3. General Relativity — Equivalence Principle

What do we require of a theory of gravitation ?

- it should reproduce the known classical physics
  - the Weak Equivalence Principle (WEP)
    - $*$ inertial mass equals gravitational mass
  - with Special Relativity we noticed:
    - $*$ mass is just a form of energy

- it should generalize the WEP
  - uniform acceleration is similar to an extended gravitational field
    - $*$ a free falling observer cannot detect the gravitational field

  $\Rightarrow$ the free falling observer replaces the inertial frame of SR

$\Rightarrow$ Einsteins Equivalence Principle:

”In small enough regions of spacetime, we only need SR;
it is impossible to detect the gravitational field”

How can we "derive" General Relativity ?

- without gravity a test particle should move on a straight line
  - like in Newtonian mechanics

- but what is a "straight line" in a curved spacetime?
  ⇒ a curve $x(\tau)$ with tangent vector $\frac{dx}{d\tau}$, constant along the curve

⇒ geodesic equation: $0 = \nabla_V V$ with $V = \frac{dx}{d\tau}$ or

$$\frac{dx^\mu}{d\tau}\left(\partial_\mu \frac{dx^\rho}{d\tau} + \Gamma^\rho_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1}$$

- locally we can always choose an orthonormal coordinate system
  - the Christoffel symbols vanish at the point P: $\Gamma^\rho_{\mu\nu}|_P = 0$
  ⇒ at P we get for the "straight line"

$$\left.\frac{d^2 x^\mu}{d\tau^2}\right|_P = 0 \qquad \text{with solution:} \qquad x^\mu = x^\mu_0 + v^\mu \tau \tag{2}$$

3. General Relativity — Einstein equations

How can we "derive" General Relativity ?

- Newtonian gravity has to be a limiting case for
  - constant and weak gravitational fields
  - slow moving test particles

$\Rightarrow$ the metric should have Minkovski form with a small perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{with} \qquad |h_{\mu\nu}| \ll 1 \tag{3}$$

- slow moving means $|\frac{dx^i}{d\tau}| \ll \frac{dx^0}{d\tau} = \frac{dt}{d\tau}$

  $\Rightarrow$ the geodesic equation becomes a perturbation series:

$$\frac{d^2t}{d\tau^2} = 0 \quad \text{and} \quad \frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\frac{dt}{d\tau}\frac{dt}{d\tau} = (\frac{dt}{d\tau})^2(\frac{d^2x^i}{dt^2} + \frac{1}{2}\eta^{ij}(-\partial_j h_{00})) = 0 \tag{4}$$

  - and the second looks like Newtons equation for gravity:

$$\frac{d^2x^i}{dt^2} = a^i = -\frac{d}{dx^i}\Phi = -\frac{d}{dx^i}(-\frac{GM}{r}) \tag{5}$$

$\Rightarrow$ we can identify the Newtonian limit for the metric as

$$g_{00} = 1 - 2\frac{GM}{r} \qquad \text{and} \qquad g_{ii} = -1 \tag{6}$$

3. General Relativity — Einstein equations

How can we "derive" General Relativity ?

- generalizing the Poisson equation for gravity

$$\nabla^2\Phi = 4\pi G\rho \tag{7}$$

  - we need second derivatives of the potential, i.e. the metric

    $\Rightarrow$ the Riemann curvature tensor ... or contractions of it

  - a generalization for the density

    $\Rightarrow$ the stress-energy tensor $T_{\mu\nu}$ with $T_{00} = \rho$

    * which is conserved: $\nabla^\mu T_{\mu\nu} = 0$

$\Rightarrow$ we have to find a conserved tensor, made out of $R^\rho{}_{\sigma\mu\nu}$

  - contracting the second Bianchi identity:

$$0 = g^{\mu\rho}g^{\nu\lambda}(R_{\mu\nu\rho\sigma;\lambda} + R_{\mu\nu\lambda\rho;\sigma} + R_{\mu\nu\sigma\lambda;\rho}) = R_{\nu\sigma;\lambda}g^{\nu\lambda} - R_{;\sigma} + R_{\mu\sigma;\rho}g^{\mu\rho} = 2\nabla^\mu R_{\mu\sigma} - \nabla_\sigma R \tag{8}$$

  $\Rightarrow$ tells us that the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is conserved

$\Rightarrow$ Einstein equations $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{9}$$

3. General Relativity — Vacuum solutions

Ricci flatness

- contracting the Einstein equations

$$g^{\mu\nu}(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}) = R(1 - \tfrac{4}{2}) = -R = 8\pi G g^{\mu\nu}T_{\mu\nu} = 8\pi G T \tag{10}$$

  we can rewrite them as

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} + \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}) \tag{11}$$

- vacuum means $T_{\mu\nu} = 0$. So we get $R_{\mu\nu} = 0$
  - solutions with this behaviour are called Ricci flat
    - * but that does not require $R^{\mu}_{\ \nu\rho\sigma} = 0$
- including the electro-magnetic field as $T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$

