4. General Relativity — non-vacuum solutions

assuming the solution to be homogeneous and isotropic

- the space part of the curvature tensor has to be maximally symmetric:

$$R^{(3)}_{jk\ell m} = k(g^{(3)}_{j\ell} g^{(3)}_{km} - g^{(3)}_{jm} g^{(3)}_{k\ell}) \qquad \Rightarrow \qquad R^{(3)}_{km} = 2k g^{(3)}_{km} \tag{1}$$

- using Frobenius theorem, we can write the metric as

$$ds^2 = g_{00}(t')dt'^2 - g^{(3)}_{jk}(t')dx^j dx^k = dt^2 - a^2(t) g^{(3)}_{jk} dx^j dx^k \tag{2}$$

- isotropic and homogeneous definitely includes spherically symmetric

⇒ we can use Frobenius theorem again to write

$$g^{(3)}_{jk} dx^j dx^k = g_{rr}(r)dr^2 + r^2 d^2\Omega = e^{2\beta} dr^2 + r^2 d^2\Omega \tag{3}$$

- we can use our calculation of the Schwarzschild metric for the space part
  - by setting $\alpha(t,r) = 0$ and $\beta = \beta(r)$ we get

$$R^{(3)}_{rr} = \tfrac{2}{r}\partial_r\beta \qquad \text{and} \qquad R^{(3)}_{\varphi\varphi} = R^{(3)}_{\vartheta\vartheta} \sin^2\vartheta = (1 - e^{-2\beta}[-r(\partial_r\beta) + 1]) \sin^2\vartheta \tag{4}$$

  - using $R^{(3)}_{km} = 2k g^{(3)}_{km}$ we get from (rr): $2ke^{2\beta} = \frac{2}{r}\partial_r\beta$ and from $(\vartheta\vartheta)$:

$$2kr^2 = 1 - e^{-2\beta}[-r(\partial_r\beta) + 1] = 1 + e^{-2\beta}[r(kre^{2\beta}) - 1] = 1 + kr^2 - e^{-2\beta} \tag{5}$$

⇒ $e^{-2\beta} = 1 - kr^2$ and we get the Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d^2\Omega\right] \tag{6}$$

4. General Relativity — Robertson-Walker metric

features of
$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d^2\Omega\right] = dt^2 - a^2(t)d\sigma^2 \qquad (7)$$

- k can be from $\{-1, 0, +1\}$
  - the substitutions $k \to \frac{k}{|k|}$, $a \to a\sqrt{|k|}$, $r \to \frac{r}{\sqrt{|k|}}$ leave the metric invariant
- $d\sigma^2$, the space part of the metric, describes constant curvature:
  - $k = 1$ is called closed (de Sitter)
    - the substitution $r \to \sin\chi$ gives the metric of S^3: $d\sigma^2 = d\chi^2 + \sin^2\chi\, d^2\Omega$
  - $k = 0$ is called flat (Euclidean)
    - one has the metric of R^3: $d\sigma^2 = dr^2 + r^2 d^2\Omega = dx^2 + dy^2 + dz^2$
  - $k = -1$ is called open (anti-de Sitter)
    - the substitution $r \to \sinh\xi$ gives the metric: $d\sigma^2 = d\xi^2 + \sinh^2\xi\, d^2\Omega$
- the only function not determined by symmetry is the scale factor $a(t)$
  - $a(t)$ will be determined by the Einstein equations
    - that means: the energy content determines the size of the curvature

calculating the non-vanishing curvature functions

- using the definition $\dot{a} = \partial_t a$

$$\Gamma^t_{\ rr} = \tfrac{a\dot{a}}{1-kr^2} \qquad \Gamma^t_{\ \vartheta\vartheta} = a\dot{a}r^2 \qquad \Gamma^t_{\ \varphi\varphi} = a\dot{a}r^2 \sin^2\vartheta \qquad \Gamma^r_{\ tr} = \Gamma^\vartheta_{\ t\vartheta} = \Gamma^\varphi_{\ t\varphi} = \tfrac{\dot{a}}{a}$$
$$\Gamma^r_{\ rr} = \tfrac{kr}{1-kr^2} \qquad \Gamma^r_{\ \vartheta\vartheta} = -r(1-kr^2) \qquad \Gamma^r_{\ \varphi\varphi} = -r(1-kr^2)\sin^2\vartheta \qquad \Gamma^\vartheta_{\ r\vartheta} = \Gamma^\varphi_{\ r\varphi} = \tfrac{1}{r} \tag{8}$$
$$\Gamma^\vartheta_{\ \varphi\varphi} = -\sin\vartheta\,\cos\vartheta \qquad \Gamma^\varphi_{\ \vartheta\varphi} = \tfrac{\cos\vartheta}{\sin\vartheta}$$

