\frac{1}{g} = -g\frac{\delta g}{g^2} = -\frac{\delta g}{g}$

$$\Rightarrow \quad \delta\sqrt{-g} = \frac{-\delta g}{2\sqrt{-g}} = -\frac{1}{2\sqrt{-g}}(-g)g_{\mu\nu}\delta g^{\mu\nu} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} \quad (10)$$

  - the third term has already the wanted differential $\delta g^{\mu\nu}$
  - the fourth term gives a total divergence (see next slide)

  $\Rightarrow$ it does not contribute to the equations of motion we can derive Einsteins equations also from an action:

- the variation of the Ricci tensor is the contracted variation of the Riemann tensor:

$$\delta R_{\alpha\beta} = \delta R^{\lambda}{}_{\alpha\lambda\beta} = \delta(\delta^{\kappa}_{\lambda} R^{\lambda}{}_{\alpha\kappa\beta}) = \delta^{\kappa}_{\lambda}\delta[\partial_{\kappa}\Gamma^{\lambda}_{\alpha\beta} + \Gamma^{\lambda}_{\kappa\rho}\Gamma^{\rho}_{\alpha\beta} - (\kappa \leftrightarrow \beta)] \tag{11}$$

- the trick in the calculation is to realize, that $\delta\Gamma$ is a difference of two connections

  $\Rightarrow$ it is a tensor and we can calculate the covariant derivative:

$$\nabla_{\kappa}\delta\Gamma^{\lambda}_{\alpha\beta} = \partial_{\kappa}\delta\Gamma^{\lambda}_{\alpha\beta} + \Gamma^{\lambda}_{\kappa\nu}\delta\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\mu}_{\kappa\alpha}\delta\Gamma^{\lambda}_{\mu\beta} - \Gamma^{\mu}_{\kappa\beta}\delta\Gamma^{\lambda}_{\alpha\mu} \tag{12}$$

  – the antisymmetric part in $(\kappa \leftrightarrow \beta)$ gives

$$\begin{aligned}
\nabla_{\kappa}\delta\Gamma^{\lambda}_{\alpha\beta} - \nabla_{\beta}\delta\Gamma^{\lambda}_{\alpha\kappa} &= \partial_{\kappa}\delta\Gamma^{\lambda}_{\alpha\beta} + \Gamma^{\lambda}_{\kappa\nu}\delta\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\mu}_{\kappa\alpha}\delta\Gamma^{\lambda}_{\mu\beta} - \Gamma^{\mu}_{\kappa\beta}\delta\Gamma^{\lambda}_{\alpha\mu} \\
&\quad -\partial_{\beta}\delta\Gamma^{\lambda}_{\alpha\kappa} - \Gamma^{\lambda}_{\beta\nu}\delta\Gamma^{\nu}_{\alpha\kappa} + \Gamma^{\mu}_{\beta\alpha}\delta\Gamma^{\lambda}_{\mu\kappa} + \Gamma^{\mu}_{\beta\kappa}\delta\Gamma^{\lambda}_{\alpha\mu} \\
&= \partial_{\kappa}\delta\Gamma^{\lambda}_{\alpha\beta} + \Gamma^{\lambda}_{\kappa\mu}\delta\Gamma^{\mu}_{\alpha\beta} + \delta\Gamma^{\lambda}_{\kappa\mu}\Gamma^{\mu}_{\alpha\beta} - (\kappa \leftrightarrow \beta) \\
&= \delta[\partial_{\kappa}\Gamma^{\lambda}_{\alpha\beta} + \Gamma^{\lambda}_{\kappa\mu}\Gamma^{\mu}_{\alpha\beta}] - (\kappa \leftrightarrow \beta) \;=\; \delta R^{\lambda}{}_{\alpha\kappa\beta}
\end{aligned} \tag{13}$$

- so the term $g^{\alpha\beta}(\delta R_{\alpha\beta})$ can be written as

$$\begin{aligned}
g^{\alpha\beta}\delta R_{\alpha\beta} &= g^{\alpha\beta}(\nabla_{\lambda}\delta\Gamma^{\lambda}_{\alpha\beta} - \nabla_{\beta}\delta\Gamma^{\lambda}_{\alpha\lambda}) = \nabla^{\kappa}g_{\kappa\lambda}g^{\alpha\beta}\delta\Gamma^{\lambda}_{\alpha\beta} - \nabla^{\alpha}\delta\Gamma^{\lambda}_{\alpha\lambda} \\
&= \nabla^{\alpha}[g_{\alpha\beta}g^{\mu\nu}\delta\Gamma^{\beta}_{\mu\nu} - \delta\Gamma^{\lambda}_{\alpha\lambda}] = \nabla^{\alpha}V_{\alpha}
\end{aligned} \tag{14}$$

