

2. General Relativity — the metric

mathematical definition:

- a metric in M is a symmetric, bilinear, non-degenerate function

$$g_P : T_P \times T_P \rightarrow \mathbf{R} \quad (1)$$

- the metric acts on (contravariant) vectors (tensor indices)
- symmetric means $g(X, Y) = g(Y, X) \in \mathbf{R}$ for $X, Y \in T_P$
- bilinear means for $X, Y, Z \in T_P$ and $a, b, c \in \mathbf{R}$

$$g(aX + bY, cZ) = ac g(X, Z) + bc g(Y, Z) \in \mathbf{R} \quad (2)$$

- non-degenerate: $g(X, Y) = 0$ for all $Y \in T_P$ only if $X = 0$
- acting on the basis vectors $\vec{e}_{(\mu)}$ gives the **metric tensor**

$$g_{\mu\nu}(P) = g(\vec{e}_{(\mu)}, \vec{e}_{(\nu)}) \Big|_P \quad (3)$$

- the metric allows length and angle measurements

$$\|X\| := \sqrt{|g(X, X)|} \quad \text{and} \quad \cos \varphi = \frac{g(X, Y)}{\|X\| * \|Y\|} \quad (4)$$

2. General Relativity — the metric

examples for the metric

- in Euclidean space the metric is the normal dot-product:

$$\vec{a} \cdot \vec{b} = g_{jk} a^j b^k = \delta_{jk} a^j b^k \quad \text{so} \quad g_{j=k} = 1 \quad \text{and} \quad g_{j \neq k} = 0 \quad (5)$$

- we can use this metric of the embedding to get the **induced** metric in S^2

$$g_{jk} = g(\vec{e}_{(j)}, \vec{e}_{(k)}) = \vec{e}_{(j)} \cdot \vec{e}_{(k)} \quad \text{so} \quad g_{11} = s_{\vartheta_0}^2 \quad g_{12} = g_{21} = 0 \quad g_{22} = 1 \quad (6)$$

⇒ as we can see, the metric tensor depends on the position

- in Minkovsky space we had $g_{00} = 1$, $g_{ii} = -1$, and $g_{\mu \neq \nu} = 0$

- we can generalize the line element

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = g_{\mu\nu} \Delta x^\mu \otimes \Delta x^\nu \quad (7)$$

- the "differentials" Δx here are **not** multiplied with their "natural" wedge product

- one can define the **inverse metric** $g^{\mu\nu}$ by

$$g^{\mu\nu} g_{\mu\rho} = \delta_\rho^\nu \quad (8)$$

- the inverse metric is the metric of the cotangent space

2. General Relativity — the metric

connection and the metric

- the metric gives angles
 - ⇒ so we can relate angles between **vector fields** all over the manifold
 - but we have not specified the vectorfield, that defines parallel transport
 - that is done with the connection / gauge field
 - the metric has to be compatible with this choice
 - as seen with the Lie bracket, parallel transport has to do with derivation
 - the Lie derivative was a map $\mathcal{L}(T_P) : T_P \rightarrow T_P$
 - * leaving the differentiated object in its own representation
 - the directional derivative was a map $D(T_P) : \mathbf{R} \rightarrow T_P$
 - ⇒ this is, what we want
 - * but it does not connect the different points in M
- ⇒ including an affine connection Γ to make a **covariant derivative** ∇
- * like in gauge theories: $\nabla = \partial + \Gamma = d + \omega$

2. General Relativity — Levi Civita connection

definition for a covariant derivative:

- the covariant derivative ∇_U in the direction of U
 - should map tensors to tensors
 - be linear in the direction: $\nabla_{fU+gV}T = f\nabla_U T + g\nabla_V T$
 - be linear in the argument: $\nabla_U(T + S) = \nabla_U T + \nabla_U S$
 - follow the Leibniz rule: $\nabla_U(T \otimes S) = (\nabla_U T) \otimes S + T \otimes (\nabla_U S)$
 - commute with contractions: $\nabla_U \delta_\rho^\lambda = 0$
 - * this means in index notation $\nabla_\mu(T^\lambda_{\lambda\nu}) = \nabla_\mu(\delta_\rho^\lambda T^\rho_{\lambda\nu}) = \delta_\rho^\lambda \nabla_\mu(T^\rho_{\lambda\nu}) = (\nabla T)^\lambda_{\lambda\nu;\mu}$
 - be the partial derivative on scalars: $\nabla_U \phi = U(\phi) = (U)^\mu \partial_\mu \phi$

definition of the Levi Civita connection:

- the covariant derivative ∇_U in the direction of U
 - is metric compatible: $\nabla_U g(X, Y) = 0$
 - is torsion free: $(\nabla_X Y - \nabla_Y X) = [X, Y]$
 - * this is also easier explained in index notation ...

