Cosmology

Lectures: NFTMC B435, Monday 15^{00} to 17^{00} (catch-up) ; Tuesday 13^{00} to 15^{00} (lecture)

09/04 Overview, Introduction; description of the course

09/05 Special Relativity [1, 2]

09/11 Special Relativity: Astronomical relevance [1]

09/12 GR 1: General covariance [3, 6, 7]

09/18-19 *no lecture:* conference "Matter to the Deepest"

09/26 GR 2: Riemannian geometry & Einstein equations [3, 6, 7]

10/03 GR 3: Vacuum solutions: Schwarzschild, Reissner-Nordström, Kerr-Newman [3, 6, 7]

10/09-10 *no lecture:* 3rd CERN Baltic Conference

10/17 GR 4: symmetric solutions: de Sitter, anti de Sitter, FLRW; time evolution [3, 6, 7]

10/24 GR 5: Big Bang [3, 6, 7, 8, 9]

10/31 GR 6: Inflation [3, 6, 7, 8, 9]

11/07 GR 7: CMB [6, 7, 8, 9]

11/14 GR 8: Gravitational Waves [13, 14]

11/21 Special Relativity: Algebra of the Poincaré group

11/28 The Standard Model: Particle content

12/05 Supersymmetry (SUSY) & Dark Matter from SUSY

12/12 Particle detection, DM Searches

12/18 Repetition of the Homework

12/19+ Exam

The exam will be a written test, that I want to discuss afterwards with the student;
if needed also on Saturday or online ...

Homework is given to get familiar with concepts and notation.
can be discussed in the discussion / catch-up meetings ...

Students presentations are practicing the soft skill of presenting. The presentations should not last longer than 10-15 minutes, but also be longer than a pep-talk of 2-3 minutes. They allow the students to choose a subject of their interest that is *somehow* connected to GR and/or Cosmology (Astronomy).

References

[1] Lecture notes by David Hogg:
http://cosmo.nyu.edu/hogg/sr/sr.pdf

[2] David Griffiths, *Introduction to Elementary Particles*,
John Wiley & Sons, Inc.; ISBN 0-471-60386-4 (1987)

[3] S. M. Carroll, *Spacetime and geometry: An introduction to general relativity*,
San Francisco, USA: Addison-Wesley (2004) 513 p
free preprint: *Lecture Notes on General Relativity*, 1st section
gr-qc/9712019

[4] B. F. Schutz,
A First Course In General Relativity,
Cambridge, Uk: Univ. Pr. (1985) 376p

[5] C. W. Misner, K. S. Thorne and J. A. Wheeler,
Gravitation,
San Francisco 1973, 1279p

[6] T. P. Cheng, *Relativity, Gravitation, And Cosmology: A Basic Introduction*,
Oxford, UK: Univ. Pr. (2010) 435 p

[7] P. J. E. Peebles, *Principles of physical cosmology*,
Princeton, USA: Univ. Pr. (1993) 718 p

[8] D. H. Perkins, *Particle astrophysics*,
Oxford, UK: Univ. Pr. (2003) 256 p

[9] S. Dodelson, *Modern cosmology*,
Amsterdam, Netherlands: Academic Pr. (2003) 440 p

[10] The particle adventure:
http://www.particleadventure.org/

[11] Warren Siegel, *Fields*
http://insti.physics.sunysb.edu/~siegel/plan.html (2021)

[12] J. L. Cervantes-Cota, S. Galindo-Uribarri and G. F. Smoot,
A Brief History of Gravitational Waves
Universe **2** (2016) no.3, 22 doi:10.3390/universe2030022 [arXiv:1609.09400 [physics.hist-ph]].

[13] F. D'Ambrosio, S. D. B. Fell, L. Heisenberg, D. Maibach, S. Zentarra and J. Zosso,
Gravitational Waves in Full, Non-Linear General Relativity,
[arXiv:2201.11634 [gr-qc]].