- and the integral over it gives only the boundary terms

$$\int_{\Omega} d^4x\sqrt{-g}\,\nabla^{\alpha}[g_{\alpha\beta}g^{\mu\nu}\delta\Gamma^{\beta}_{\mu\nu} - \delta\Gamma^{\lambda}_{\alpha\lambda}] = \left[g_{\alpha\beta}g^{\mu\nu}\delta\Gamma^{\beta}_{\mu\nu} - \delta\Gamma^{\lambda}_{\alpha\lambda}\right]_{\partial\Omega} \to 0 \tag{15}$$

6. General Relativity — Lagrangian formulation of gravity

we can derive Einsteins equations also from an action:

- varying the matter Lagrangian with respect to $\delta g^{\mu\nu}$

$$\frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g^{\mu\nu}} - \partial_\rho \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g^{\mu\nu}{}_{,\rho}} =: \frac{\sqrt{-g}}{2} T_{\mu\nu} \tag{16}$$

  gives the definition of the symmetric Hilbert (stress energy) tensor !

- the canonical stress energy tensor (using $\Phi_{,\mu} := \partial_\mu \Phi$)

$$T^\mu{}_\nu := \frac{\partial \mathcal{L}_m}{\partial \Phi_{,\mu}} \Phi_{,\nu} - \mathcal{L}_m \delta^\mu_\nu \tag{17}$$

  is not necessarily symmetric (when both indices are up or down)

- the Belinfante-Rosenfeld (stress-energy) tensor

$$T_B^{\mu\nu} = T^\mu{}_\lambda g^{\lambda\nu} + \partial_\lambda (S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu}) \tag{18}$$

  - adds a divergence of the spin part $S^{\mu\nu\lambda}$ to make it symmetric
  - it is equivalent to the Hilbert tensor

we can derive Einsteins equations also from an action:

- putting the parts of the variation with respect to $\delta g^{\mu\nu}$ together

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \;=\; \frac{\sqrt{-g}}{2} T_{\mu\nu} - (-\tfrac{1}{2}\sqrt{-g}\, g_{\mu\nu}) \frac{R}{16\pi G} - \sqrt{-g} \frac{R_{\mu\nu}}{16\pi G}$$
$$\;=\; -\tfrac{\sqrt{-g}}{16\pi G} \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R - 8\pi G T_{\mu\nu} \right) \qquad (19)$$

$\Rightarrow$ we get Einsteins equations

- we get also the field equations in the curved space time
  - they are the normal Euler-Lagrange equations

$$\frac{\delta S}{\delta \Phi} = 0 = \sqrt{-g} \frac{\partial \mathcal{L}_m}{\partial \Phi} - \partial_\rho \left(\sqrt{-g} \frac{\partial \mathcal{L}_m}{\partial \Phi_{,\rho}} \right) = \sqrt{-g} \left(\frac{\partial \mathcal{L}_m}{\partial \Phi} - \nabla_\rho \frac{\partial \mathcal{L}_m}{\partial \Phi_{,\rho}} \right) \qquad (20)$$

    * the last equality only holds, if $\frac{\partial \mathcal{L}_m}{\partial \Phi_{,\rho}} = V^\rho$ is a vector

complex scalar Lagrangian:

- using only first derivatives of the complex scalar field ϕ
  - the flat space Lagrangian

$$\mathcal{L}_\phi = (\partial^\mu \phi^*)(\partial_\mu \phi) - m_\phi^2 \phi^* \phi - V(\phi^* \phi) \tag{21}$$

  - can be written with covariant derivatives as

$$\mathcal{L}_\phi = g^{\mu\nu}(\nabla_\mu \phi^*)(\nabla_\nu \phi) - m_\phi^2 \phi^* \phi - V(\phi^* \phi) \tag{22}$$

