# 1. General Relativity — vectors

## What is a vector?

- a quantity that has a size and a direction
  - the size of a vector $\vec{a}$ can be written as $|\vec{a}|$
  - $|\vec{a}|$ is usually a real number, but not necessarily positive
  - the direction is ”optional”
- mathematical definition:
  - a vector is an element of a vector space
- examples
  - the real number $x$
  - the vector pointing from a point $A$ to a point $B$
  - an $n$-tuple of numbers $\vec{x} = (x_1, x_2, \ldots, x_n)$
  - the RGB color value $(r, g, b)$
- in Geometric Algebra
  - vectors are objects, that do not necessarily commute: $ab \neq ba$

mathematical definition:

- A vector space over a field (corpus) $F$ is a set $V$ with "$+$" and "$*$"
  - $(V, +)$ forms an abelian group: for $\vec{x}, \vec{y}, \vec{z} \in V$
    - $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$
    - $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
    - there exists $\vec{0} \in V$ with $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$
    - for all $\vec{x} \in V$ there exists $(-\vec{x}) \in V$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
  - $(F, +, *)$ forms a field (corpus)
    - $(F, +)$ forms an abelian group
    - $(F \backslash \{0\}, *)$ forms an abelian group
  - for $a, b \in F$
    - $a * (\vec{x} + \vec{y}) = a * \vec{x} + a * \vec{y}$ and $(a + b) * \vec{x} = a * \vec{x} + b * \vec{x}$
    - $a * (b * \vec{x}) = (a * b) * \vec{x}$
    - $1 * \vec{x} = \vec{x}$ for "1" being the unit element of $(F \backslash \{0\}, *)$

## 1. General Relativity — vector space

examples:

- the real numbers over the real numbers
  - the real numbers are a field, so they form also an abelian group ...
- vectors in $\mathbf{R}^3$ (three dimensional Euclidean space) over $\mathbf{R}$
- $m \times n$ real (complex) matrices over the real (complex) numbers
- tensors with arbitrary, but fixed index structure
  - example: all tensors of the form ${t_i}^j{}_k$
- real analytic functions in $[0, 1]$ over $\mathbf{R}$
- complex holomorphic functions over $\mathbf{C}$
- solutions to homogeneous linear differential equations over $\mathbf{R}$ or $\mathbf{C}$

# 1. General Relativity — manifolds

loose mathematical definition:

- a manifold is a space that is **locally** similar to an Euclidean space
  - every point of an n-dimensional manifold has a neighborhood homeomorphic to an open subset of the n-dimensional space $\mathbf{R}^n$.
    - ''A is homeomorphic to B'' = A can be continuously deformed to resemble B ( actually, there has only to be a continuous invertible mapping between A and B )
  
  $\Rightarrow$ a ball is homeomorphic to a cube, but not to a torus

$\Rightarrow$ we can use our understanding of Euclidean space
  - including all our mathematical tools
  - and look, where the manifold differs from $\mathbf{R}^n$

- we can make local maps (charts) of the manifold
  - in regions where maps overlap, they should be compatible
  - compatible maps that cover the whole manifold form an atlas

- example: the surface of the earth
  - we need at least two maps to show the whole surface

## 1. General Relativity — manifolds

### why manifolds?

- they generalize normal Euclidean space
  - example: the surface of the earth is not really Euclidean

- they preserve the local information

⇒ one can discuss causality
  - no need for ”action at a distance”

- one can study wave phenomena on a manifold
  - for that one needs differential equations – wave equations
  
  ⇒ differentiable manifolds

- if we can measure distances and angles

  ⇒ (Pseudo-) Riemannian manifolds
    * they are also equipped with a metric

## differentiable manifolds

- have a globally defined differential structure
  - one can define differentiation similar to Euclidean space
    * even multivariable differentiation

- allow a globally defined differentiable tangent space
  - one can define differentiable
    * functions
    * vectors
    * tensors

- calculus on differential manifolds

  $\Rightarrow$ exterior derivative (Ėlie Cartan)
  - generalizes the differential of a function to forms of higher degree