2. General Relativity — Levi Civita connection

using the natural basis

- the covariant derivative ∇_μ goes into the direction of $\partial_{(\mu)}$
- the covariant derivative in the direction of U is $\nabla_U = U^\mu \nabla_\mu$
 - linearity in the direction is obvious from $U \in T_P(M)$

writing the Levi Civita connection as:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma_{\mu\rho}^\nu A^\rho \quad \text{and} \quad \nabla_\mu a_\nu = \partial_\mu a_\nu + \Gamma_{\mu\nu}^{\prime\rho} a_\rho \quad (9)$$

- commutation with contractions means:
 - with $B = A^\mu a_\mu = \delta_\rho^\lambda A^\rho a_\lambda$ a scalar:

$$\begin{aligned} \nabla_\mu B &= \delta_\rho^\lambda [(\partial_\mu A^\rho + \Gamma_{\mu\nu}^\rho A^\nu) a_\lambda + A^\rho (\partial_\mu a_\lambda + \Gamma_{\mu\lambda}^{\prime\nu} a_\nu)] \\ &= (\partial_\mu A^\rho) a_\rho + A^\rho (\partial_\mu a_\rho) + [\Gamma_{\mu\nu}^\rho A^\nu a_\rho + A^\lambda \Gamma_{\mu\lambda}^{\prime\nu} a_\nu] \\ &= \partial_\mu (A^\nu a_\nu) + A^\nu a_\rho [\Gamma_{\mu\nu}^\rho + \Gamma_{\mu\nu}^{\prime\rho}] \end{aligned} \quad (10)$$

$$\Rightarrow \Gamma_{\mu\nu}^{\prime\rho} = -\Gamma_{\mu\nu}^\rho$$

* covariant indices transform opposite from contravariant ones

2. General Relativity — Levi Civita connection

writing the Levi Civita connection as:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma_{\mu\rho}^\nu A^\rho \quad \text{and} \quad \nabla_\mu a_\nu = \partial_\mu a_\nu - \Gamma_{\mu\nu}^\rho a_\rho \quad (11)$$

- torsion free means:

$$\begin{aligned} 0 &= X^\mu \nabla_\mu Y^\nu - Y^\mu \nabla_\mu X^\nu - [X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu] \\ &= X^\mu \Gamma_{\mu\rho}^\nu Y^\rho - Y^\mu \Gamma_{\mu\rho}^\nu X^\rho = X^\mu Y^\rho (\Gamma_{\mu\rho}^\nu - \Gamma_{\rho\mu}^\nu) \end{aligned} \quad (12)$$

⇒ the Christoffel symbols $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$ are symmetric

- metric compatible means:

$$0 = \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma_{\rho\mu}^\lambda g_{\lambda\nu} - \Gamma_{\rho\nu}^\lambda g_{\lambda\mu} \quad (13)$$

— adding $\nabla_\mu g_{\nu\rho} + \nabla_\nu g_{\mu\rho} - \nabla_\rho g_{\mu\nu}$ we get

$$\begin{aligned} 0 &= \partial_\mu g_{\nu\rho} - \Gamma_{\mu\nu}^\lambda g_{\lambda\rho} - \Gamma_{\mu\rho}^\lambda g_{\lambda\nu} + \partial_\nu g_{\mu\rho} - \Gamma_{\nu\mu}^\lambda g_{\lambda\rho} - \Gamma_{\nu\rho}^\lambda g_{\lambda\mu} - \partial_\rho g_{\mu\nu} + \Gamma_{\rho\mu}^\lambda g_{\lambda\nu} + \Gamma_{\rho\nu}^\lambda g_{\lambda\mu} \\ &= \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} - 2\Gamma_{\mu\nu}^\lambda g_{\lambda\rho} \end{aligned} \quad (14)$$

⇒ the Christoffel symbols can be calculated from the metric:

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \quad (15)$$

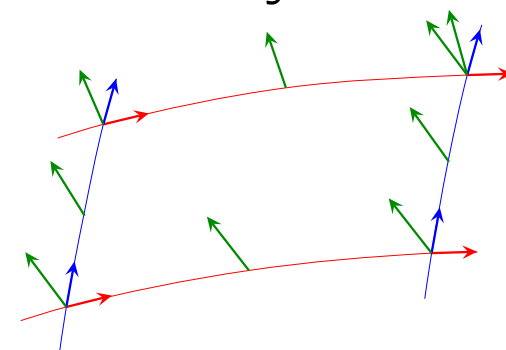
2. General Relativity — Riemann curvature tensor

The difference between a Riemannian manifold and flat space

- parallel transport of a vector V along a curve $C(t)$ is defined by

$$\nabla_{\frac{d}{dt}C(t)} V = 0$$

- taking the curve as a Levi Civita parallelogramoid
 - along the curves A and B



$$P_{00} \xrightarrow{A(\Delta s)} P_{10} \xrightarrow{B'(\Delta t)} P_{11} \xrightarrow{A'(-\Delta s)} P_{01} \xrightarrow{B(-\Delta t)} P_{00} \quad (16)$$