- the canonical stress energy tensor

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}_\phi}{\partial \phi_{,\mu}} \phi_{,\nu} + \frac{\partial \mathcal{L}_\phi}{\partial \phi^*_{,\mu}} \phi^*_{,\nu} - \mathcal{L}_\phi \delta^\mu_\nu \tag{23}$$

$$= g^{\lambda\mu}[(\nabla_\lambda \phi^*)\phi_{,\nu} + (\nabla_\lambda \phi)\phi^*_{,\nu}] - \delta^\mu_\nu [g^{\lambda\kappa}(\nabla_\lambda \phi^*)(\nabla_\kappa \phi) - m_\phi^2 \phi^* \phi - V(\phi^* \phi)]$$

  is already symmetric (when both indices are up or down)
  - and the same as the Hilbert tensor (since $\nabla_\mu \phi = \partial_\mu \phi = \phi_{,\mu}$)

6. General Relativity — examples of matter Lagrangians

Maxwell Lagrangian:

- using the vector potential A_μ and its fieldstrength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
  - the flat space Lagrangian is $\mathcal{L}_A = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}F^2$ (24)
    - ∗ the vector potential can be written as a one-form $A = A_\mu dx^\mu$
    - ∗ the fieldstrength as a two-form $F = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu = dA$

    $\Rightarrow$ there is no metric dependence in $F_{\mu\nu}$
  - the Lagrangian on a curved manifold is just

$$\mathcal{L}_A = -\frac{1}{4}g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} = -\frac{1}{2}A^{\beta,\alpha}(A_{\beta,\alpha} - A_{\alpha,\beta}) \tag{25}$$

- the canonical stress energy tensor

$$\begin{aligned} T^\mu{}_\nu &= \frac{\partial \mathcal{L}_A}{\partial A_{\rho,\mu}}A_{\rho,\nu} - \delta^\mu_\nu \mathcal{L}_A \\ &= [-\tfrac{1}{2}g^{\beta\rho}g^{\alpha\mu}(A_{\beta,\alpha} - A_{\alpha,\beta}) - \tfrac{1}{2}A^{\beta,\alpha}(\delta^\rho_\beta\delta^\mu_\alpha - \delta^\rho_\alpha\delta^\mu_\beta)]A_{\rho,\nu} + \tfrac{1}{4}\delta^\mu_\nu F^2 \\ &= -F^{\mu\rho}A_{\rho,\nu} + \delta^\mu_\nu \tfrac{1}{4}F^2 \end{aligned} \tag{26}$$

  is not symmetric (when both indices are up or down)

6. General Relativity — examples of matter Lagrangians

Maxwell Lagrangian:

- using the generators of Lorentz transformations that act on four-vectors

$$(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} = i(g^{\alpha\mu}\delta^{\nu}_{\beta} - g^{\alpha\nu}\delta^{\mu}_{\beta}) \tag{27}$$

- the spin tensor

$$\begin{aligned} S^{\mu\nu\lambda} &= \tfrac{i}{2}\tfrac{\partial\mathcal{L}_A}{\partial A^{\alpha}_{,\mu}}(\mathcal{J}^{\nu\lambda})^{\alpha}{}_{\beta}A^{\beta} = -\tfrac{1}{2}\,g_{\rho\alpha}F^{\rho\mu}(g^{\alpha\nu}\delta^{\lambda}_{\beta} - g^{\alpha\lambda}\delta^{\nu}_{\beta})A^{\beta} \\ &= -\tfrac{1}{2}(F^{\nu\mu}A^{\lambda} - F^{\lambda\mu}A^{\nu}) = \tfrac{1}{2}(F^{\mu\nu}A^{\lambda} - F^{\mu\lambda}A^{\nu}) = -S^{\mu\lambda\nu} \end{aligned} \tag{28}$$

- and its derivative $\partial_{\lambda}(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} + S^{\lambda\mu\nu})$

$$\begin{aligned} &= \tfrac{1}{2}\partial_{\lambda}(F^{\mu\nu}A^{\lambda} - F^{\mu\lambda}A^{\nu} + F^{\nu\mu}A^{\lambda} - F^{\nu\lambda}A^{\mu} + F^{\lambda\mu}A^{\nu} - F^{\lambda\nu}A^{\mu}) \\ &= \partial_{\lambda}(F^{\lambda\mu}A^{\nu}) = -(\partial_{\lambda}F^{\lambda\mu})A^{\nu} - F^{\lambda\mu}A^{\nu}{}_{,\lambda} = -g_{\alpha\beta}F^{\alpha\mu}A^{\nu,\beta} \end{aligned} \tag{29}$$