- gives

$$R_{trtr} = \tfrac{a\ddot{a}}{1-kr^2} \qquad R_{t\vartheta t\vartheta} = a\ddot{a}r^2 \qquad R_{t\varphi t\varphi} = a\ddot{a}r^2\sin^2\vartheta \tag{9}$$
$$R_{r\vartheta r\vartheta} = -\tfrac{a^2(\dot{a}^2+k)r^2}{1-kr^2} \qquad R_{r\varphi r\varphi} = -\tfrac{a^2(\dot{a}^2+k)r^2\sin^2\vartheta}{1-kr^2} \qquad R_{\vartheta\varphi\vartheta\varphi} = -a^2(\dot{a}^2+k)r^4\sin^2\vartheta$$

- and contracting gives

$$R_{tt} = -3\frac{\ddot{a}}{a} \tag{10}$$
$$R_{rr} = \tfrac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} = -g_{rr}\tfrac{a\ddot{a}+2\dot{a}^2+2k}{a^2} \tag{11}$$
$$R_{\vartheta\vartheta} = (a\ddot{a}+2\dot{a}^2+2k)r^2 = -g_{\vartheta\vartheta}\tfrac{a\ddot{a}+2\dot{a}^2+2k}{a^2} \tag{12}$$
$$R_{\varphi\varphi} = (a\ddot{a}+2\dot{a}^2+2k)r^2\sin^2\vartheta = -g_{\varphi\varphi}\tfrac{a\ddot{a}+2\dot{a}^2+2k}{a^2} \tag{13}$$

$\Rightarrow$ there are only two independent components of the Ricci tensor:

$$R_{00} = -3\frac{\ddot{a}}{a} \qquad \text{and} \qquad R_{ii} = \frac{a\ddot{a}+2\dot{a}^2+2k}{a^2} \tag{14}$$

4. General Relativity — energy stress tensor

cosmological forms of matter

- taking stars and galaxies: we see no interaction except gravity
- the electromagnetic field has interaction
- assuming both of them in their rest frame
  - the average motion vanishes
    - the galaxy rotates, but does not move;
    - other galaxies move away, but summing over all of them gives no net motion
    - the sun emits light, but in all directions the same
  - they can be described as a cosmological "perfect fluid"
    - with density ρ : like the mass of a particle $\Rightarrow$ timelike
    - and pressure $\mathbf{p}$: pushing things away $\Rightarrow$ spacelike
- writing it covariantly with the timelike comoving coordinates U^μ
  - and $h_{\mu\nu} = g_{\mu\nu} - U_\mu U_\nu$, projecting to the spacelike hypersurface, orthogonal to U^μ

$$T_{\mu\nu} = \rho U_\mu U_\nu - \mathbf{p} h_{\mu\nu} = (\rho + \mathbf{p}) U_\mu U_\nu - \mathbf{p} g_{\mu\nu} \tag{15}$$

4. General Relativity — Friedmann equations

using the stress energy tensor of the perfect fluid

- we get its trace as: $T = g^{\mu\nu} T_{\mu\nu} = (\rho + \mathbf{p}) U^2 - 4\mathbf{p} = \rho - 3\mathbf{p}$ (16)
- Einstein equations in terms of the Ricci tensor are

$$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \tfrac{1}{2} T g_{\mu\nu}) = 8\pi G ((\rho + \mathbf{p}) U_\mu U_\nu - \mathbf{p} g_{\mu\nu} - \tfrac{1}{2}(\rho - 3\mathbf{p}) g_{\mu\nu}) \quad (17)$$

or in components

$$R_{00} = -3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3\mathbf{p}) \quad (18)$$

$$R_{ii} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{a^2} = 4\pi G(\rho - \mathbf{p}) \quad (19)$$

