# 5. General Relativity — physical cosmology

we have learned so far:

- SR as the background to non-gravitational physics
  - Lorentz transformations
  - their application to astronomical observations:
    - relativistic Doppler effect $\sim$ astronomical redshift
    - relativistic "beaming"
- GR as the covariant inclusion of gravity into SR
  - mathematical background of Riemannian geometry
    - curvature as an intrisic property of a manifold
  - validity of a local Lorentz frame:
    - $\Rightarrow$ physics on Earth is SR physics
- vacuum solutions for GR ... and their properties
- Friedmann-Lemaître models for GR ... and their properties
  - relation between scale parameter, time, distance, and energy content
- but no observations ... yet

# 5. General Relativity — physical cosmology

## what can we observe?

- light ... and only very recently also cosmic electrons, protons, and neutrinos
  - electromagnetic radiation
    - in different wavelengths: 8 pm (40 EHz) to 670 $\mu$m (450GHz) up to 2m (144kHz)
    - in different intensities: apparent magnitude from -27 (sun) to +32 (limit of HST)
    - from different directions: "the whole sky" ($4\pi$ spherical surface)

$\Rightarrow$ differentiating the light into different "identified" objects

- solar system objects: sun, planets, and moons
- stars
  - in our own galaxy (the Milky Way)
  - supernovae: also in other galaxies ... "standard candles"
- galaxies: different types of galaxies
- nebulae (clouds): interstellar dust
- cosmic (microwave) background radiation (CMB) or (CBR)

# 5. General Relativity — physical cosmology

## what do we conclude?

- understanding the solar sytem observations
  - ⇒ Keplers laws and Newtonian gravity ... Copernicus principle
- understanding the spectral lines of hydrogen
  - measuring the redshift of the light + SR
  - ⇒ velocity of the emitter
    - Cepheids (variable stars) ⇒ Hubbles law
    - parts of galaxies ⇒ rotation curves
    - supernovae Ia ⇒ modern measurement of Hubbles law
- measuring the CMB
  - perfect blackbody radiation, scale invariant, isotropic
    - with a dipole part: movement against the isotropic background
    - we calculated the effect in SR2 ... but more later
- galaxy surveys ⇒ universe seems *roughly* isotropic

⇒ confidence in homogeneity (from theory) and isotropy (observational)

⇒ confidence in Friedmann-Lemaître models ... as a first approximation

## Cosmological units

- the basic cosmological unit is the Hubble parameter

$$H_0 \sim 69.32 \pm 0.80 \frac{\text{km/s}}{\text{Mpc}} \sim 2.24 \times 10^{-18}/\text{s} \quad \text{measured 2012 by WMAP} \tag{1}$$

  - derived from that is the Hubble time

$$t_H = 1/H_0 \sim 4.46 \times 10^{17}\text{s} = 14.125 \times 10^9\text{yr} = 14.125\,\text{Gyr} \tag{2}$$

  - derived from that is the Hubble length

$$\ell_H = ct_H \sim 1.336 \times 10^{26}\text{m} = 4331\,\text{Mpc} \tag{3}$$

- using Newton's gravitational constant $G_N = 6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$
  - one defines the Planck mass $M_P = \sqrt{\frac{\hbar c}{G_N}}$ or the Planck energy

$$E_P = M_P c^2 = \sqrt{\hbar c^5/G_N} = 1.22 \times 10^{19}\text{GeV} \tag{4}$$

  - the Planck length

$$\ell_P = \hbar c/E_P = \sqrt{\hbar G_N/c^3} = 1.62 \times 10^{-35}\,\text{m} \tag{5}$$

  - and the Planck time

$$t_P = \ell_P/c = \hbar/E_P = \sqrt{\hbar G_N/c^5} = 5.39 \times 10^{-44}\,\text{s} \tag{6}$$

## mass density

- the critical density

$$\rho_c = \frac{3H_0^2}{8\pi G_N} \sim 9.7 \times 10^{-27}\,\frac{\text{kg}}{\text{m}^3} \sim 5.45\frac{\text{GeV}/c^2}{\text{m}^3} \sim 5.8\,\text{protons}/\text{m}^3 \tag{7}$$