- we get the Belinfante-Rosenfeld tensor

$$\begin{aligned} T^{\mu\nu}_{B} &= T^{\mu}{}_{\lambda}g^{\lambda\nu} + \partial_{\lambda}(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} + S^{\lambda\mu\nu}) \\ &= g_{\alpha\beta}F^{\alpha\mu}A^{\beta,\nu} + \tfrac{1}{4}g^{\mu\nu}F^{2} - g_{\alpha\beta}F^{\alpha\mu}A^{\nu,\beta} = -g_{\alpha\beta}F^{\alpha\mu}F^{\beta\nu} + \tfrac{1}{4}g^{\mu\nu}F^{2} \end{aligned} \tag{30}$$

- the Hilbert tensor

$$\begin{aligned} T_{\mu\nu} &= 2\tfrac{\partial}{\partial g^{\mu\nu}}[-\tfrac{1}{4}g^{\alpha\rho}g^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma}] - g_{\mu\nu}\mathcal{L}_A = -\tfrac{1}{2}[\delta^{\alpha}_{\mu}g^{\rho}_{\nu}g^{\beta\sigma} + g^{\alpha\rho}\delta^{\beta}_{\mu}g^{\sigma}_{\nu}]F_{\alpha\beta}F_{\rho\sigma} - g_{\mu\nu}\mathcal{L}_A \\ &= -\tfrac{1}{2}[F_{\mu\beta}F_{\nu\sigma}g^{\beta\sigma} + F_{\alpha\mu}F_{\rho\nu}g^{\alpha\rho}] - g_{\mu\nu}\mathcal{L}_A = -F_{\mu\beta}F_{\nu\sigma}g^{\beta\sigma} + \tfrac{1}{4}g_{\mu\nu}F^{2} \end{aligned} \tag{31}$$

  gives the same result

Dirac Lagrangian:

- using the Dirac spinor ψ and its adjoint $\bar{\psi} = \psi^\dagger \gamma^0$
  - the flat space Lagrangian $\mathcal{L}_\psi = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ (32)
  - can be written with a covariant derivative as $\mathcal{L}_\psi = \bar{\psi}(i g^{\mu\nu}\gamma_\mu \nabla_\nu - m)\psi$ (33)

  $\Rightarrow$ the covariant derivative has to use the spin connection !

- the canonical stress energy tensor is not symmetric (when both indices are up or down)

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}_\psi}{\partial \psi_{,\mu}} \psi_{,\nu} + \bar{\psi}_{,\nu} \frac{\partial \mathcal{L}_\phi}{\partial \bar{\psi}_{,\mu}} - \mathcal{L}_\psi \delta^\mu_\nu = \bar{\psi} i \gamma^\mu \psi_{,\nu} + \bar{\psi}_{,\nu} \cdot 0 - \delta^\mu_\nu \bar{\psi}(i\gamma^\lambda \nabla_\lambda - m)\psi \quad (34)$$

$\Rightarrow$ we need the Belinfante-Rosenfeld tensor $T_B^{\mu\nu} = T^\mu{}_\lambda g^{\lambda\nu} + \partial_\lambda (S^{\mu\nu\lambda} + S^{\nu\mu\lambda} + S^{\lambda\mu\nu})$ (35)

  - with the spintensor $S^{\lambda\mu\nu} = \frac{i}{2} \frac{\partial \mathcal{L}_\psi}{\partial \psi^\alpha_{,\lambda}} (\mathcal{J}^{\mu\nu})^\alpha{}_\beta \psi^\beta$ (36)
    * α is the spinor index of the Dirac spinor ψ
    * $(\mathcal{J}^{\mu\nu})^\alpha{}_\beta = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]^\alpha{}_\beta$ generates the Lorentz transformation on the spinors

- for the Hilbert tensor $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_\psi)}{\delta g^{\mu\nu}} = 2\frac{\delta \mathcal{L}_\psi}{\delta g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_\psi$ (37)
  - we have to include the dependence of the spin connections on the metric

6. General Relativity — paradigm of inflation

most models of inflation assume one or more scalar fields

- taking real scalar fields ϕ_k with the abbreviation $\phi^2 = \sum_k \phi_k^2$

$$S_\phi = \int d^4x\sqrt{-g}\,\mathcal{L}_\phi = \int d^4x\sqrt{-g}\,\tfrac{1}{2}\sum_k g^{\mu\nu}(\partial_\mu\phi_k)(\partial_\nu\phi_k) - V(\phi^2) \tag{38}$$