  ⇒ gives the electro-vacuum solutions
    - * they are not Ricci flat
- including the cosmological constant Λ as $T_{\mu\nu} = -g_{\mu\nu}\frac{\Lambda}{8\pi G}$

  ⇒ gives the ''Lambda-vacuum'' solutions
    - * de Sitter space (dS_4) with $\Lambda > 0$
    - * anti-de Sitter space (AdS_4) with $\Lambda < 0$

3. General Relativity — Vacuum solutions

symmetries can classify the possible solutions

- a vanishing $R^{\mu}{}_{\nu\rho\sigma}$ gives the maximally symmetric solution
  - Minkovski spacetime has the constant metric $\eta_{\mu\nu}$
  - $\Rightarrow$ allows invariance under all Lorentz transformations
- the spherically symmetric solution is the Schwarzschild metric
  - it does not change with time $\Rightarrow$ static
  - it is invariant under rotations around the central mass
  - inclusion of the EM field: the static Reissner-Nordström metric
- rotational symmetry around an axis: the stationary Kerr metric
  - inclusion of the EM field: the stationary Kerr-Newman metric
- since these solutions are static or stationary
  - there are no problems with time evolution and stability
  - $\Rightarrow$ used for studying features of spacetime: black holes, singularities

How can we express symmetries?

- We know from Noether's theorem :
  - symmetries are connected to conserved quantities
  - the derivative of conserved quantities vanishes

- with the Lie derivative one studies the change along the vector
  - if the Lie derivative $\mathcal{L}_V(X)$ vanishes

    $\Rightarrow$ X is invariant along V

  $\Rightarrow$ the vector V generates a symmetry transformation for X

- if $\mathcal{L}_V(g_{\mu\nu}) = 0$ $\Rightarrow$ V generates a symmetry for M
  - such a vectorfield V is called Killing vectorfield
  - the equation is called Killing equation $(\mathcal{L}_V(g_{\mu\nu}) = 0)$

$$0 = V^\rho(\nabla_\rho g_{\mu\nu}) + g_{\lambda\nu}(\nabla_\mu V^\lambda) + g_{\mu\lambda}(\nabla_\nu V^\lambda) = (\nabla_\mu V_\nu) + (\nabla_\nu V_\mu) \tag{12}$$

  - here we have used the definition of the Lie derivative on covectors from

$$\begin{aligned} \mathcal{L}_V(A^\mu\omega_\mu) = V^\nu\nabla_\nu(A^\mu\omega_\mu) &= V^\nu[(\nabla_\nu A^\mu)\omega_\mu + A^\mu(\nabla_\nu\omega_\mu)] \\ = \mathcal{L}_V(A^\mu)\omega_\mu + A^\mu\mathcal{L}_V(\omega_\mu) &= [V^\nu(\nabla_\nu A^\mu) - A^\nu(\nabla_\nu V^\mu)]\omega_\mu + A^\mu\mathcal{L}_V(\omega_\mu) \end{aligned} \tag{13}$$

3. General Relativity — Symmetries

Killing vectorfields express the symmetries

- for S^2 we have three Killing vector fields $V^{(1)}$, $V^{(2)}$, $V^{(3)}$
  - they "close" under commutation: $[V^{(j)}, V^{(k)}] = \epsilon^{jk\ell} V^{(\ell)}$

    $\Rightarrow$ they form the rotation group $SO(3)$
  - S^2 is maximally symmetric
- for a spherically symmetric M
  - we have to have three Killing vector fields $V^{(j)}$
  - these $V^{(j)}$ transport points around *within the same sphere*

    $\Rightarrow$ they foliate M (like onion shells)
- Frobenius theorem tells us:
  - we can pick coordinates b^k in the space spanned by $V^{(j)}$
  - and coordinates a^K outside the space spanned by $V^{(j)}$
  - then the metric in M can be written as

$$ds^2 = g_{JK}(a) da^J da^K + g_{jk}(a) db^j db^k \qquad (14)$$

Spherical symmetry

- choosing the coordinates on S^2 as (ϑ, φ) with $d^2\Omega = d^2\vartheta + \sin^2\vartheta d^2\varphi$
- choosing the coordinates outside S^2 as (a, b) we get the metric

$$ds^2 = g_{aa}da^2 + g_{ab}(dadb + dbda) + g_{bb}db^2 - r^2(a,b)d^2\Omega \tag{15}$$