* with B' parallel transported along A , so $\nabla_A B = 0$, and $B' = B$

- we can compare the vector at P_{11} between the two different paths

$$P_{00} \xrightarrow{A(\Delta s)} P_{10} \xrightarrow{B(\Delta t)} P_{11} \quad \text{and} \quad P_{00} \xrightarrow{B(\Delta t)} P_{01} \xrightarrow{A(\Delta s)} P_{11} \quad (17)$$

- and the difference of the parallel transported vector at P_{11} is

$$\nabla_B \nabla_A V - \nabla_A \nabla_B V \quad (18)$$

2. General Relativity — Riemann curvature tensor

The difference between a Riemannian manifold and flat space

- evaluating the difference in the natural basis ∂_μ we get

$$\begin{aligned}\nabla_B \nabla_A V - \nabla_A \nabla_B V &= B^\sigma \nabla_\sigma A^\rho \nabla_\rho V^\mu \partial_\mu - A^\rho \nabla_\rho B^\sigma \nabla_\sigma V^\mu \partial_\mu \\ &= A^\rho B^\sigma (\nabla_\sigma \nabla_\rho - \nabla_\rho \nabla_\sigma) V^\mu \partial_\mu\end{aligned}\quad (19)$$

— due to the condition $\nabla_A B = \nabla_B A = 0$

- using the affine connection we get

$$\begin{aligned}(\nabla_\sigma \nabla_\rho - \nabla_\rho \nabla_\sigma) V^\mu &= \nabla_\sigma (\partial_\rho V^\mu + \Gamma_{\rho\nu}^\mu V^\nu) - \nabla_\rho (\partial_\sigma V^\mu + \Gamma_{\sigma\nu}^\mu V^\nu) \\ &= \partial_\sigma (\partial_\rho V^\mu + \Gamma_{\rho\nu}^\mu V^\nu) - \Gamma_{\sigma\rho}^\lambda (\partial_\lambda V^\mu + \Gamma_{\lambda\nu}^\mu V^\nu) + \Gamma_{\sigma\lambda}^\mu (\partial_\rho V^\lambda + \Gamma_{\rho\nu}^\lambda V^\nu) \\ &\quad - \partial_\rho (\partial_\sigma V^\mu + \Gamma_{\sigma\nu}^\mu V^\nu) + \Gamma_{\rho\sigma}^\lambda (\partial_\lambda V^\mu + \Gamma_{\lambda\nu}^\mu V^\nu) - \Gamma_{\rho\lambda}^\mu (\partial_\sigma V^\lambda + \Gamma_{\sigma\nu}^\lambda V^\nu) \\ &= (\partial_\sigma \Gamma_{\rho\nu}^\mu - \partial_\rho \Gamma_{\sigma\nu}^\mu - \Gamma_{\rho\lambda}^\mu \Gamma_{\sigma\nu}^\lambda + \Gamma_{\sigma\lambda}^\mu \Gamma_{\rho\nu}^\lambda) V^\nu - (\Gamma_{\sigma\rho}^\lambda - \Gamma_{\rho\sigma}^\lambda) (\partial_\lambda V^\mu + \Gamma_{\lambda\nu}^\mu V^\nu) \\ &=: R_{\nu\sigma\rho}^\mu V^\nu - T_{\sigma\rho}^\nu (\nabla_\nu V^\mu)\end{aligned}\quad (20)$$

\Rightarrow the Riemann curvature tensor $R_{\nu\sigma\rho}^\mu$ and the Torsion tensor $T_{\sigma\rho}^\nu$

— $T_{\rho\sigma}^\nu$ vanishes for the Christoffel symbols $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$

2. General Relativity — Riemann curvature tensor

The difference between a Riemannian manifold and flat space

- in Euclidean (or Minkovsky) flat space one can use
 - Cartesian coordinates, where the metric is constant
 - ⇒ the connection and the Riemann tensor vanish identically
 - curvilinear coordinates, where the metric is not constant
 - ⇒ the connection is non zero
 - * but the Riemann tensor still vanishes identically
- example: spherical coordinates in \mathbf{R}^3
 - with unit vectors $e_{(r)} = (s_\vartheta c_\varphi, s_\vartheta s_\varphi, c_\vartheta)$, $e_{(\vartheta)} = (c_\vartheta c_\varphi, c_\vartheta s_\varphi, -s_\vartheta)$, and $e_{(\varphi)} = (-s_\varphi, c_\varphi, 0)$
 - the metric is $ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2$
 - * with non-zero coefficients $g_{rr} = 1$, $g_{\vartheta\vartheta} = r^2$, and $g_{\varphi\varphi} = r^2 \sin^2 \vartheta$
 - the Christoffel symbols are

$$\Gamma_{\mu\nu}^r = \frac{1}{2}g^{rr}(\partial_\mu g_{\nu r} + \partial_\nu g_{\mu r} - \partial_r g_{\mu\nu}) \Rightarrow \Gamma_{\vartheta\vartheta}^r = -r \quad \Gamma_{\varphi\varphi}^r = -r \sin^2 \vartheta \quad (21)$$

$$\Gamma_{\mu\nu}^\vartheta = \frac{1}{2}g^{\vartheta\vartheta}(\partial_\mu g_{\nu\vartheta} + \partial_\nu g_{\mu\vartheta} - \partial_\vartheta g_{\mu\nu}) \Rightarrow \Gamma_{r\vartheta}^\vartheta = r^{-1} \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta \quad (22)$$

$$\Gamma_{\mu\nu}^\varphi = \frac{1}{2}g^{\varphi\varphi}(\partial_\mu g_{\nu\varphi} + \partial_\nu g_{\mu\varphi} - \partial_\varphi g_{\mu\nu}) \Rightarrow \Gamma_{r\varphi}^\varphi = r^{-1} \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta \quad (23)$$