  - we get the field equations

$$\begin{aligned}\frac{\delta S_\phi}{\delta\phi_j(y)} = 0 &= \int d^4x\sqrt{-g}\sum_k g^{\mu\nu}(\partial_\mu\delta_k^j\delta(x-y))(\partial_\nu\phi_k) - V'(\phi^2)2\phi_j(x)\delta(x-y)\\ &= -\int d^4x\delta(x-y)\partial_\mu[\sqrt{-g}\,g^{\mu\nu}(\partial_\nu\phi_j)] + \sqrt{-g}\,V'(\phi^2)2\phi_j(y)\\ &= -\sqrt{-g}\,\left(\nabla_\mu[g^{\mu\nu}(\partial_\nu\phi_j)] + 2\phi_j V'\right) = -\sqrt{-g}\,\left(\nabla^\nu(\partial_\nu\phi_j) + 2\phi_j V'\right)\end{aligned} \tag{39}$$

  - and the stress energy tensor

$$T_{\mu\nu} = 2\frac{\partial\mathcal{L}_\phi}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_m = \sum_b(\partial_\mu\phi_b)(\partial_\nu\phi_b) - g_{\mu\nu}[\tfrac{1}{2}\sum_b(\partial^\rho\phi_b)(\partial_\rho\phi_b) - V(\phi^2)] \tag{40}$$

$$\begin{aligned}T_{00} &= \tfrac{1}{2}\textstyle\sum_b[\dot\phi_b^2 + (\vec\partial\phi_b)^2] + V(\phi^2) = H(\phi) & T_{0i} &= \textstyle\sum_b\dot\phi_b(\partial_i\phi_b)\\ T_{ii} &= \textstyle\sum_b[(\partial_i\phi_b)^2 + \frac{a^2}{2}\dot\phi_b^2 - \frac{a^2}{2}(\vec\partial\phi_b)^2] - a^2V(\phi^2) & T_{jk} &= \textstyle\sum_b(\partial_j\phi_b)(\partial_k\phi_b)\end{aligned} \tag{41}$$

  - for a homogeneous and isotropic field, we can set $(\partial_j\phi_b) \to 0$ so $(\vec\partial\phi_b) \to 0$, too)
  - that gives us $\rho = \frac{1}{2}\sum_b\dot\phi_b^2 + V$ and $\mathbf{p} = \frac{1}{2}\sum_b\dot\phi_b^2 - V$

6. General Relativity — paradigm of inflation

using only a single scalar field

- with a sizeable, but slowly varying potential $V(\phi^2)$
- and ϕ slowly varying, i.e. $\dot{\phi} \ll V$
  - we get the conditions like with the cosmological constant:
    - $\rho > 0$ and $\mathbf{p} < -\frac{1}{3}\rho$

$\Rightarrow$ the scalar field does not act like "normal" matter

- using the Friedmann equations for the Robertson-Walker metric

$$\tfrac{1}{2}R_{ii} + \tfrac{1}{6}R_{00} = \frac{\dot{a}^2 + k}{a^2} = \frac{4\pi G}{3}(\dot{\phi}^2 + 2V) \qquad \tfrac{1}{3}R_{00} = -\frac{\ddot{a}}{a} = \frac{8\pi G}{3}(\dot{\phi}^2 - V) \tag{42}$$

- together with the field equations, remembering $(\partial_j \phi_b) \to 0$,

$$0 = g^{\mu\nu}\nabla_\mu(\partial_\nu\phi) + 2\phi V' = g^{\mu\nu}\partial_\mu(\partial_\nu\phi) + g^{\mu\nu}\Gamma^\rho_{\mu\nu}(\partial_\rho\phi) + 2\phi V' = \ddot{\phi} - 3\frac{\dot{a}}{a}\dot{\phi} + 2\phi V' \tag{43}$$

- making the ansatz $a = ce^{H_{\text{infl}}t}$ we get

$$H^2_{\text{infl}} + \frac{k}{a^2} = \frac{4\pi G}{3}(2V + \dot{\phi}^2) \qquad H^2_{\text{infl}} = \frac{4\pi G}{3}(2V - 2\dot{\phi}^2) \tag{44}$$

$\Rightarrow$ $k = 4\pi G a^2 \dot{\phi}^2 \geq 0$ $\quad\Rightarrow$ only flat or de Sitter ... consistent with measurements

- and $H_{\text{infl}} \sim \frac{8\pi}{3}\frac{V}{M_P^2}$ for $H_{\text{infl}}t \sim 60$ $\Rightarrow$ $V \sim \frac{45}{2\pi}\frac{t_P}{t}M_P^2 \sim 8 \times 10^{-12} M_P^2 \sim (3.45 \times 10^{13}\text{GeV})^2$

how does inflation stop?