  - inverting $r(a, b)$ to $b(a, r)$ we can replace b by r
  - finding $t(a, r)$ so that cross terms in (t, r) vanishes: $dt = \frac{\partial t}{\partial a}da + \frac{\partial t}{\partial r}dr$
  - making an ansatz with functions $m(t, r)$ and $n(t, r)$

$$\begin{aligned} ds^2 &= mdt^2 + ndr^2 - r^2d^2\Omega \\ &= m[(\tfrac{\partial t}{\partial a})^2da^2 + (\tfrac{\partial t}{\partial a})(\tfrac{\partial t}{\partial r})(dadr + drda) + (\tfrac{\partial t}{\partial r})^2dr^2] + ndr^2 - r^2d^2\Omega \end{aligned} \tag{16}$$

  - gives three equations

$$m(\tfrac{\partial t}{\partial a})^2 = g_{aa} \qquad m(\tfrac{\partial t}{\partial a})(\tfrac{\partial t}{\partial r}) = g_{ar} \qquad m(\tfrac{\partial t}{\partial r})^2 + n = g_{rr} \tag{17}$$

  - can be solved for m, n, and t, which gives then $a(t, r)$

- looking at the flat Minkovsky metric $ds^2 = dt^2 - dr^2 - r^2d^2\Omega$
  - we assume $m = e^{2\alpha(t,r)}$ positive and $n = -e^{2\beta(t,r)}$ negative

⇒ we get the metric $$ds^2 = e^{2\alpha(t,r)}dt^2 - e^{2\beta(t,r)}dr^2 - r^2d^2\Omega \tag{18}$$

  - now we have to solve the Einstein equations

3. General Relativity — spherical vacuum solutions

calculating the non-vanishing Christoffel symbols

- $\Gamma^t_{\mu\nu} = \frac{1}{2}g^{tt}(\partial_\mu g_{\nu t} + \partial_\nu g_{\mu t} - \partial_t g_{\mu\nu}) = \frac{1}{2}e^{-2\alpha}[(\partial_\mu \delta^t_\nu + \partial_\nu \delta^t_\mu)e^{2\alpha} - \partial_t g_{\mu\nu}]$ gives

$$\Gamma^t_{tt} = \tfrac{1}{2}e^{-2\alpha}[2\partial_t e^{2\alpha} - \partial_t e^{2\alpha}] = \partial_t\alpha \quad (19)$$
$$\Gamma^t_{tr} = \tfrac{1}{2}e^{-2\alpha}[\partial_r e^{2\alpha}] = \partial_r\alpha \quad (20)$$
$$\Gamma^t_{rr} = \tfrac{1}{2}e^{-2\alpha}[-\partial_t(-e^{2\beta})] = e^{-2(\alpha-\beta)}\partial_t\beta \quad (21)$$

- $\Gamma^r_{\mu\nu} = \frac{1}{2}g^{rr}(\partial_\mu g_{\nu r} + \partial_\nu g_{\mu r} - \partial_r g_{\mu\nu}) = -\frac{1}{2}e^{-2\beta}[(\partial_\mu \delta^r_\nu + \partial_\nu \delta^r_\mu)(-e^{2\beta}) - \partial_r g_{\mu\nu}]$ gives

$$\Gamma^r_{tt} = \tfrac{1}{2}e^{-2\beta}[\partial_r e^{2\alpha}] = e^{2(\alpha-\beta)}\partial_r\alpha \quad (22)$$
$$\Gamma^r_{tr} = \tfrac{1}{2}e^{-2\beta}[\partial_t e^{2\beta}] = \partial_t\beta \quad (23)$$
$$\Gamma^r_{rr} = \tfrac{1}{2}e^{-2\beta}[2\partial_r e^{2\beta} + \partial_r(-e^{2\beta})] = \partial_r\beta \quad (24)$$
$$\Gamma^r_{\vartheta\vartheta} = \tfrac{1}{2}e^{-2\beta}[\partial_r(-r^2)] = -e^{-2\beta}r \quad (25)$$
$$\Gamma^r_{\varphi\varphi} = \tfrac{1}{2}e^{-2\beta}[\partial_r(-r^2\sin^2\vartheta)] = -e^{-2\beta}r\sin^2\vartheta \quad (26)$$

- $\Gamma^\vartheta_{\mu\nu} = \frac{1}{2}g^{\vartheta\vartheta}(\partial_\mu g_{\nu\vartheta} + \partial_\nu g_{\mu\vartheta} - \partial_\vartheta g_{\mu\nu}) = -\frac{1}{2}r^{-2}[(\partial_\mu \delta^\vartheta_\nu + \partial_\nu \delta^\vartheta_\mu)(-r^2) - \partial_\vartheta g_{\mu\nu}]$ gives

$$\Gamma^\vartheta_{r\vartheta} = -\tfrac{1}{2}r^{-2}[\partial_r(-r^2)] = r^{-1} \quad (27)$$
$$\Gamma^\vartheta_{\varphi\varphi} = -\tfrac{1}{2}r^{-2}[-\partial_\vartheta(-r^2\sin^2\vartheta)] = -\sin\vartheta\cos\vartheta \quad (28)$$