- the calculus on differential manifolds is differential geometry

## 1. General Relativity — differential geometry

describes the properties of the manifold ''from within''

- a smooth (i.e. differentiable) curve $C$ on a manifold $M$ can be understood as a smooth multidimensional function:

$$C : \mathbf{R} \to M \tag{1}$$

  – example: the 3D-helix $C(t) = (r \cos t, r \sin t, bt)$

- studying the smooth curves through a point $P$:
  – differentiation with respect to $t$ gives the tangent vector:

$$\vec{v}_P = \frac{d}{dt} C(t)\Big|_P = (-r \sin t, r \cos t, b)|_P \tag{2}$$

  – all tangent vectors in $P$ form the tangent space $T_P$

- the tangent space has the same dimension as $M$
  – but is isomorphic (identical in structure) to $\mathbf{R}^n$

## $N$ dimensional tangent space $T_P$

- since we can reparametrize the curves $C(t) \to C(s) = C(t(s))$
  - the length of the tangent vector changes, but not its direction

- one can determine $N$ linearly independent vectors

  $\Rightarrow$ basis vectors of the tangent space $T_P$

- with coordinate functions $X_{(\mu)}(t)$ on the manifold
  - we can define a "natural" basis by

$$\vec{e}_{(\mu)} = \frac{d}{dt} X_{(\mu)}(t)\Big|_P = \text{''}\partial_{(\mu)}\text{''} \tag{3}$$

- any vector in $T_P$ is a linear combination of the basis vectors:

$$\vec{A} = \sum_{\mu}^{N} A^{\mu} \vec{e}_{(\mu)} \qquad \text{or} \qquad A = A^{\mu} \partial_{(\mu)} \tag{4}$$

  - $A^{\mu}$ are the (contravariant) components of the vector $A$

## 1. General Relativity — tangent space

### $N$ dimensional tangent space $T_P$

- using the coordinates $X_{(\mu)}$ on the manifold
  - we can define smooth functions of the manifold: $f(X) \in \mathbf{R}$

  $\Rightarrow$ then vectors $\vec{v}$ of the tangent space $T_P$
  can be seen as directional derivatives

$$D_{\vec{v}} f(\vec{x}) = \frac{d}{dt} f(\vec{x} + \vec{v}t) = \sum_{\mu}^{N} v^{\mu} \frac{\partial}{\partial x^{\mu}} f(\vec{x}) \tag{5}$$

  - again, the derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ form ''natural'' basis directions

- one can define the cotangent space $T_P^*$ as the dual to $T_P$:
  - every element $\omega \in T_P^*$ is a linear map $\omega : T_P \to \mathbf{R}$

  $\omega(aV + bW) = a\omega(V) + b\omega(W) \in \mathbf{R}$ for $V, W \in T_P$ and $\omega \in T_P^*$

  - and the dual vector space to $T_P^*$ is $T_P$: $V : T_P^* \to \mathbf{R}$

  $V(a\omega + b\eta) = aV(\omega) + bV(\eta) \in \mathbf{R}$ for $\omega, \eta \in T_P^*$ and $V \in T_P$

## $N$ dimensional cotangent space $T_P^*$

- using the basis vectors $\vec{e}_{(\mu)}$ of $T_P$
  - we can define basis vectors $\vec{\theta}^{(\mu)}$ in $T_P^*$ by
$$\vec{\theta}^{(\nu)}(\vec{e}_{(\mu)}) = \vec{e}_{(\mu)}(\vec{\theta}^{(\nu)}) = \delta_\mu^\nu \tag{6}$$
  - and write a covector $\omega$ in terms of the $N$ basis vectors $\vec{\theta}^{(\mu)}$:
$$\omega = \omega_\mu \vec{\theta}^{(\mu)} \tag{7}$$
- the map $T_P \times T_P^* \to \mathbf{R}$ can be expressed as:
$$\omega(V) = \omega_\mu V^\nu \vec{\theta}^{(\mu)}(\vec{e}_{(\nu)}) = \omega_\mu V^\nu \delta_\nu^\mu = \omega_\mu V^\mu \in \mathbf{R} \tag{8}$$