- rearranging gives the Friedmann equations (also found by Lemaître)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{k}{a^2} \qquad \frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3\mathbf{p}) \quad (20)$$

- $H = \frac{\dot{a}}{a}$ is the Hubble parameter
  - it describes the change of the scale parameter $a(t)$
  - the value of H for the universe today is $H_0 \sim 72 \frac{\text{km/s}}{\text{Mpc}} \sim 2.3 \times 10^{-18}/\text{s}$

4. General Relativity — solving Friedmann equations

Time evolution

- assuming "regular" matter and energy we have $\rho > 0$ and $\mathbf{p} \geq 0$
  - that ignores the cosmological constant
- then we get $\ddot{a} = -\frac{4\pi}{3} a G(\rho + 3\mathbf{p}) < 0$
  - since the scale factor $a(t)$ is positive, $\ddot{a}(t)$ is negative

  $\Rightarrow$ $\dot{a}$ gets smaller with time $\Rightarrow$ deceleration
- today t_0 we measure $\dot{a}_0 := \dot{a}(t_0) = H(t_0)a(t_0) =: H_0 a_0 > 0$
  - $a(t)$ gets smaller when we "go back" in time
  - going back far enough we reach a time, when $a(t)$ was zero!

  $\Rightarrow$ everything we see now had at that time no distance

  $\Rightarrow$ we can estimate the age of the universe as $t_{\text{Universe}} < a_0/\dot{a}_0 = 1/H_0$

  * we do not know the value of the scale factor today, though
- we see a naked spacelike singularity: the Big Bang
  - all timelike curves have their origin in that singularity

Energy conservation

- for a better understanding we have to look at the behaviour of $T_{\mu\nu}$
  - $T_{\mu\nu}$ is conserved, which means $\nabla^\mu T_{\mu\nu} = \nabla_\mu T^\mu_\nu = 0$
  - for the energy component $T^\mu_0 = g^{\mu\nu} T_{\nu 0} = \delta^\mu_0 \rho$ we get

$$0 = \nabla_\mu T^\mu_0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\rho} T^\rho_0 - \Gamma^\rho_{\mu 0} T^\mu_\rho \tag{21}$$

    * for the summed terms we have $\Gamma^\mu_{\mu 0} = \Gamma^r_{rt} + \Gamma^\vartheta_{\vartheta t} + \Gamma^\varphi_{\varphi t} = 3\frac{\dot a}{a}$ and

$$\begin{aligned} \Gamma^\rho_{\mu 0} T^\mu_\rho &= \Gamma^r_{rt} T^r_r + \Gamma^\vartheta_{\vartheta t} T^\vartheta_\vartheta + \Gamma^\varphi_{\varphi t} T^\varphi_\varphi = \tfrac{\dot a}{a}(g^{rr} T_{rr} + g^{\vartheta\vartheta} T_{\vartheta\vartheta} + g^{\varphi\varphi} T_{\varphi\varphi}) = \tfrac{\dot a}{a}(g^{\mu\nu} T_{\mu\nu} - g^{tt} T_{tt}) \\ &= \tfrac{\dot a}{a}(T - T_{00}) = \tfrac{\dot a}{a}(\rho - 3\mathbf{p} - \rho) = -3\tfrac{\dot a}{a}\mathbf{p} \end{aligned} \tag{22}$$

- for a perfect fluid we can write an equation of state: $\mathbf{p} = w\rho$
  - since we assume the laws of physics do not change, we get w independent of time

$\Rightarrow$ so energy conservation gives us $0 = \dot\rho + 3\dfrac{\dot a}{a}(\rho + \mathbf{p}) = \dot\rho + 3\dfrac{\dot a}{a}(1 + w)\rho$ (23)

  - this can be integrated: $\dfrac{\dot\rho}{\rho} = -3\,(1+w)\,\dfrac{\dot a}{a} \quad \Rightarrow \quad \rho \propto a^{-3(1+w)}$ (24)