2. General Relativity — Riemann curvature tensor

The difference between a Riemannian manifold and flat space

- example: spherical coordinates in \mathbf{R}^3

- the Riemann tensor $R^\mu{}_{\nu\sigma\rho} = (\partial_\sigma \Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\sigma\nu}) - (\rho \leftrightarrow \sigma)$
- * in components $R^\mu{}_{\nu\sigma\rho} = (\partial_\sigma \Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\rho\lambda} \Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\rho\vartheta} \Gamma^\vartheta_{\sigma\nu} - \Gamma^\mu_{\rho\varphi} \Gamma^\varphi_{\sigma\nu}) - (\rho \leftrightarrow \sigma)$

$$\begin{aligned} R^r{}_{\sigma\rho} &= (\partial_\sigma \Gamma^r_{\rho r} - \Gamma^r_{\rho r} \Gamma^r_{\sigma r} - \Gamma^r_{\rho\vartheta} \Gamma^\vartheta_{\sigma r} - \Gamma^r_{\rho\varphi} \Gamma^\varphi_{\sigma r}) - (\rho \leftrightarrow \sigma) \\ &= (\partial_\sigma 0 - 00 - (-r)\delta_\rho^\vartheta (r^{-1})\delta_\sigma^\vartheta - (-r \sin^2 \vartheta)\delta_\rho^\varphi (r^{-1})\delta_\sigma^\varphi) - (\rho \leftrightarrow \sigma) \\ &= (\delta_\rho^\vartheta \delta_\sigma^\vartheta + \sin^2 \vartheta \delta_\rho^\varphi \delta_\sigma^\varphi) - (\rho \leftrightarrow \sigma) = 0 \end{aligned} \quad (24)$$

$$\begin{aligned} R^r{}_{\vartheta\sigma\rho} &= (\partial_\sigma \Gamma^r_{\rho\vartheta} - 0\Gamma^r_{\sigma\vartheta} - \Gamma^r_{\rho\vartheta} \Gamma^\vartheta_{\sigma\vartheta} - \Gamma^r_{\rho\varphi} \Gamma^\varphi_{\sigma\vartheta}) - (\rho \leftrightarrow \sigma) \\ &= (\partial_\sigma (-r)\delta_\rho^\vartheta - (-r)\delta_\rho^\vartheta (r^{-1})\delta_\sigma^r - (-r \sin^2 \vartheta)\delta_\rho^\varphi (\cot \vartheta)\delta_\sigma^\varphi) - (\rho \leftrightarrow \sigma) \\ &= (-\delta_\sigma^r \delta_\rho^\vartheta + \delta_\rho^\vartheta \delta_\sigma^r) - (\rho \leftrightarrow \sigma) = 0 \end{aligned} \quad (25)$$

$$\begin{aligned} R^r{}_{\varphi\sigma\rho} &= (\partial_\sigma \Gamma^r_{\rho\varphi} - 0\Gamma^r_{\sigma\varphi} - \Gamma^r_{\rho\vartheta} \Gamma^\vartheta_{\sigma\varphi} - \Gamma^r_{\rho\varphi} \Gamma^\varphi_{\sigma\varphi}) - (\rho \leftrightarrow \sigma) \\ &= (\partial_\sigma (-rs_\vartheta^2)\delta_\rho^\varphi - (-r)\delta_\rho^\vartheta (-s_\vartheta c_\vartheta)\delta_\sigma^\varphi - (-rs_\vartheta^2)\delta_\rho^\varphi [(r^{-1})\delta_\sigma^r + (c_\vartheta/s_\vartheta)\delta_\sigma^\vartheta]) - (\rho \leftrightarrow \sigma) \\ &= (-[\delta_\sigma^r s_\vartheta^2 + r2s_\vartheta c_\vartheta \delta_\sigma^\vartheta]\delta_\rho^\varphi - rs_\vartheta c_\vartheta \delta_\rho^\vartheta \delta_\sigma^\varphi + s_\vartheta^2 \delta_\rho^\varphi \delta_\sigma^r + rs_\vartheta c_\vartheta \delta_\rho^\varphi \delta_\sigma^\vartheta) - (\rho \leftrightarrow \sigma) \\ &= -rs_\vartheta c_\vartheta (2\delta_\rho^\varphi \delta_\sigma^\vartheta + \delta_\sigma^\varphi \delta_\rho^\vartheta - \delta_\rho^\varphi \delta_\sigma^\vartheta) - (\rho \leftrightarrow \sigma) = 0 \end{aligned} \quad (26)$$