- $\Gamma^\varphi_{\mu\nu} = \frac{1}{2}g^{\varphi\varphi}(\partial_\mu g_{\nu\varphi} + \partial_\nu g_{\mu\varphi} - \partial_\varphi g_{\mu\nu}) = -\frac{1}{2}r^{-2}\sin^{-2}\vartheta[(\partial_\mu \delta^\varphi_\nu + \partial_\nu \delta^\varphi_\mu)(-r^2\sin^2\vartheta)]$ gives

$$\Gamma^\varphi_{r\varphi} = \tfrac{1}{2}r^{-2}\sin^{-2}\vartheta[\partial_r r^2\sin^2\vartheta] = r^{-1} \quad (29)$$
$$\Gamma^\varphi_{\vartheta\varphi} = \tfrac{1}{2}r^{-2}\sin^{-2}\vartheta[\partial_\vartheta r^2\sin^2\vartheta] = \frac{\cos\vartheta}{\sin\vartheta} \quad (30)$$

3. General Relativity — spherical vacuum solutions

calculating from the Γs the non-vanishing Riemann tensor components

- $R_{tr\sigma\rho} = g_{tt}(\partial_\sigma \Gamma^t_{\rho r} - \Gamma^t_{\rho\lambda}\Gamma^\lambda_{\sigma r}) - (\rho \leftrightarrow \sigma) = e^{2\alpha}(\partial_\sigma \Gamma^t_{\rho r} - \Gamma^t_{\rho t}\Gamma^t_{\sigma r} - \Gamma^t_{\rho r}\Gamma^r_{\sigma r} - \partial_\rho \Gamma^t_{\sigma r} + \Gamma^t_{\sigma t}\Gamma^t_{\rho r} + \Gamma^t_{\sigma r}\Gamma^r_{\rho r})$ gives

$$\begin{aligned} R_{trtr} &= e^{2\alpha}(\partial_t \Gamma^t_{rr} - \Gamma^t_{rt}\Gamma^t_{tr} - \Gamma^t_{rr}\Gamma^r_{tr} - \partial_r \Gamma^t_{tr} + \Gamma^t_{tt}\Gamma^t_{rr} + \Gamma^t_{tr}\Gamma^r_{rr}) \\ &= e^{2\alpha}(\partial_t(e^{-2(\alpha-\beta)}\partial_t\beta) - (\partial_r\alpha)^2 - (e^{-2(\alpha-\beta)}\partial_t\beta)(\partial_t\beta) - \partial_r(\partial_r\alpha) + (\partial_t\alpha)(e^{-2(\alpha-\beta)}\partial_t\beta) + (\partial_r\alpha)(\partial_r\beta)) \\ &= e^{2\beta}[\partial_t^2\beta - (\partial_t\alpha)(\partial_t\beta) + (\partial_t\beta)^2] - e^{2\alpha}[\partial_r^2\alpha + (\partial_r\alpha)^2 - (\partial_r\alpha)(\partial_r\beta)] \end{aligned} \tag{31}$$

- $R_{t\vartheta\sigma\rho} = g_{tt}(\partial_\sigma \Gamma^t_{\rho\vartheta} - \Gamma^t_{\rho\lambda}\Gamma^\lambda_{\sigma\vartheta}) - (\rho \leftrightarrow \sigma) = -e^{2\alpha}(\Gamma^t_{\rho r}\Gamma^r_{\sigma\vartheta} - \Gamma^t_{\sigma r}\Gamma^r_{\rho\vartheta})$ gives

$$\begin{aligned} R_{t\vartheta t\vartheta} &= -e^{2\alpha}(\Gamma^t_{\vartheta r}\Gamma^r_{t\vartheta} - \Gamma^t_{tr}\Gamma^r_{\vartheta\vartheta}) = e^{2\alpha}(\partial_r\alpha)(-e^{-2\beta}r) \\ &= -e^{2(\alpha-\beta)}r(\partial_r\alpha) \end{aligned} \tag{32}$$

$$\begin{aligned} R_{t\vartheta r\vartheta} &= e^{2\alpha}(\Gamma^t_{rr}\Gamma^r_{\vartheta\vartheta}) = e^{2\alpha}(e^{-2(\alpha-\beta)}\partial_t\beta)(-e^{-2\beta}r) \\ &= -r(\partial_t\beta) \end{aligned} \tag{33}$$