  $\Rightarrow$ one can use components without specifying a basis
  - $*$ assuming that the bases of $T_P$ and $T_P^*$ are related by $\vec{\theta}^{(\nu)}(\vec{e}_{(\mu)}) = \delta_\mu^\nu$
- all $T_P$ with $P \in M$ give the $2N$ dimensional vector bundle $T_P(M)$
- all $T_P^*$ with $P \in M$ give the $2N$ dimensional cotangent bundle $T_P^*(M)$

# 1. General Relativity — tangent space

example $T_P(S^2)$: the 2-sphere $S^2$ with $\mathbf{R}^2$ attached to each point

- looking at the embedding in $\mathbf{R}^3$ we can choose coordinates $(x, y, z)$
  - for the sphere they have to satisfy $x^2 + y^2 + z^2 = r^2 = 1$
  - going to spherical coordinates we have on the sphere $(r = 1, \vartheta, \varphi)$
    - with $x = \sin\vartheta\cos\varphi$, $y = \sin\vartheta\sin\varphi$, and $z = \cos\vartheta$
  
  $\Rightarrow$ so for a coordinate patch (a map) we can use $(\vartheta, \varphi)$

- the point $P_0 = (\vartheta_0, \varphi_0)$ in $S^2$ is embedded in $\mathbf{R}^3$:
  - considering the curves through $P_0$

$$C_1(s) = (\sin\vartheta_0\cos s, \sin\vartheta_0\sin s, \cos\vartheta_0) \tag{9}$$
$$C_2(t) = (\sin t\cos\varphi_0, \sin t\sin\varphi_0, \cos t) \tag{10}$$

  - the tangent plane at $P_0$ has tangent vectors in $\mathbf{R}^3$

$$\vec{e}_{(1)} = \frac{d}{ds}C_1(s)\Big|_{P_0} = \sin\vartheta_0(-\sin\varphi_0, \cos\varphi_0, 0) \tag{11}$$
$$\vec{e}_{(2)} = \frac{d}{dt}C_2(t)\Big|_{P_0} = (\cos\vartheta_0\cos\varphi_0, \cos\vartheta_0\sin\varphi_0, -\sin\vartheta_0) \tag{12}$$

  - since $\vec{e}_{(1)}$ and $\vec{e}_{(2)}$ are linearly independent: $\nexists\lambda$ with $\vec{e}_{(1)} = \lambda\vec{e}_{(2)}$

  $\Rightarrow$ they form a basis in $T_{P_0}$

- using this basis, we can write any vector $V$ in $T_{P_0}$ as $V = V^1\vec{e}_{(1)} + V^2\vec{e}_{(2)} = V^i\vec{e}_{(i)}$

$\Rightarrow$ the coordinates of $V$ in $T_P(S^2)$ are $(\vartheta_0, \varphi_0, V^1, V^2)$

example $T_P(S^2)$: the 2-sphere $S^2$ with $\mathbf{R}^2$ attached to each point

- picking different curves through $P_0$ is the same as changing the basis of $T_{P_0}$
  - $V \to V'$ or $(\vartheta_0, \varphi_0, V^1, V^2) \to (\vartheta_0, \varphi_0, V'^1, V'^2)$
  - this change is a normal coordinate transformation in $\mathbf{R}^2$
    * which can also stretch and rotate $T_{P_0}$
- going to a point $P_1 = (\vartheta_1, \varphi_1)$
  - is a normal translation on $S^2$
  - but it can also change the basis of $T_P$: $(\vec{e}_{(1)}, \vec{e}_{(2)}) \to (\vec{e}\,'_{(1)}, \vec{e}\,'_{(2)})$
  - in $\mathbf{R}^2$ (and $\mathbf{R}^n$) we have an understanding, what is parallel
  - but $\vec{e}_{(i)}$ and $\vec{e}\,'_{(i)}$ are not necessarily parallel
    * in the embedding space $\mathbf{R}^3$ they would be