4. General Relativity — solving Friedmann equations

equation of state

- the equation of state can be derived from $T_{\mu\nu}$
  - from the consideration of the perfect fluid we have $T = \rho - 3\mathbf{p}$
  - from the description of matter with a Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{25}$$

we can calculate $T_{\mu\nu}$:

$$T_{\mu\nu} = \frac{i}{2}\bar{\psi}\{\gamma_\mu, D_\nu\}\psi - g_{\mu\nu}\bar{\psi}(i\not{D} - m)\psi - F_\mu^{\ \lambda}F_{\lambda\nu} + \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \tag{26}$$

when applying the equation of motion $(i\not{D} - m)\psi = 0$, contraction gives

$$T = \frac{i}{2}\bar{\psi}2\not{D}\psi - 4\bar{\psi}(i\not{D} - m)\psi - F^{\lambda\nu}F_{\lambda\nu} + \frac{1}{4}4F^{\rho\sigma}F_{\rho\sigma} = m\bar{\psi}\psi \sim \rho \tag{27}$$

- that gives the equation of state for
  - photons: $\rho - 3\mathbf{p} = 0$ $\Rightarrow$ $w_\gamma = \frac{1}{3}$ ... radiation
  - fermions: $\rho - 3\mathbf{p} = \rho$ $\Rightarrow$ $w_d = 0$ ... dust

4. General Relativity — solving Friedmann equations

using the equation of state

- the scale factor shrinks when we go back
  - the energy density increases: $\rho_\gamma \propto a^{-4}$ and $\rho_m \propto a^{-3}$
    - when $a \to 0$ as we go back $\rho_\gamma, \rho_m \to \infty$
    - this is the singularity! (not that $a \to 0$)
- today (1998) we have a ratio $\rho_m/\rho_\gamma \sim 10^6$
  - but going back in time radiation was more important than today
- considering the total energy in a volume cube of length a: ρa^3

$$\frac{d}{dt}\rho a^3 = \dot{\rho} a^3 + 3\rho \dot{a} a^2 = a^3(\dot{\rho} + 3\rho\frac{\dot{a}}{a}) = a^3(-3\mathbf{p}\frac{\dot{a}}{a}) = -3\mathbf{p}\dot{a}a^2 \leq 0 \tag{28}$$

$\Rightarrow$ ρa^3 cannot increase with time $\Rightarrow$ $\rho a^2 \to 0$ as $a \to \infty$

- with the first Friedman equation $\dot{a}^2 = \frac{8\pi}{3} G \rho a^2 - k$
  - for $k \leq 0$: $\dot{a}^2 \xrightarrow[a\to\infty]{} 0$ or $1 \geq 0$ ... the expansion slows down
  - for $k = 1$: $\dot{a}^2 \xrightarrow[a\to\infty]{} -1 < 0$ ... a contradiction

  $\Rightarrow$ there has to be a_{max} and then the universe contracts again

4. General Relativity — Time evolution

a dust-only universe

- has $w = 0$ and we can write $\rho = ma^{-3}$
  - the Friedmann equations with the abbreviation $b = \frac{4\pi}{3}Gm$

$$\dot{a}^2 = 2ba^{-1} - k \qquad \ddot{a} = -ba^{-2} \tag{29}$$

can be solved parametrically by

$$\begin{array}{ccc} \text{for } k=-1 & \text{for } k=0 & \text{for } k=+1 \\ \begin{aligned} a &= b\,(\cosh\phi - 1) \\ t &= b\,(\sinh\phi - \phi) \end{aligned} & a = \left(\frac{9b}{2}\right)^{1/3} t^{2/3} & \begin{aligned} a &= b\,(1 - \cos\phi) \\ t &= b\,(\phi - \sin\phi) \end{aligned} \end{array} \tag{30}$$

    - ∗ for calculating $\dot{a}$ one has to use the chain-rule $\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi}$
    - ∗ and calculate $\frac{d\phi}{dt} = (\frac{dt}{d\phi})^{-1}$
  - for $k = 1$ we have $a_{\text{max}} = 2b$ after a finite time t_{max}
    - ∗ and after the finite time $2t_{\text{max}}$ we get $a(2t_{\text{max}}) = 0 \quad \Rightarrow \quad$ "Big Crunch"
- expanding into a Taylor series around $t = 0$, one sees in all solutions:

$$a \sim \phi^2 \qquad t \sim \phi^3 \qquad \Rightarrow \qquad a \propto t^{2/3} \qquad \text{for small } t \tag{31}$$

a radiation-only universe

- has $w = \frac{1}{3}$ and we can write $\rho = Ea^{-4}$
  - the Friedmann equations with the abbreviation $b^2 = \frac{8\pi}{3} GE$