- $R_{t\varphi\sigma\rho} = g_{tt}(\partial_\sigma \Gamma^t_{\rho\varphi} - \Gamma^t_{\rho\lambda}\Gamma^\lambda_{\sigma\varphi}) - (\rho \leftrightarrow \sigma) = -e^{2\alpha}(\Gamma^t_{\rho r}\Gamma^r_{\sigma\varphi} - \Gamma^t_{\sigma r}\Gamma^r_{\rho\varphi})$ gives

$$\begin{aligned} R_{t\varphi t\varphi} &= -e^{2\alpha}(\Gamma^t_{\varphi r}\Gamma^r_{t\varphi} - \Gamma^t_{tr}\Gamma^r_{\varphi\varphi}) = e^{2\alpha}(\partial_r\alpha)(-e^{-2\beta}r\sin^2\vartheta) \\ &= -e^{2(\alpha-\beta)}r\sin^2\vartheta\,(\partial_r\alpha) \end{aligned} \tag{34}$$

$$\begin{aligned} R_{t\varphi r\varphi} &= e^{2\alpha}(\Gamma^t_{rr}\Gamma^r_{\varphi\varphi}) = e^{2\alpha}(e^{-2(\alpha-\beta)}\partial_t\beta)(-e^{-2\beta}r\sin^2\vartheta) \\ &= -r\sin^2\vartheta\,(\partial_t\beta) \end{aligned} \tag{35}$$

3. General Relativity — spherical vacuum solutions

calculating from the Γs the non-vanishing Riemann tensor components

- $R_{r\vartheta\sigma\rho} = g_{rr}(\partial_\sigma \Gamma^r_{\rho\vartheta} - \Gamma^r_{\rho\lambda}\Gamma^\lambda_{\sigma\vartheta}) - (\rho \leftrightarrow \sigma) = -e^{2\beta}(\partial_\sigma \Gamma^r_{\rho\vartheta} - \Gamma^r_{\sigma r}\Gamma^r_{\rho\vartheta} - \Gamma^r_{\sigma\vartheta}\Gamma^\vartheta_{\rho\vartheta} - \Gamma^r_{\sigma\varphi}\Gamma^\varphi_{\rho\vartheta}) - (\rho \leftrightarrow \sigma)$ gives

$$\begin{aligned} R_{r\vartheta r\vartheta} &= -e^{2\beta}(\partial_r \Gamma^r_{\vartheta\vartheta} - \Gamma^r_{rr}\Gamma^r_{\vartheta\vartheta} - \Gamma^r_{r\vartheta}\Gamma^\vartheta_{\vartheta\vartheta} - \Gamma^r_{r\varphi}\Gamma^\varphi_{\vartheta\vartheta} - \partial_\vartheta \Gamma^r_{r\vartheta} + \Gamma^r_{\vartheta r}\Gamma^r_{r\vartheta} + \Gamma^r_{\vartheta\vartheta}\Gamma^\vartheta_{r\vartheta} + \Gamma^r_{\vartheta\varphi}\Gamma^\varphi_{r\vartheta}) \\ &= -e^{2\beta}(\partial_r(-e^{-2\beta}r) - (\partial_r\beta)(-e^{-2\beta}r) + (-e^{-2\beta}r)(r^{-1})) = -2r(\partial_r\beta) + 1 + r(\partial_r\beta) - 1 \\ &= -r(\partial_r\beta) \end{aligned} \tag{36}$$

- $R_{r\varphi\sigma\rho} = g_{rr}(\partial_\sigma \Gamma^r_{\rho\varphi} - \Gamma^r_{\rho\lambda}\Gamma^\lambda_{\sigma\varphi}) - (\rho \leftrightarrow \sigma) = -e^{2\beta}(\partial_\sigma \Gamma^r_{\rho\varphi} - \Gamma^r_{\sigma r}\Gamma^r_{\rho\varphi} - \Gamma^r_{\sigma\vartheta}\Gamma^\vartheta_{\rho\varphi} - \Gamma^r_{\sigma\varphi}\Gamma^\varphi_{\rho\varphi}) - (\rho \leftrightarrow \sigma)$ gives

$$\begin{aligned} R_{r\varphi r\varphi} &= -e^{2\beta}(\partial_r \Gamma^r_{\varphi\varphi} - \Gamma^r_{rr}\Gamma^r_{\varphi\varphi} - \Gamma^r_{r\vartheta}\Gamma^\vartheta_{\varphi\varphi} - \Gamma^r_{r\varphi}\Gamma^\varphi_{\varphi\varphi} - \partial_\varphi \Gamma^r_{r\varphi} + \Gamma^r_{\varphi r}\Gamma^r_{r\varphi} + \Gamma^r_{\varphi\vartheta}\Gamma^\vartheta_{r\varphi} + \Gamma^r_{\varphi\varphi}\Gamma^\varphi_{r\varphi}) \\ &= -e^{2\beta}(\partial_r(-e^{-2\beta}r\sin^2\vartheta) - (\partial_r\beta)(-e^{-2\beta}r\sin^2\vartheta) + (-e^{-2\beta}r\sin^2\vartheta)(r^{-1})) \\ &= -r\sin^2\vartheta\,(\partial_r\beta) \end{aligned} \tag{37}$$