- and similar for $R^\vartheta{}_{\nu\sigma\rho}$ and $R^\varphi{}_{\nu\sigma\rho}$

$\Rightarrow \mathbf{R}^3$ is flat also in spherical coordinates

2. General Relativity — Riemann curvature tensor

The difference between a Riemannian manifold and flat space

- example: S^2 with coordinates $(x^1 = \varphi, x^2 = \vartheta)$ and basis vectors $e_{(1)}, e_{(2)}$
 - embedded in \mathbf{R}^3 with $(x, y, z) = r(s_\vartheta c_\varphi, s_\vartheta s_\varphi, c_\vartheta)$ and r constant
 - the metric is $ds^2 = (dx^1)^2 + (dx^2)^2 = r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2$
 - * with non-zero coefficients $g_{\vartheta\vartheta} = r^2$ and $g_{\varphi\varphi} = r^2 \sin^2 \vartheta$
 - the Christoffel symbols are

$$\Gamma_{\mu\nu}^\vartheta = \frac{1}{2}g^{\vartheta\vartheta}(\partial_\mu g_{\nu\vartheta} + \partial_\nu g_{\mu\vartheta} - \partial_\vartheta g_{\mu\nu}) \Rightarrow \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta \quad (27)$$

$$\Gamma_{\mu\nu}^\varphi = \frac{1}{2}g^{\varphi\varphi}(\partial_\mu g_{\nu\varphi} + \partial_\nu g_{\mu\varphi} - \partial_\varphi g_{\mu\nu}) \Rightarrow \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta \quad (28)$$

- the Riemann tensor in components $R^\mu{}_{\nu\sigma\rho} = (\partial_\sigma \Gamma_{\rho\nu}^\mu - \Gamma_{\rho\vartheta}^\mu \Gamma_{\sigma\nu}^\vartheta - \Gamma_{\rho\varphi}^\mu \Gamma_{\sigma\nu}^\varphi) - (\rho \leftrightarrow \sigma)$

$$R_{\vartheta\sigma\rho}^\vartheta = (\partial_\sigma \Gamma_{\rho\vartheta}^\vartheta - \Gamma_{\rho\vartheta}^\vartheta \Gamma_{\sigma\vartheta}^\vartheta - \Gamma_{\rho\varphi}^\vartheta \Gamma_{\sigma\vartheta}^\varphi) - (\rho \leftrightarrow \sigma) = (-(-\sin \vartheta \cos \vartheta) \delta_\rho^\varphi (\cot \vartheta) \delta_\sigma^\varphi) - (\rho \leftrightarrow \sigma) = 0 \quad (29)$$

$$\begin{aligned} R_{\varphi\sigma\rho}^\vartheta &= (\partial_\sigma \Gamma_{\rho\varphi}^\vartheta - \Gamma_{\rho\vartheta}^\vartheta \Gamma_{\sigma\varphi}^\vartheta - \Gamma_{\rho\varphi}^\vartheta \Gamma_{\sigma\varphi}^\varphi) - (\rho \leftrightarrow \sigma) = (\partial_\sigma (-\sin \vartheta \cos \vartheta) \delta_\rho^\varphi - (-\sin \vartheta \cos \vartheta) \delta_\rho^\varphi (\cot \vartheta) \delta_\sigma^\vartheta) - (\rho \leftrightarrow \sigma) \\ &= ([-\cos^2 \vartheta + \sin^2 \vartheta] + \cos^2 \vartheta) \delta_\rho^\varphi \delta_\sigma^\vartheta - (\rho \leftrightarrow \sigma) = \sin^2 \vartheta (\delta_\rho^\varphi \delta_\sigma^\vartheta - \delta_\sigma^\varphi \delta_\rho^\vartheta) \neq 0 \end{aligned} \quad (30)$$

$$\begin{aligned} R_{\vartheta\sigma\rho}^\varphi &= (\partial_\sigma \Gamma_{\rho\vartheta}^\varphi - \Gamma_{\rho\vartheta}^\varphi \Gamma_{\sigma\vartheta}^\vartheta - \Gamma_{\rho\varphi}^\varphi \Gamma_{\sigma\vartheta}^\varphi) - (\rho \leftrightarrow \sigma) = (\partial_\sigma (\cot \vartheta) \delta_\rho^\varphi - (\cot \vartheta) \delta_\rho^\vartheta (\cot \vartheta) \delta_\sigma^\varphi) - (\rho \leftrightarrow \sigma) \\ &= ([-\sin^{-2} \vartheta] \delta_\rho^\varphi \delta_\sigma^\vartheta - \cot^2 \vartheta \delta_\rho^\vartheta \delta_\sigma^\varphi) - (\rho \leftrightarrow \sigma) = -(\delta_\rho^\varphi \delta_\sigma^\vartheta - \delta_\sigma^\varphi \delta_\rho^\vartheta) \neq 0 \end{aligned} \quad (31)$$