$$\vec{e}_{(1)} = s_{\vartheta_0}(-s_{\varphi_0}, c_{\varphi_0}, 0) \quad \text{not parallel to} \quad \vec{e}\,'_{(1)} = s_{\vartheta_1}(-s_{\varphi_1}, c_{\varphi_1}, 0) \tag{13}$$
$$\vec{e}_{(2)} = (c_{\vartheta_0}c_{\varphi_0}, c_{\vartheta_0}s_{\varphi_0}, -s_{\vartheta_0}) \quad \text{not parallel to} \quad \vec{e}\,'_{(2)} = (c_{\vartheta_1}c_{\varphi_1}, c_{\vartheta_1}s_{\varphi_1}, -s_{\vartheta_1}) \tag{14}$$

    * in $T_P(S^2)$ we only see

$$\vec{e}_{(1)} = (\vartheta_0, \varphi_0, 1, 0) \quad \ldots \quad \vec{e}_{(2)} = (\vartheta_0, \varphi_0, 0, 1) \tag{15}$$
$$\vec{e}\,'_{(1)} = (\vartheta_1, \varphi_1, 1, 0) \quad \ldots \quad \vec{e}\,'_{(2)} = (\vartheta_1, \varphi_1, 0, 1) \tag{16}$$

- how can we compare them from inside ?
  - we need a connection

connections allow the definition of parallel transport on $T_P(M)$

- for $P_0, P_1 \in M$ the affine connection relates $T_{P_0}$ and $T_{P_1}$
  - so that tangent vector fields can be differentiated
    - i.e. compared between the points $P_0$ and $P_1$
    - this comparison usually uses the concepts of pullback and pushforward
    - these concepts become too mathematically abstract for this lecture ...
- for our example it means, that by using an affine connection
  - we can effectively reduce the dimension of $T_P(M)$ back to $T_P$ (or $M$)
  - and the basis vectors in $T_P$ are smooth functions of the point:

$$\vec{e}_{(1)}(\vartheta, \varphi) \qquad \text{and} \qquad \vec{e}_{(2)}(\vartheta, \varphi) \tag{17}$$

- the Cartan connection uses the Lie group structure of $M$
  - and transports (coordinate) frames without specifying a metric
    - uses the exterior derivative
- the Levi-Civita connection uses the additional structure of a metric

  $\Rightarrow$ Riemannian manifolds ... will be assumed for the rest of this lecture formalizes the known differential of a multiparametric function

$$df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{18}$$

- choosing coordinates $(x,y,z) = (x^1, x^2, x^3)$
- the quantities $dx$, $dy$, and $dz$ are understood as covectors or 1-forms
  - forming a basis of $T_P^*$ with $\vec{\theta}^{(\nu)} = dx^\nu$
- the wegde product "$\wedge$" defines the multiplication of forms
  - it is antisymmetric: $dx \wedge dy = -dy \wedge dx$

    $\Rightarrow$ it is nilpotent: $dx \wedge dx = 0$
  - for obvious differentials $dx$ the $\wedge$ can be omitted: $dx\, dy := dx \wedge dy$
  - a one-form in this natural basis: $\omega = \omega_\mu dx^\mu$
  - a $p$-form in this natural basis: $\Omega = \Omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$
    - obviously, $\Omega_{\mu_1 \ldots \mu_p}$ is completely antisymmetric in its indices
    - and $p \leq N$, the dimension of $M$

# 1. General Relativity — exterior derivative

## features of the exterior derivative $d$

- increases the grade of a form
  - defining the smooth functions $f$ or $x^\mu$ as 0-forms
  - we have 1-forms $df$ or $dx^\mu$
  - a 2-form $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$ gives a 3-form $dF = (\partial_\rho F_{\mu\nu}) dx^\rho \wedge dx^\mu \wedge dx^\nu$
  - for a $N$-form $I = I_{\mu_1 \dots \mu_N} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_N}$ we get $dI = 0$
- defines the natural basis by

$$df(V) = (\partial_\mu f) dx^\mu (V^\nu \partial_\nu) = V^\nu (\partial_\mu f) \delta^\mu_\nu = V^\mu \partial_\mu (f) = V(f) \tag{19}$$