- the density of luminous baryonic matter (i.e. stars) is estimated
  - by the product of the luminosity density $\rho_L \approx 2 \times 10^8 L_\odot/\text{Mpc}^3$
    - estimated from galaxy counts
  - and the mass over luminosity ratio $M/L \approx 4M_\odot/L_\odot$
    - averaged over different samples of stars and galaxies

$$\begin{aligned} \rho_{\text{lum}} &\sim 2 \times 10^8 L_\odot/\text{Mpc}^3 \times 4M_\odot/L_\odot \sim 8 \times 10^8 \cdot 1.99 \times 10^{30}\text{kg}/(30.857 \times 10^{15}\text{m})^3 \\ &\sim 5.4 \times 10^{-29}\,\frac{\text{kg}}{\text{m}^3} \sim 0.03\frac{\text{GeV}/c^2}{\text{m}^3} \sim 0.032\,\text{protons}/\text{m}^3 \end{aligned} \tag{8}$$

  $\Rightarrow$ that gives a density $\Omega_{\text{lum}} \sim 0.0056$

- there is also interstellar nonluminous gas
  - mainly hydrogen and helium
  - hydrogen absoption lines were seen in Quasar spectra, indicating $\Omega_{\text{gas}} \sim 0.04$

  $\Rightarrow$ that gives a baryonic density $\Omega_{\text{B}} = \Omega_{\text{lum}} + \Omega_{\text{gas}} \sim 0.0056 + 0.04 = 0.0456$
  - a consistent estimate comes from Big Bang nucleosynthesis: $\Omega_{\text{B}} \sim 0.043$

# 5. General Relativity — cosmological measurements

## radiation

- obviously from stars
- also from the cosmic background radiation (CBR)
  - a blackbody radiation with a temperature of $T = 2.725$ K
  - counting the number density of photons we get

$$dn = \frac{g}{h^3c^3} \cdot \frac{d^3p}{e^{E/k_BT} - 1} = \frac{4\pi g}{(2\pi)^3\hbar^3c^3} \cdot \frac{E^2\,dE}{e^{E/k_BT} - 1} \tag{9}$$

    * $g$ counts the possible states: for the photon $g = 2$ polarization states

  - using $\int_0^\infty \frac{x^2 \mathrm{d}x}{e^x - 1} = 2\,\zeta(3) \sim 2.40411$ we can integrate to get

$$n_\gamma = \frac{\zeta(3)}{\pi^2} g \left(\frac{k_BT}{\hbar c}\right)^3 \sim 4.1 \times 10^8/m^3 \tag{10}$$

- the energy density of these photons is

$$d\rho_\gamma = \frac{g}{h^3c^3} \cdot \frac{Ed^3p}{e^{E/k_BT} - 1} = \frac{4\pi g}{h^3c^3} \cdot \frac{E^3\,dE}{e^{E/k_BT} - 1} \quad \text{or} \quad \rho_\gamma = \frac{\hbar c\, g}{2\pi^2}\frac{\pi^4}{15}\left(\frac{k_BT}{\hbar c}\right)^4 \sim 0.264\frac{\mathrm{MeV}}{\mathrm{m}^3} \tag{11}$$

$\Rightarrow$ $\Omega_\gamma = \rho_\gamma/\rho_c \sim \frac{0.264\mathrm{MeV}}{5.45\mathrm{GeV}} \sim 4.85 \times 10^{-5}$

  - three orders of magnitude smaller than $\Omega_\mathrm{B}$

# 5. General Relativity — Temperature

## radiation

- we know from the conservation of the stress energy tensor
  - $\rho_\gamma$ scales with the scale factor $a$ as $\rho_\gamma \propto a^{-4} \propto (1+z)^4$
- we saw from the Stefan-Boltzmann law $\rho_\gamma \propto T^4$
  - derived before by integrating the energy density according to Bose-Einstein statistics

$\Rightarrow$ the temperature $T$ scales like the inverse scale factor $T \propto a^{-1}$
  - for a radiation dominated universe