- $R_{\vartheta\varphi\sigma\rho} = g_{\vartheta\vartheta}(\partial_\sigma \Gamma^\vartheta_{\rho\varphi} - \Gamma^\vartheta_{\rho\lambda}\Gamma^\lambda_{\sigma\varphi}) - (\rho \leftrightarrow \sigma) = -r^2(\partial_\sigma \Gamma^\vartheta_{\rho\varphi} - \Gamma^\vartheta_{\rho r}\Gamma^r_{\sigma\varphi} - \Gamma^\vartheta_{\rho\vartheta}\Gamma^\vartheta_{\sigma\varphi} - \Gamma^\vartheta_{\rho\varphi}\Gamma^\varphi_{\sigma\varphi}) - (\rho \leftrightarrow \sigma)$ gives

$$\begin{aligned} R_{\vartheta\varphi\vartheta\varphi} &= -r^2(\partial_\vartheta \Gamma^\vartheta_{\varphi\varphi} - \Gamma^\vartheta_{\varphi r}\Gamma^r_{\vartheta\varphi} - \Gamma^\vartheta_{\varphi\vartheta}\Gamma^\vartheta_{\vartheta\varphi} - \Gamma^\vartheta_{\varphi\varphi}\Gamma^\varphi_{\vartheta\varphi} - \partial_\varphi \Gamma^\vartheta_{\vartheta\varphi} + \Gamma^\vartheta_{\vartheta r}\Gamma^r_{\varphi\varphi} + \Gamma^\vartheta_{\vartheta\vartheta}\Gamma^\vartheta_{\varphi\varphi} + \Gamma^\vartheta_{\vartheta\varphi}\Gamma^\varphi_{\varphi\varphi}) \\ &= -r^2(\partial_\vartheta(-\sin\vartheta\,\cos\vartheta) - (-\sin\vartheta\,\cos\vartheta)(\tfrac{\cos\vartheta}{\sin\vartheta}) + (r^{-1})(-e^{-2\beta}r\sin^2\vartheta)) \\ &= r^2(\cos^2\vartheta - \sin^2\vartheta - \cos^2\vartheta + e^{-2\beta}\sin^2\vartheta) \\ &= r^2\sin^2\vartheta\,(e^{-2\beta} - 1) \end{aligned} \tag{38}$$

3. General Relativity — spherical vacuum solutions

contracting gives the Ricci tensor components, which have to be zero

- $R_{t\mu} = g^{tt} R_{ttt\mu} + g^{rr} R_{rtr\mu} + g^{\vartheta\vartheta} R_{\vartheta t \vartheta \mu} + g^{\varphi\varphi} R_{\varphi t \varphi \mu} = g^{rr} R_{tr\mu r} + g^{\vartheta\vartheta} R_{t\vartheta\mu\vartheta} + g^{\varphi\varphi} R_{t\varphi\mu\varphi}$ gives

$$\begin{aligned} R_{tt} &= -e^{-2\beta}(e^{2\beta}[\partial_t^2\beta - (\partial_t\alpha)(\partial_t\beta) + (\partial_t\beta)^2] - e^{2\alpha}[\partial_r^2\alpha + (\partial_r\alpha)^2 - (\partial_r\alpha)(\partial_r\beta)]) \\ &\quad -r^{-2}[-e^{2(\alpha-\beta)} r(\partial_r\alpha)] - r^{-2}\sin^{-2}\vartheta\,[-e^{2(\alpha-\beta)} r\sin^2\vartheta\,(\partial_r\alpha)] \\ &= -[\partial_t^2\beta - (\partial_t\alpha)(\partial_t\beta) + (\partial_t\beta)^2] + e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - (\partial_r\alpha)(\partial_r\beta) + 2r^{-1}(\partial_r\alpha)] \end{aligned} \tag{39}$$

$$R_{tr} = -r^{-2}[-r(\partial_t\beta)] - r^{-2}\sin^{-2}\vartheta\,[-r\sin^2\vartheta\,(\partial_t\beta)] = 2r^{-1}(\partial_t\beta) \tag{40}$$

- $R_{r\mu} = g^{tt} R_{trt\mu} + g^{rr} R_{rrr\mu} + g^{\vartheta\vartheta} R_{\vartheta r \vartheta \mu} + g^{\varphi\varphi} R_{\varphi r \varphi \mu} = g^{tt} R_{trt\mu} + g^{\vartheta\vartheta} R_{r\vartheta\mu\vartheta} + g^{\varphi\varphi} R_{r\varphi\mu\varphi}$ gives

$$\begin{aligned} R_{rr} &= e^{-2\alpha}(e^{2\beta}[\partial_t^2\beta - (\partial_t\alpha)(\partial_t\beta) + (\partial_t\beta)^2] - e^{2\alpha}[\partial_r^2\alpha + (\partial_r\alpha)^2 - (\partial_r\alpha)(\partial_r\beta)]) \\ &\quad -r^{-2}[-r(\partial_r\beta)] - r^{-2}\sin^{-2}\vartheta\,[-r\sin^2\vartheta\,(\partial_r\beta)] \\ &= e^{2(\beta-\alpha)}[\partial_t^2\beta - (\partial_t\alpha)(\partial_t\beta) + (\partial_t\beta)^2] - \partial_r^2\alpha - (\partial_r\alpha)^2 + (\partial_r\alpha)(\partial_r\beta) + 2r^{-1}(\partial_r\beta) \end{aligned} \tag{41}$$