$$\begin{aligned} R_{\varphi\sigma\rho}^\varphi &= (\partial_\sigma \Gamma_{\rho\varphi}^\varphi - \Gamma_{\rho\vartheta}^\varphi \Gamma_{\sigma\varphi}^\vartheta - \Gamma_{\rho\varphi}^\varphi \Gamma_{\sigma\varphi}^\varphi) - (\rho \leftrightarrow \sigma) \\ &= (\delta_\rho^\vartheta \partial_\vartheta (\cot \vartheta) \delta_\sigma^\varphi - (\cot \vartheta) \delta_\rho^\varphi (-\sin \vartheta \cos \vartheta) \delta_\sigma^\varphi - (\cot \vartheta)^2 \delta_\rho^\vartheta \delta_\sigma^\vartheta) - (\rho \leftrightarrow \sigma) = 0 \end{aligned} \quad (32)$$

$\Rightarrow S^2$ is **not flat**

2. General Relativity — Riemann curvature tensor

Properties of the Riemann curvature tensor

- the first Bianchi identity

$$R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\sigma\nu} + R^\mu{}_{\sigma\nu\rho} = 0 \quad \Leftrightarrow \quad R^\mu{}_{[\nu\rho\sigma]} = 0 \quad (33)$$

- with the abbreviation $\nabla_\lambda X =: X_{;\lambda}$, the second Bianchi identity

$$R^\mu{}_{\nu\rho\sigma;\lambda} + R^\mu{}_{\nu\lambda\rho;\sigma} + R^\mu{}_{\nu\sigma\lambda;\rho} = 0 \quad \Leftrightarrow \quad R^\mu{}_{\nu[\rho\sigma;\lambda]} = 0 \quad (34)$$

- lowering the first index $R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma}$

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} = R_{\rho\sigma\mu\nu} \quad (35)$$

- contraction gives the symmetric Ricci tensor and the Ricci scalar

$$R^\lambda{}_{\mu\lambda\nu} = g^{\lambda\rho} R_{\lambda\mu\rho\nu} = R_{\mu\nu} = R_{\nu\mu} \quad g^{\mu\nu} R_{\mu\nu} = R \quad (36)$$

- this allows the decomposition $R_{\mu\nu\rho\sigma} = S_{\mu\nu\sigma\rho} + E_{\nu\mu\rho\sigma} + C_{\rho\sigma\mu\nu}$

- into a scalar part $S_{\mu\nu\sigma\rho} = \frac{R}{n(n-1)}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})$
- using the traceless Ricci tensor $S_{\mu\nu} = R_{\mu\nu} - \frac{R}{n}g_{\mu\nu}$ into a semi traceless part

$$E_{\mu\nu\sigma\rho} = \frac{1}{n-2}(g_{\mu\sigma}S_{\nu\rho} - g_{\mu\rho}S_{\nu\sigma} + g_{\nu\rho}S_{\mu\sigma} - g_{\nu\sigma}S_{\mu\rho})$$

- and into the fully traceless Weyl tensor $C_{\mu\nu\sigma\rho}$

2. General Relativity — Riemann curvature tensor

Curvature in general

- for a curve we can imagine, how it bends on the plane or in space
 - this is the **extrinsic** curvature
 - the **intrinsic curvature** of a curve is zero
 - ⇒ it is similar to a straight line
- for a two dimensional surface we can imagine its bending in 3D space
 - its **intrinsic curvature** is defined independently from the embedding
 - * in our example of the sphere S^2 we calculated the Riemann curvature tensor

$$R_{\varphi\vartheta\varphi}^{\vartheta} = -R_{\varphi\varphi\vartheta}^{\vartheta} = \sin^2 \vartheta \quad R_{\vartheta\vartheta\varphi}^{\varphi} = -R_{\vartheta\varphi\vartheta}^{\varphi} = -1 \quad (37)$$

- * this gives the Ricci tensor $R_{\vartheta\vartheta} = R_{\vartheta\varphi\varphi}^{\varphi} = 1$, $R_{\varphi\varphi} = R_{\varphi\vartheta\vartheta}^{\vartheta} = \sin^2 \vartheta$
- * and the Ricci scalar $R = g^{\vartheta\vartheta} R_{\vartheta\vartheta} + g^{\varphi\varphi} R_{\varphi\varphi} = r^{-2} \cdot 1 + r^{-2} \sin^2 \vartheta \cdot \sin^2 \vartheta = \frac{2}{r^2}$
- ⇒ which gives the radius of the sphere ...