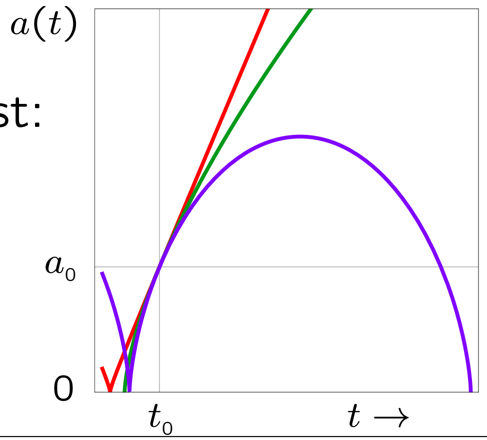
$$\dot{a}^2 = b^2 a^{-2} - k \qquad \ddot{a} = -b^2 a^{-3} \tag{32}$$

can be solved by

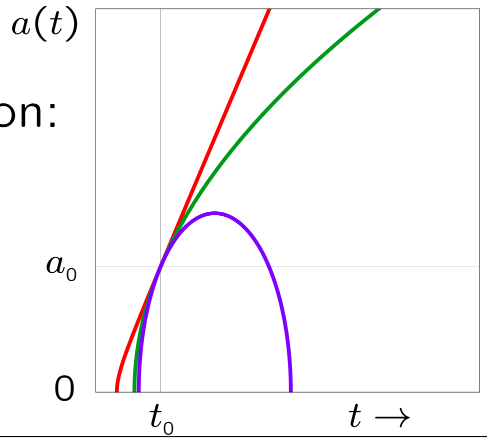
$$\begin{array}{ccc} \text{for } k = -1 & \text{for } k = 0 & \text{for } k = +1 \\ a = [(2b + t)t]^{1/2} & a = (4b)^{1/4}\, t^{1/2} & a = [(2b - t)t]^{1/2} \end{array} \tag{33}$$

  - for $k = 1$ we have a limited time $t < 2b$ $\Rightarrow$ "Big Crunch"
- for early times $t \ll b$ we have for all solutions: $a \propto t^{1/2}$

Dust:



Radiation:



4. General Relativity — Time evolution

a vaccum-only universe

- has $w = -1$ and we can write $\rho = \frac{\Lambda}{8\pi G}$
  - then either density ρ or pressure $\mathbf{p}$ become negative !
    - opposite to the assumptions of ”normal matter”
- the metric has a larger symmetry: the full Lorentz group
- the first Friedmann equation $\dot{a}^2 = \frac{\Lambda}{3} a^2 - k$ tells (34)
  - for $\Lambda < 0$ we have to have $k = -1$ (anti-de Sitter)
    - with $\frac{\Lambda}{3} = -b^2$ we get $a = b^{-1} \sin bt$ $\Rightarrow$ ”Big Crunch”
    - consistent with the second Friedmann equation $\ddot{a} = -b^2 a$
  - for $\Lambda > 0$ we can write $\frac{\Lambda}{3} = b^2$ and get

$$\begin{array}{ccc} \text{for } k = -1 & \text{for } k = 0 & \text{for } k = +1 \\ a = b^{-1} \sinh bt & a = b^{-1} e^{bt} & a = b^{-1} \cosh bt \end{array} \tag{35}$$

    - consistent with the second Friedmann equation $\ddot{a} = b^2 a$
    - we have exponential growth $a \propto e^{bt}$ ... Inflation

today it seems that $\ddot{a} > 0$

$\Rightarrow$ the universe does not contain only normal matter

- conservation of the stress-energy tensor still holds
  - $\Rightarrow$ the scaling of the energy density stays the same:

$$\rho_\gamma \propto a^{-4} \qquad \rho_m \propto a^{-3} \qquad \rho_\Lambda \propto a^0 \tag{36}$$