- Thermodynamics tells
  - interacting systems try to reach thermal equilibrium
  - baryonic matter and radiation can interact

  $\Rightarrow$ $T_{\text{radiation}} \approx T_{\text{baryonic matter}}$ in equilibrium

    * we can call this radiation temperature the temperature of the universe

- when the scale factor shrinks (back in time) the temperature grows

  $\Rightarrow$ a Hot Big Bang

## 5. General Relativity — baryonic matter

cosmological baryonic matter today is described as dust

- no interaction between its "molecules"
  - these molecules are
    - hydrogen atoms of interstellar clouds
    - stars
    - galaxies, galaxy clusters, . . .
- their interaction to radiation is very weak, too
  - stars and galaxies emit light
    - but the absorbtion does not change their state
  - interstellar clouds (nebulae) emit/rescatter the light they absorb
    - ⇒ there it makes more sense to talk about their temperature

⇒ description of cosmological baryonic matter as a perfect fluid without pressure is justified

- only radiation interacts and has pressure in this picture

# 5. General Relativity — baryonic matter

## inhomogeneities in the dust

- Einstein's equations are local
  - ⇒ they describe parts of the universe, too
    - for instance a gas cloud . . .
- "handwaving" description:
  - outer atoms are attracted to the center of the cloud
  - they gain kinetic energy from their fall in the gravitational potential
  - they scatter ⇒ energy distributes ⇒ gas heats up ⇒ gas pressure
  - ⇒ thermodynamic description of the gas cloud when pressure stops the collaps
    - ⇒ Einstein–Boltzmann equations for the exact description
- what happens to a gas cloud?
  - Newton: denser regions experience stronger gravitational attraction
    - a dilute gas of hydrogen has a low density ⇒ weak gravitational field
    - being initially at rest ⇒ low velocity of the atoms
    - ⇒ Newtonian limit applicable
  - ⇒ inhomogeneities grow through gravitation: gravitational collaps

# 5. General Relativity — structure formation

## description of the gas cloud

- Newtonian limit allowed (initially) $\Rightarrow$ linearized Einstein equations
  - linearizing the metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$
  - assuming homogeneity and isotropy
    - they are only slightly perturbed

  $\Rightarrow$ use Robertson-Walker line element (for the locally flat background)

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \approx dt^2 - a^2(t)\eta_{jk}\, dx^j\, dx^k + \mathcal{O}(h) \tag{12}$$

    - $\eta_{jk} = \delta_{jk}$ describes the Euclidean metric of flat space
  - a multipole expansion of the metric perturbations $h_{\mu\nu}$ along the $\hat{z}$-axis gives
    - scalar perturbations: $h_{00} = 2\Psi$ and $h_{jk} = \eta_{jk}\, 2\Phi\, a^2$
    - vector perturbations: only $h_{13}$ and $h_{23}$ are non-zero
    - tensor perturbations: only $h_{11} = -h_{22} =: h_+$ and $h_{12} =: h_\times$ are non-zero

- a density contrast $\delta$ on top of a background density $\rho_{\text{bg}}$

$$\rho(t, \vec{x}) = \rho_{\text{bg}}(t, \vec{x})[1 + \delta(t, \vec{x})] \tag{13}$$

# 5. General Relativity — structure formation

## stress energy of the gas cloud

- the stress energy tensor in comoving coordinates

$$T_{\mu\nu} = \rho\, U_\mu U_\nu - \mathbf{p}(g_{\mu\nu} - U_\mu U_\nu) = g_{\mu\lambda}[\rho\, U^\lambda U_\nu - \mathbf{p}(\delta^\lambda_\nu - U^\lambda U_\nu)] \tag{14}$$

  – was composed from its components

$$T^0{}_0 = \rho = \rho_{\text{bg}}(1+\delta) \quad \text{and} \quad T^j{}_k = -\mathbf{p}\,\delta^j_k \tag{15}$$

- using the equation of state $\mathbf{p} = w\rho$

  $\Rightarrow$ we only need the density and not anymore the pressure

## conservation of the stress energy $\nabla^\mu T_{\mu\nu} = 0$

- requires the Christoffel symbols in terms of the metric perturbations $h_{\mu\nu}$
- evaluating the conservation law to zero order in $\delta$ and $h$