- for higher dimensional (hyper)surfaces the Ricci scalar is the generalisation of the curvature radius

2. General Relativity — orthonormal coordinates

using the metric we can change our basis to make it orthonormal:

- defining a new set of local coordinates

$$e_{(m)}(x) = e_m^\mu(x) \partial_{(\mu)} \quad (38)$$

- the greek indices μ indicate the natural basis $\partial_{(\mu)}$
 - * obtained from the coordinate functions $X(x)$ on M
- the latin indices m indicate the orthonormal basis $e_{(m)}(x)$
 - * defined by the relation $g(e_{(m)}(x), e_{(n)}(x))(x) = \eta_{mn}$
 - * the flat Minkovsky metric $\eta_{00} = 1$, $\eta_{ii} = -1$, and $\eta_{m \neq n} = 0$
- $e_{(m)}(x)$ is an orthonormal coordinate system
- $e_m^\mu(x)$ are called "vierbein" or tetrad
 - its inverse $e_\mu^m(x)$ is defined by $e_m^\mu e_\nu^\mu = \delta_\nu^\mu$ and $e_m^\mu e_\mu^n = \delta_m^n$
 - we can write the metric tensor in natural coordinates as

$$g_{\mu\nu}(x) = e_\mu^m(x) e_\nu^n(x) \eta_{mn} \quad (39)$$

2. General Relativity — orthonormal coordinates

using the tetrad

- similar like for the natural coordinate basis $\partial_{(\mu)}$
 - we can introduce a dual basis in T_P^* by requiring $e^m(e_n) = \delta_n^m$
 - this dual basis can also be written as $e^m = e_\mu^m dx^{(\mu)}$
- with the tetrad we can change any index into the orthonormal frame
 - the vectors are just represented in a different basis
 - * tensors can even have mixed components: $T_{\rho\sigma}^\mu = e_m^\mu T_{\rho\sigma}^m = e_m^\mu e_\rho^r T_{r\sigma}^m = e_m^\mu e_\rho^r e_\sigma^s T_{rs}^m$
- changing the coordinates $x \rightarrow x'$ changes components of a tensor by

$$T_{\rho'\sigma'}^{\mu'}(x') = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\sigma}{\partial x^{\sigma'}} T_{\rho\sigma}^\mu(x) \quad (40)$$

⇒ this is called a **general coordinate transformation (GCT)**

- changing the orthonormal basis $e_n \rightarrow e'_n$ to another orthonormal one
 - the metric **does not change** $\eta_{mn} = \eta'_{mn}$
- ⇒ the transformation is a **local Lorentz transformation (LLT)** :

$$e_n(x) \rightarrow e'_n(x) = \Lambda^m_n(x) e_m(x) \quad (41)$$

2. General Relativity — orthonormal coordinates

a connection is not a tensor

⇒ a transformation of its indices might not lead to a connection

- parametrizing the connection in the orthonormal indices as

$$\nabla_\mu T^a_b = \partial_\mu T^a_b + \omega_\mu^a{}_c T^c_b - \omega_\mu^c{}_b T^a_c \quad (42)$$

- studying the covariant derivative of a vector helps:

$$\begin{aligned} \nabla_{dx} V &= dx^\mu (\nabla_\mu V)^c e_c = (\partial_\mu V^c + \omega_\mu^a{}_b V^b) dx^\mu \otimes (e_c^\sigma \partial_{(\sigma)}) = e_c^\sigma (\partial_\mu (e_\nu^c V^\nu) + \omega_\mu^c{}_b e_\rho^b V^\rho) dx^\mu \otimes \partial_{(\sigma)} \\ &= e_c^\sigma e_\nu^c ((\partial_\mu V^\nu) + e_a^\nu (\partial_\mu e_\rho^a) V^\rho + e_a^\nu e_\rho^b \omega_\mu^a{}_b V^\rho) dx^\mu \otimes \partial_{(\sigma)} \\ &= (\partial_\mu V^\nu + e_a^\nu (\partial_\mu e_\rho^a) V^\rho + e_a^\nu e_\rho^b \omega_\mu^a{}_b V^\rho) dx^\mu \otimes \partial_{(\nu)} = (\partial_\mu V^\nu + \Gamma_{\mu\rho}^\nu V^\rho) dx^\mu \otimes \partial_{(\nu)} \end{aligned} \quad (43)$$

$$\Rightarrow \Gamma_{\mu\rho}^\nu = e_a^\nu (\partial_\mu e_\rho^a) + e_a^\nu e_\rho^b \omega_\mu^a{}_b \quad \text{or} \quad \omega_\mu^a{}_b = e_\lambda^a e_b^\rho \Gamma_{\mu\rho}^\lambda - e_b^\rho (\partial_\mu e_\rho^a) \quad (44)$$

- multiplying with e_ν^b this can be written as the "tetrad postulate"

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\lambda e_\lambda^a + \omega_\mu^a{}_b e_\nu^b = 0 \quad (45)$$

- $\omega_\mu^a{}_b$ is called the spin connection

— since it transports an index that transforms under Lorentz transformations

* and Lorentz transformations act also on spinors

2. General Relativity — orthonormal coordinates

the spin connection

- transforms as a **one-form** under GCTs
- but inhomogenously under LLTs

$$\omega_{\mu}^a{}_b \rightarrow \omega'_{\mu}{}^a{}_b = \Lambda^a{}_c \Lambda^d{}_b \omega_{\mu}^c{}_d - \Lambda^c{}_b \partial_{\mu} \Lambda^a{}_c$$