  - $\Rightarrow$ Λ was smaller (= less important) in earlier times

- introducing the "critical density" $\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$
- the "density parameter" $\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{\text{crit}}}$
  - the first Friedmann equation becomes

$$H^2 = \frac{\rho H^2}{\rho_{\text{crit}}} - \frac{k}{a^2} = \Omega H^2 - \frac{k}{a^2} \qquad \text{or} \qquad \Omega - 1 = \frac{k}{a^2 H^2} \tag{37}$$

- the different components of the stress energy tensor
  - can be written as dimensionless densities

$$\rho_{\text{total}} = \rho_\gamma + \rho_m + \rho_\Lambda = \rho_{\text{crit}}(\Omega_\gamma + \Omega_m + \Omega_\Lambda) = \rho_{\text{crit}}\Omega \tag{38}$$

measuring deceleration / acceleration

- defining the deceleration parameter $q = -\frac{\ddot{a}a}{\dot{a}^2}$
  - we can relate q to Ω using the second Friedmann equation

$$\begin{aligned} q &= -\frac{\ddot{a}a}{\dot{a}^2} = -H^{-2}\frac{\ddot{a}}{a} = \frac{4\pi G}{3H^2}(\rho + 3\mathbf{p}) = \frac{1}{2\rho_{\text{crit}}}\rho(1+3w) \\ &= \frac{1}{2}(1+3w)\Omega \end{aligned} \quad (39)$$

  - w describes the overall state of the universe
    - we know the values of w for radiation, dust, and vacuum
    - but what is the right mixture?

- how to measure q and Ω in a FLRW universe ?
  - there is no timelike Killing vector
    - that would give conserved quantities

measuring deceleration / acceleration

- there is a Killing tensor $K_{\mu\nu} = a^2 h_{\mu\nu} = a^2(g_{\mu\nu} - U_\mu U_\nu)$ (40)

  $\Rightarrow$ $K^2 := -K_{\mu\nu}V^\mu V^\nu$ is constant along a geodesic

- for a massive particle we have $V^\mu = \frac{1}{m}p^\mu$ and $V^2 = V^\mu V_\mu = 1$

$$K^2 = -a^2(V^2 - (U.V)^2) = -a^2((V^0)^2 - |\vec{V}|^2 - (V^0)^2) = a^2|\vec{V}|^2 \tag{41}$$

  $\Rightarrow$ $|\vec{V}| = \frac{1}{m}|\vec{p}| = \frac{K}{a}$ decreases as the universe expands

  $\Rightarrow$ a gas cools down

- for a photon we have $p^2 = p^\mu p_\mu = 0$
  - the comoving observer measures its frequency $\omega = (U.p)$

$$K^2 = -a^2(p^2 - (U.p)^2) = a^2\omega^2 \quad \Rightarrow \quad \omega = \frac{K}{a} \tag{42}$$

  - emitted with the frequency ω_1 at the scale factor a_1
  - we get the cosmological redshift

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\omega_1}{\omega_0} - 1 = \frac{a_0}{a_1} - 1 \tag{43}$$

4. General Relativity — measuring cosmological parameters

how can we measure time and distance?

- proper distance is measured between two events $\Delta s^2 = -(A-B)^2$
  - infinitesimally we can go along a radial line from A to B: $\Delta s = \int_A^B ds$

$$-(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d^2\Omega\right] \tag{44}$$

    * on a radial connection between A and B we have $dt = 0$ and $d^2\Omega = 0$

$$(ds)^2 = a^2(t)\frac{1}{1-kr^2}dr^2 \qquad \text{or} \qquad ds = a(t)\frac{1}{\sqrt{1-kr^2}}dr \tag{45}$$

  - so formally we can write the distance as

$$\Delta s = a(t)\int_{r_A}^{r_B} \frac{dr}{\sqrt{1-kr^2}} \tag{46}$$

$\Rightarrow$ Δs increases with time due to the expansion

- distance measurements in SR are done with light signals:
  - by the travel time of a light signal from A to B (and back): $ds = a(t) c\, dt$
  - but for large distances: which $a(t)$ should we take?

how can we measure time and distance?