  $\Rightarrow$ the normal evolution of the background density $\rho_{\text{bg}} \propto a^{-3(1+w)}$

  $=$ consistency condition

## 5. General Relativity — structure formation

### conservation of the stress energy $\nabla^\mu T_{\mu\nu} = 0$

- evaluating the conservation law to first order in $\delta$ and $h$
  - a relation between derivatives of the parameters
    $\Rightarrow$ the perturbation expansions in $\delta$ and $h$ are linked:

$$\mathcal{O}(\delta^n) = \mathcal{O}(h^n) \tag{16}$$

  $=$ only a single perturbation expansion

### linearized Einstein equations

- solving the perturbation series in $\delta$ and $h$
  - zero order $\Rightarrow$ normal Friedmann equations with $\rho_{\text{bg}}$
  - first order $\Rightarrow$ evolution of the density contrast $\delta$

$\Rightarrow$ seeds for structure formation
  - cosmic Walls
  - cosmic Strings
  - point-like (spherically symmetric)

# 5. General Relativity — structure formation

## calculating the time for a gas cloud to collapse

- taking a spherical cloud of constant mass $M$ and initial radius $r_0$
- let a shell of thickness $dr$ contract under the gravity of the cloud
  - its mass $$m = 4\pi\rho r^2\, dr \tag{17}$$
  - its Newtonian gravitational potential $$V(r) = -\frac{GMm}{r} \tag{18}$$
- falling from $r_0$ to $r$ converts the potential difference
$$\Delta V = -\frac{GMm}{r_0} + \frac{GMm}{r} \tag{19}$$
  into kinetic energy $$\Delta E = E(r) - E(r_0) = \tfrac{m}{2}[v^2 - v_0^2] \tag{20}$$
  - in the Newtonian limit we get $\Delta V = \Delta E$ or $$v^2 = 2GM[\tfrac{1}{r} - \tfrac{1}{r_0}] + v_0^2 \tag{21}$$

$\Rightarrow$ velocity (from the start $v_0 = 0$) $$v(r) = \frac{dr}{dt} = [2GM(r^{-1} - r_0^{-1})]^{1/2} \tag{22}$$
gives the free falling time $$t_{ff} = \int_0^{t_{ff}} dt = \int_{r_0}^{r} \frac{dr}{dr/dt} = \int_{r_0}^{r} \frac{dr}{v(r)} \tag{23}$$

## 5. General Relativity — structure formation

### calculating the time for a gas cloud to collapse

- the free falling time

$$t_{ff} = \int_{r_0}^{r} \frac{dr'}{[2GM(1/r' - 1/r_0)]^{1/2}} \xrightarrow{r\to 0} \frac{\pi}{2}\left[\frac{r_0^3}{2GM}\right]^{1/2} = \left[\frac{3\pi}{32G\rho}\right]^{1/2} \tag{24}$$

  - taking the density in terms of the critical density $\rho = \Omega\rho_c$

$$t_{ff} = \sqrt{3\pi}\left[32G\,\Omega\frac{3H_0^2}{8\pi G}\right]^{-1/2} = \frac{\pi}{2H_0\sqrt{\Omega}} = \frac{\pi\, t_H}{2\sqrt{\Omega}} \tag{25}$$

- before collapsing into a point, the thermodynamic pressure might stop the collaps
  - igniting of a star ... the first stars formed $\sim t_H/30$

$$\Rightarrow \frac{\pi\, t_H}{2\sqrt{\Omega}} \sim \frac{t_H}{30} \quad \text{or} \quad (15\pi)^2 \sim \Omega = \Omega_{\text{bg}}[1+\delta] \quad \text{or} \quad 1+\delta \sim \frac{(15\pi)^2}{\Omega_{\text{bg}}} \tag{26}$$

  - using the baryonic density $\Omega_{\text{B}} = 0.0456 \quad \Rightarrow \quad 1+\delta \sim 5 * 10^5 \gg 1$