- allows the implementation of a covariant exterior derivative:
 - by writing forms we can suppress the tensor index of the natural basis:
 - * EM potential $A = A_{\mu} dx^{\mu}$, field strength $F = \frac{1}{2} F_{\mu\nu} dx^{\mu} dx^{\nu} = \frac{1}{2} (dA)_{\mu\nu} dx^{\mu} dx^{\nu}$
 - having a vector valued (or group valued) one-form $A^a = A^a_{\mu} dx^{\mu}$,
 - we can extend the exterior derivative to a covariant exterior derivative

$$\begin{aligned} 2(\nabla A)^a &= (\nabla A)^a_{\mu\nu} dx^{\mu} dx^{\nu} = (dA)^a_{\mu\nu} dx^{\mu} dx^{\nu} + (\omega \wedge A)^a_{\mu\nu} dx^{\mu} dx^{\nu} \\ &= (\partial_{\mu} A^a_{\nu} + \omega_{\mu}^a{}_b A^b_{\nu}) dx^{\mu} dx^{\nu} \end{aligned}$$

- writing the torsion as a vector valued two form:

$$\begin{aligned} T^a &= \frac{1}{2} T^a_{\mu\nu} dx^{\mu} dx^{\nu} = \frac{1}{2} e^a_{\rho} (\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}) dx^{\mu} dx^{\nu} = (\partial_{\mu} e^a_{\nu} + \omega_{\mu}^a{}_b e^b_{\nu}) dx^{\mu} dx^{\nu} \\ &= (\nabla_{\mu} e^a_{\nu}) dx^{\mu} dx^{\nu} = \nabla e^a \end{aligned}$$

2. General Relativity — orthonormal coordinates

the spin connection

- calculating the curvature tensor as a tensor valued two form

$$\begin{aligned}
 & \frac{1}{2}(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^a e_a^\rho dx^\mu dx^\nu \partial_{(\rho)} = e_a^\rho \nabla \nabla V^a \partial_{(\rho)} = e_a^\rho \nabla (dV^a + \omega_b^a V^b) \partial_{(\rho)} \\
 &= e_a^\rho (d(dV^a + \omega_b^a V^b) + \omega_c^a \wedge (dV^c + \omega_b^c V^b)) \partial_{(\rho)} \\
 &= (d^2 V^a + (d\omega_b^a) V^b - \omega_b^a \wedge dV^b + \omega_c^a \wedge dV^c + \omega_c^a \wedge \omega_b^c V^b) e_a^\rho \partial_{(\rho)} \\
 &= (d\omega_b^a + \omega_c^a \wedge \omega_b^c) V^b e_a = R^a{}_b V^b e_a
 \end{aligned} \tag{46}$$

⇒ gives the Maurer-Cartan structure equations

$$T^a = de^a + \omega_b^a \wedge e^b \quad \text{and} \quad R^a{}_b = d\omega_b^a + \omega_c^a \wedge \omega_b^c \tag{47}$$

- this allows an easy "proof" of the Bianchi identities ($R^{\rho}{}_{[\sigma\mu\nu]} = 0$ and $R^{\rho}{}_{\sigma[\mu\nu;\lambda]} = 0$)

$$\begin{aligned}
 \nabla T^a &= d(de^a + \omega_b^a \wedge e^b) + \omega_c^a \wedge (de^b + \omega_b^c \wedge e^c) \\
 &= (d\omega_b^a) \wedge e^b - \omega_b^a \wedge (de^b) + \omega_b^a \wedge (de^b) + \omega_b^a \wedge \omega_c^b \wedge e^c = R^a{}_b \wedge e^b = \frac{1}{2} R^a{}_{b\mu\nu} \wedge e^b \wedge dx^\mu \wedge dx^\nu
 \end{aligned} \tag{48}$$

$$\begin{aligned}
 \nabla R^a{}_b &= d(d\omega_b^a + \omega_c^a \wedge \omega_b^c) + \omega_c^a \wedge (d\omega_b^c + \omega_d^c \wedge \omega_b^d) - (d\omega_d^a + \omega_c^a \wedge \omega_d^c) \wedge \omega_b^d \\
 &= (d\omega_c^a) \wedge \omega_b^c - \omega_c^a \wedge (d\omega_b^c) + \omega_c^a \wedge (d\omega_b^c) + \omega_c^a \wedge (\omega_d^c \wedge \omega_b^d) - (d\omega_d^a) \wedge \omega_b^d - (\omega_c^a \wedge \omega_d^c) \wedge \omega_b^d \equiv 0
 \end{aligned} \tag{49}$$

- metric compatibility $0 = \nabla_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega_{\mu a}^c \eta_{cb} - \omega_{\mu b}^c \eta_{ac} = -\omega_{\mu ba} - \omega_{\mu ab}$

⇒ gives an antisymmetric spin connection $\omega_{\mu ab} = -\omega_{\mu ba}$