- even with the "slowly rolling" inflaton field, there is a small change
  $\Rightarrow$ the field value approaches its minimum
- the inflaton field dominates, but there are the other fields, too
  $\Rightarrow$ it can decay into the other fields
- with the increasing scale factor, the temperature drops
  - the effective potential can decrease to the minimum value
  - assuming for example a form like the SM Higgs potential
    * the value of the inflaton field stays large: ϕ is heavy
    * but the value of the potential can go to zero: $H_{\text{infl}} \rightarrow 0$

    $\Rightarrow$ inflation stops
  - including supersymmetry
    * the value of the potential is bounded from below, mostly positive
- the heavy inflaton decays into the other fields: reheating

6. General Relativity — paradigm of inflation

consequences of inflation

- the universe appears as flat, homogeneous, and isotropic
  - as is seen in the CMB
- the seeds for structure formation can be understood
  - as quantum fluctuations blown up to cosmic scales
- the primordial particle spectrum is thermal
  - from the decay of the inflaton

problems with inflation

- how ''natural'' are the conditions for inflation ?
- how can we understand the ''ordered'' state after inflation
  - coming from an ''unordered'' state before inflaton ?
    - it seems the initial conditions for inflation have to be more fine tuned than the conditions of the accelerating universe we see now

6. General Relativity — paradigm of inflation

current research issues regarding inflation

- the simplest assumptions are too restrictive
  - minimal coupling (ϕ is only used as the source in $T_{\mu\nu}$)
  - single field
  - $\partial_i \phi \sim 0$

$\Rightarrow$ generalized G-inflation (Galileon inflation):

- more terms in the Lagrangian

$$S = \int d^4x \sqrt{-g} \left(\sum_{i=1}^{5} \mathcal{L}_i - \frac{R}{16\pi G} \right) \qquad (45)$$

- the first term, $\mathcal{L}_1$, being a SM-like Higgs Lagrangian:

$$\mathcal{L}_1 = |D_\mu \phi|^2 + \lambda(|\phi|^2 - v^2)^2 \qquad (46)$$

  * ϕ the inflaton field and D_μ its covariant derivative
  * λ the self-coupling of the inflaton
  * v the vacuum expectation value of the inflaton field

6. General Relativity — paradigm of inflation

current research issues regarding inflation

- other terms in generalized G-inflation:

$$\begin{aligned} \mathcal{L}_2 &= K(\phi, X) \qquad\qquad \mathcal{L}_3 = -G_3 \Box \phi \\ \mathcal{L}_4 &= -G_4 R + G_{4X} \left[(\Box\phi)^2 - (\nabla^2\phi)^2 \right] \\ \mathcal{L}_5 &= -G_5 G_{\mu\nu} (\nabla^\mu \nabla^\nu \phi) - \tfrac{1}{6} G_{5X} \left[(\Box\phi)^3 - 3(\Box\phi)(\nabla^2\phi)^2 + 2(\nabla^2\phi)^3 \right] \end{aligned} \tag{47}$$

  - where the kinetic term of the inflaton is $X = -\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi)$
  - $K(\phi, X)$ is the Kähler potential
  - $G_i = G_i(\phi, X)$ is a paramterizing function with its derivative $G_{iX} = \frac{\partial G_i}{\partial X}$
  - and the abbreviations

$$\begin{aligned} (\Box\phi) &= (\nabla_\mu \nabla^\mu \phi) \qquad (\nabla^2\phi)^2 = (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) \\ (\nabla^2\phi)^3 &= (\nabla_\mu \nabla^\nu \phi)(\nabla_\nu \nabla^\rho \phi)(\nabla_\rho \nabla^\mu \phi) \end{aligned} \tag{48}$$

- modifies the allowed potential for the ''Higgs'' field
  - can easier accommodate initial conditions and end of inflation
  - at the ''expense'' of several additional functions
    - * thereby being again less predictive