- follows the Leibnitz rule
  - for a $p$-form $\alpha$ and a $q$-form $\beta$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{20}$$

- is useful for the discussion of fiber bundles $G(M)$
  - with the Lie group $G$ defined at every point of $M$
  - the tangent bundle $T_P(M)$ is a fiber bundle with the tangent space $T_P$ as the structure given at each point in $M$
- all forms on $M$ form a vector space of dimension $2^N$

vector fields as derivatives of functions

- $df(V) = V(f)$ is again a function on $M$ for $V \in T_P$
- for $X, Y \in T_P$ we can define the Lie derivative or Lie bracket

$$\mathcal{L}_X(Y)(f) = [X, Y](f) := X(Y(f)) - Y(X(f)) \quad \in \mathbf{R} \tag{21}$$

- the Lie bracket
  - is bilinear and antisymmetric
  - fulfills the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \tag{22}$$

  - $\mathcal{L}_X(Y)$ can be seen as the directional derivative of $Y$ along $X$

    $\Rightarrow$ it allows to define a parallel transport

    * for $[X, Y](f) = 0$, $Y(f)$ stays constant along the flow of $X$

- the Lie bracket endows $M$ with the algebraic structure of a Lie algebra mathematical definition:

- a metric in $M$ is a symmetric, bilinear, non-degenerate function

$$g_P : T_P \times T_P \to \mathbf{R} \tag{23}$$

  - the metric acts on (contravariant) vectors (tensor indices)
  - symmetric means $g(X,Y) = g(Y,X) \in \mathbf{R}$ for $X, Y \in T_P$
  - bilinear means for $X, Y, Z \in T_P$ and $a, b, c \in \mathbf{R}$

$$g(aX + bY, cZ) = ac\, g(X,Z) + bc\, g(Y,Z) \in \mathbf{R} \tag{24}$$

  - non-degenerate: $g(X,Y) = 0$ for all $Y \in T_P$ only if $X = 0$
  - acting on the basis vectors $\vec{e}_{(\mu)}$ gives the metric tensor

$$g_{\mu\nu}(P) = g(\vec{e}_{(\mu)}, \vec{e}_{(\nu)})\Big|_P \tag{25}$$

- the metric allows length and angle measurements

$$\|X\| := \sqrt{|g(X,X)|} \qquad \text{and} \qquad \cos\varphi = \frac{g(X,Y)}{\|X\| * \|Y\|} \tag{26}$$

## examples for the metric

- in Euclidean space the metric is the normal dot-product:

$$\vec{a} \cdot \vec{b} = g_{jk} a^j b^k = \delta_{jk} a^j b^k \qquad \text{so} \quad g_{j=k} = 1 \quad \text{and} \quad g_{j \neq k} = 0 \tag{27}$$

  – we can use this metric of the embedding to get the induced metric in $S^2$

$$g_{jk} = g(\vec{e}_{(j)}, \vec{e}_{(k)}) = \vec{e}_{(j)} \cdot \vec{e}_{(k)} \qquad \text{so} \quad g_{11} = s^2_{\vartheta_0} \qquad g_{12} = g_{21} = 0 \qquad g_{22} = 1 \tag{28}$$

  ⇒ as we can see, the metric tensor depends on the position

- in Minkovsky space we had $g_{00} = 1$, $g_{ii} = -1$, and $g_{\mu \neq \nu} = 0$
  – we can generalize the line element

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = g_{\mu\nu} \Delta x^\mu \otimes \Delta x^\nu \tag{29}$$

  – the ''differentials'' $\Delta x$ here are **not** multiplied with their ''natural'' wedge product

- one can define the inverse metric $g^{\mu\nu}$ by

$$g^{\mu\nu} g_{\mu\rho} = \delta^\nu_\rho \tag{30}$$

  – the inverse metric is the metric of the cotangent space