- $R_{\vartheta\mu} = g^{tt} R_{t\vartheta t\mu} + g^{rr} R_{r\vartheta r\mu} + g^{\vartheta\vartheta} R_{\vartheta\vartheta\vartheta\mu} + g^{\varphi\varphi} R_{\varphi\vartheta\varphi\mu} = e^{-2\alpha} R_{t\vartheta t\mu} - e^{-2\beta} R_{r\vartheta r\mu} - r^{-2}\sin^{-2}\vartheta\, R_{\vartheta\varphi\mu\varphi}$ gives

$$\begin{aligned} R_{\vartheta\vartheta} &= e^{-2\alpha}[-e^{2(\alpha-\beta)} r(\partial_r\alpha)] - e^{-2\beta}[-r(\partial_r\beta)] - r^{-2}\sin^{-2}\vartheta\,[r^2\sin^2\vartheta\,(e^{-2\beta}-1)] \\ &= 1 - e^{-2\beta}[r(\partial_r\alpha) - r(\partial_r\beta) + 1] \end{aligned} \tag{42}$$

- $R_{\varphi\mu} = g^{tt} R_{t\varphi t\mu} + g^{rr} R_{r\varphi r\mu} + g^{\vartheta\vartheta} R_{\vartheta\varphi\vartheta\mu} + g^{\varphi\varphi} R_{\varphi\varphi\varphi\mu} = e^{-2\alpha} R_{t\varphi t\mu} - e^{-2\beta} R_{r\varphi r\mu} - r^{-2}\, R_{\vartheta\varphi\vartheta\mu}$ gives

$$\begin{aligned} R_{\varphi\varphi} &= e^{-2\alpha}[e^{2\alpha}(\partial_r\alpha)(-e^{-2\beta} r\sin^2\vartheta)] - e^{-2\beta}[-r\sin^2\vartheta\,(\partial_r\beta)] - r^{-2}\,[r^2\sin^2\vartheta\,(e^{-2\beta}-1)] \\ &= -e^{-2\beta} r\sin^2\vartheta\,(\partial_r\alpha) + e^{-2\beta} r\sin^2\vartheta\,(\partial_r\beta)] - e^{-2\beta}\sin^2\vartheta + \sin^2\vartheta \\ &= (1 - e^{-2\beta}[r(\partial_r\alpha) - r(\partial_r\beta) + 1])\sin^2\vartheta = R_{\vartheta\vartheta}\sin^2\vartheta \end{aligned} \tag{43}$$

$\Rightarrow$ we get four independent equations using R_{tr} and $R_{\vartheta\vartheta}$

- $R_{tr} = 2r^{-1}(\partial_t \beta) = 0$ tells us that $\beta = \beta(r)$

- taking the derivative with respect to t of $R_{\vartheta\vartheta}$

$$\partial_t R_{\vartheta\vartheta} = -e^{-2\beta}[r(\partial_t\partial_r\alpha) - r(\partial_t\partial_r\beta)] = -e^{-2\beta}r(\partial_t\partial_r\alpha) = 0 \quad (44)$$

  tells us:

$$\alpha(t,r) = \alpha(r) + g(t) \quad (45)$$

- rescaling $t \to t' = \tilde{g}(t)$ with $dt' = e^{g(t)}dt$ (and renaming t' as t) we get

$$ds^2 = e^{2\alpha(r)}dt^2 - e^{2\beta(r)}dr^2 - r^2 d^2\Omega \quad (46)$$

$\Rightarrow$ all metric components are independent of t

  $\Rightarrow$ the metric has a timelike Killing vector ∂_0 !

    $\Rightarrow$ such a metric is called stationary

  – if ∂_0 is orthogonal to a family of hypersurfaces (like S^2)

    $\Rightarrow$ the metric is called static using R_{tt}, R_{rr}, and $R_{\vartheta\vartheta}$

- since both, R_{tt} and R_{rr}, are zero, we get

$$0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = 2r^{-1}(\partial_r\alpha) + 2r^{-1}(\partial_r\beta) \qquad (47)$$

$\Rightarrow$ $\alpha = -\beta + \text{const}$

$\Rightarrow$ but the constant can be absorbed in a constant rescaling of t

- using $R_{\vartheta\vartheta}$ again we get

$$0 = 1 - e^{-2\beta}[-2r(\partial_r\beta) + 1] = 1 - \partial_r(re^{-2\beta}) \qquad (48)$$

which has the solution

$$re^{-2\beta} = r + \mu \qquad \text{or} \qquad e^{-2\beta} = 1 + \frac{\mu}{r} = e^{2\alpha} \qquad (49)$$