  $\Rightarrow$ non-linear effects, linearized equations not enough $\Rightarrow$ full simulation

- we need additional matter for structure formation ... Dark Matter

## Dark Matter

- ''invisible'' in the sky
  - like planets, neutron stars, ...
- helps with the rotation curves of galaxies

**particle dark matter**: has to be ''invisible'', too

- neutral, only minimally coupled to light
  - ⇒ decouples earlier from the radiation
  - helps also in the initial phases of structure formation
- can not interact with nuclear of strong interactions
  - otherwise we would have seen it bound to atomic nuclei
  - ⇒ anomalous isotopes
- massive: with low kinetic energy today
  - otherwise it would wash out the density contrast

⇒ **WIMPs** (weakly interacting massive particles)

- weak interactions, like very heavy neutrinos: neutralinos, sneutrinos from SUSY
- or only gravitationally interacting (''GIMPs''): gravitinos

# 5. General Relativity — timeline in $z$

**looking back** ... assuming the best fit from $d_L(z)$ and other measurments

$$\Omega_\gamma \sim 4.85 \times 10^{-5} \qquad \Omega_m \sim 0.2736 \qquad \Omega_k \sim 0 \qquad \Omega_\Lambda \sim 0.726 \tag{27}$$

$\Rightarrow$ we can reconstruct a timeline in $z$:

| $z$ | time | T[K] | description |
|---|---|---|---|
| 0. | 13.72 Gyr | 2.725 | now |
| 0.38 | 9.56 Gyr | 3.76 | equality of matter density and cosmological constant |
| 0.6 | 8. Gyr | 4.36 | formation of the solar system |
| 1. | 6. Gyr | 5.45 | formation of superclusters, walls, and voids |
| 1.3 | 4.9 Gyr | 6.27 | formation of the thin disk of the Milky Way |
| 10.9 | .43 Gyr | 32.4 | reionization: first stars are ionizing the interstellar gas again |
| 940. | .5 Myr | 2564. | equality of baryonic matter and radiation |
| 1040. | 445. kyr | 2837. | Baryon decoupling and drag epoch |
| 1100. | 406. kyr | 3000. | Photon decoupling and last scattering (CMB) |
| 1370. | 283. kyr | 3736. | Recombination of $e^-$ and ions: roughly neutral universe |
| 3200. | 67. kyr | 8723. | Matter-radiation equality, including neutrinos |
| 5600. | 25. kyr | $15 \times 10^3$ | radiation dominance: $\rho_\gamma > \rho_m$ |
| $8.5 \times 10^8$ | 32. h | $\sim 10^7$ | Photon reheating: $e^+$-$e^-$ recombination |
| $2.6 \times 10^9$ | 10. min | $1.2 \times 10^9$ | Big Bang nucleosynthesis |

# 5. General Relativity — timeline in $t$

## starting from the Big Bang

- we can suggest a timeline in $t$:

| time until | T[GeV] | description |
|---|---|---|
| $10^{-43}$ s | $10^{19}$ GeV | Planck epoch: $t_P$, $E_P$ |
| $10^{-36}$ s | $10^{16}$ GeV | Grand unification epoch: all the forces of the SM are unified |
| $10^{-32}$ s (?) | $10^{14}$ GeV | end of inflation, Baryogenesis, Supersymmetry breaking |
| $10^{-12}$ s | $10^{3}$ GeV | electroweak symmetry breaking |
| $10^{-6}$ s | 1 GeV | quark gluon plasma |
| 0.8 s | 1.1 MeV | neutrino decoupling |
| 1 s | 1 MeV | hadronic phase |
| 10 s | 0.3 MeV | leptonic phase: photon reheating though pair annihilation |
| 20 min | 25 keV | nucleosynthesis |
| 70 kyrs | 2 eV | Matter domination, start of structure formation |
| 380 kyrs | 0.7 eV | recombination: matter becomes transparent to light $\Rightarrow$ CMB pattern |

- description of these early times needs particle physics
  $\Rightarrow$ astro particle physics ... statistical description $\Rightarrow$ Boltzmann equations