- additionally: we have to find a frame so that
  all events on the measuring grid are at the same time
  $\Rightarrow$ comoving frame

- in the comoving frame
  - ”stationary” objects stay at the same comoving distance χ
  - light needs the conformal time η to travel this distance

$$dt = a(t)\, d\eta \qquad \text{or} \qquad \eta = \int_0^t \frac{dt'}{a(t')} \tag{47}$$

    * and infinitesimally: $d\chi = c\, d\eta$
  - the farthest comoving distance light can reach in a given time:
    $\Rightarrow$ comoving horizon

how can we measure time and distance?

- the comoving distance of a light source ($a_s(t_s)$) to us ($a_0(t_0) = 1$)
  - can be integrated along the light ray:

$$\frac{\chi(a_s)}{c} = \int_{t_s}^{t_0} d\eta = \int_{t_s}^{t_0} \frac{dt}{a(t)} = \int_{a_s}^{a_0} \frac{da}{a}\frac{dt}{da} = \int_{a_s}^{1} \frac{da}{a\dot{a}} = \int_{a_s}^{1} \frac{da}{a^2\frac{\dot{a}}{a}} = \int_{a_s}^{1} \frac{da}{a^2 H(a)} \quad (48)$$

- using the scale factor – redshift relation

$$1 + z = \frac{a_0}{a(z)} = \frac{1}{a(z)} \quad , \quad \text{so} \qquad dz = -a^{-2} da \quad (49)$$

- this comoving distance χ of a light source at z_s can be expressed as

$$\chi(z_s) = \int_{a_s}^{a_0} \frac{da}{a^2 H(a)} = \int_{z(a_s)}^{z(a_0)} (-dz) \frac{1}{H(a(z))} = \int_0^{z_s} \frac{dz}{H(z)} \quad (50)$$

how can we measure time and distance?

- the luminosity distance is $d_L = \sqrt{\frac{L}{4\pi F}}$
  - $L = \frac{E_s}{\Delta t_s}$ is the known absolute luminosity of the source
  - $F = \frac{E_o}{\Delta t_o * \text{surface}} = \frac{L(d)}{4\pi d^2}$ is the flux measured by the observer
- on the comoving grid the surface ist "constant" $4\pi d^2 = 4\pi\chi^2(z_s)$
  - the ratio of emitted over observed energy is the redshift $\frac{E_s}{E_o} = 1 + z_s$
  - the ratio of observer time over emitter time is also the redshift $\frac{\Delta t_o}{\Delta t_s} = 1 + z_s$

$$d_L = \sqrt{\frac{E_s}{\Delta t_s}\frac{\Delta t_o 4\pi\chi^2(z_s)}{4\pi E_o}} = \chi(z_s)\sqrt{\frac{E_s}{E_o}\frac{\Delta t_o}{\Delta t_s}} = \chi(z_s) * (1 + z_s) = (1 + z_s)\int_0^{z_s}\frac{dz}{H(z)} \tag{51}$$

  - we can measure d_L in dependence of the redshift z_s

- but how to calculate $H(z)$?

how can we measure time and distance?

- introducing a ”curvature density” $\Omega_k = \frac{-k}{a_0^2 H_0^2}$
- and taking all densities as defined today: $\Omega_{\gamma,m} = \frac{8\pi G}{3H_0^2}\rho_{\gamma 0,m0}$
- we can write the first Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \left[\Omega_\gamma \left(\frac{a_0}{a}\right)^4 + \Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_k \left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda\right] H_0^2 \qquad (52)$$

  – since we know the cosmological redshift $\frac{a}{a_0} = \frac{1}{1+z}$

$$H^2 = \left[\Omega_\gamma (1+z)^4 + \Omega_m (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\Lambda\right] H_0^2 \qquad (53)$$

- so we get the Hubbles law (in a ”modern” formulation)

$$d_L(z) = \frac{(1+z)}{H_0}\int_0^z dz' \left[\Omega_\gamma \left(1+z'\right)^4 + \Omega_m \left(1+z'\right)^3 + \Omega_k \left(1+z'\right)^2 + \Omega_\Lambda\right]^{-\frac{1}{2}} \qquad (54)$$

$\Rightarrow$ measuring the functional form of $d_L(z)$ determines also the Ω_i