- comparing with the weak field limit for $r \to \infty$ we get $\mu = -2GM$
  – and the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2d^2\Omega \qquad (50)$$

the Schwarzschild metric describes the vacuum outside a spherical mass

- the metric is asymptotically flat:
  - for $M \to 0$ or $r \to \infty$ we recover the Minkovsky metric
- it has the physical singularity at $r \to 0$
  - can be seen from $R^{abcd}R_{abcd} = 48G^2M^2r^{-6}$
- it has a coordinate singularity at $r \to r_s = 2GM$
  - r_s is called the Schwarzschild radius
    - the Schwarzschild radius for the Earth is ~ 8.87mm; for the sun ~ 2.95km
    - but the radius of the Earth is ~ 3870km; for the sun $\sim 7 * 10^5$km
  - the coordinate singularity can be avoided by changing coordinates
    - Kruskal-Szekeres coordinates are valid up to the physical singularity
    - the radial coordinate of the Schwarzschild metric becomes timelike at r_s
- the Schwarzschild radius defines the event horizon
  - anything passing the event horizon can only move in the direction to $r = 0$

3. General Relativity — more (electro-) vacuum solutions

including the electro magnetic field

- a spherically symmetric (and hence static) solution exists
  - the static Reissner-Nordström metric:

$$ds^2 \;=\; \Delta dt^2 - \Delta^{-1} dr^2 - r^2 d^2\Omega \tag{51}$$

with

$$\Delta \;=\; 1 - \frac{2GM}{r} + \frac{G(p^2+q^2)}{r^2} = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \tag{52}$$

  - $q(p)$ is the electric (magnetic) charge of the central mass
  - $p^2 + q^2 = GM^2$ is called an "extremal black hole"
    - then $\frac{1}{2}r_s = r_Q$ and $\Delta = (1 - \frac{r_s}{2r})^2 = (1 - \frac{r_Q}{r})^2$

- this extreme Reissner-Nordström solution is used in theory to study
  - the information loss paradox of a black hole
  - the quantum gravity interpretation of a black hole
    - the electron as a charged black hole would be super-extremal with $\frac{1}{2}r_s \sim 10^{-57}\text{m} \ll r_Q \sim 10^{-36}\text{m}$

3. General Relativity — more (electro-) vacuum solutions

including angular momentum of the central mass

- it took 48 years to find a solution that includes angular momentum
  - the stationary Kerr metric in Kerr-Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{2GMr^3}{r^4+a^2z^2}k_\mu k_\nu \quad \text{with} \quad k_\mu = \left(1, \frac{rx+ay}{r^2+a^2}, \frac{ry-ax}{r^2+a^2}, \frac{z}{r}\right) \tag{53}$$

    - ∗ where r is given by the solution of $1 = \frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2}$
    - ∗ and a parametrizes the angular momentum: $J = Ma$

- including additionally the electro magnetic field
  - requires the Kerr-Newman metric

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta\, d\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left(adt - (r^2+a^2)d\phi\right)^2 - \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) \tag{54}$$

with

$$\Delta(r) = r^2 + a^2 - 2GMr + \frac{p^2+q^2}{G} = r^2 + a^2 - r_s r + r_Q^2 \tag{55}$$

$$\rho^2(r,\theta) = r^2 + a^2\sin^2\theta \tag{56}$$

  - the metric does not depend on t and ϕ $\Rightarrow$ ∂_t and ∂_ϕ are Killing vectors
  - notice the cross term between dt and $d\phi$
    - $\Rightarrow$ ∂_t is not orthogonal to S^2 hypersurfaces $\Rightarrow$ not static

features of the Kerr-Newman metric

- there are several surfaces, where the metric becomes singular
  - with the Schwarzschild metric we had only the Schwarzschild horizon
  - with the Reissner-Nordström metric there are two horizons at $r_\pm = \frac{1}{2}(r_s \pm \sqrt{r_s^2 - 4r_Q^2})$
    - going in $\partial_{(0)}$ changes its character at $r_\pm$: timelike $\xrightarrow{r_+}$ spacelike $\xrightarrow{r_-}$ timelike
  - with Kerr-Newman there is an additional Killing horizon outside of r_+
  - between the outer event horizon at r_+ and the Killing horizon is the ergosphere

- inside the ergosphere
  - everything rotates in the same direction as the central body
    - this is called dragging of inertial frames
  - the conserved energy can be negative in the ergosphere
    - following a geodesic into the ergosphere one can "throw a rock" into the black hole and emerge on a geodesic with more energy afterwards

    $\Rightarrow$ Penrose process
  - the extracted energy reduces the angular momentum of the black hole

$\Rightarrow$ analogy between black holes and thermodymics