[14] E. E. Flanagan and S. A. Hughes,
The Basics of gravitational wave theory,
New J. Phys. **7** (2005), 204 doi:10.1088/1367-2630/7/1/204 [arXiv:gr-qc/0501041 [gr-qc]].

Homework: Vector, Tensors

David Griffiths, Chapter 4, pp. 137-138, n. 4.6, and n. 4.7,
and David Griffiths, Chapter 3, pp. 100-102, n. 3.8:

4.6. Consider a vector $\vec{a}$ in two dimensions. Suppose its components with respect to Cartesian axes x, y, are (a_x, a_y). What are its components (a'_x, a'_y) in a system x', y' which is rotated, counterclockwise, by an angle θ, with respect to x, y? Express your answer in the for of a 2×2 matrix $R(\theta)$:

$$\begin{pmatrix} a'_x \\ a'_y \end{pmatrix} = R(\theta) \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

Show that R is an orthogonal matrix. What is its determinant? The set of *all* such rotations constitutes a group; what is the name of this group? By multiplying the matrices show that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$; is this an Abelian group?
0.6+0.2+0.2+0.2+0.2+0.2 POINTS

4.7. Consider the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Is it in the group $O(2)$? How about $SO(2)$? What is its effect on the vector $\vec{a}$ of Problem 4.6? Does it describe a possible rotation of the plane? 0.2+0.2+0.4+0.4 POINTS

3.8. A second–rank tensor is called *symmetric* if it is unchanged when you switch the indices ($s^{\nu\mu} = s^{\mu\nu}$); it is called *antisymmetric* if it changes sign ($a^{\nu\mu} = -a^{\mu\nu}$).

(a) How many independent elements are there in a symmetric tensor? (Since $s^{12} = s^{21}$, these would count as only *one* independent element.) 0.2 POINTS

(b) How many independent elements are there in an antisymmetric tensor?
0.2 POINTS

(c) If $s^{\mu\nu}$ is symmetric, show that $s_{\mu\nu}$ is also symmetric. If $a^{\mu\nu}$ is antisymmetric, show that $a_{\mu\nu}$ is antisymmetric. 0.2 POINTS

(d) If $s^{\mu\nu}$ is symmetric and $a^{\mu\nu}$ is antisymmetric, show that $s^{\mu\nu}a_{\mu\nu} = 0$. 0.2 POINTS

(e) Show that any second–rank tensor ($t^{\mu\nu}$) can be written as the sum of an antisymmetric part ($a^{\mu\nu}$) and a symmetric part ($s^{\mu\nu}$): ($t^{\mu\nu} = a^{\mu\nu} + s^{\mu\nu}$). Construct ($a^{\mu\nu}$) and ($s^{\mu\nu}$) explicitly, given ($t^{\mu\nu}$). 0.2 POINTS

Homework: Lorenztransformations

David Hoggs, Chapter 3, p. 14, Problems 3.3 and 3.4, and Chapter 4, p.22, Problems 4.7 and 4.8:

3.3. A rocket ship of proper length ℓ_0 travels at constant speed v in the $\hat{x}$-direction relative to a frame S. The nose of the ship passes the point $x = 0$ (in S) at time $t = 0$, and at this event a light signal is sent from the nose of the ship to the rear.

   (a) Draw a space-time diagram showing the worldlines of the nose and rear of the ship and the photon in S. 0.6 POINTS

   (b) When does the signal get to the rear of the ship in S? 0.6 POINTS

   (c) When does the rear of the ship pass $x = 0$ in S? 0.6 POINTS

3.4. At noon a rocket ship passes the Earth at speed $\beta = 0.8$. Observers on the ship and on Earth agree that it is noon. Answer the following questions and draw complete spacetime diagrams in both the Earth and rocket ship frames, showing all events and worldlines:

   (a) At 12:30 p.m., as read by a rocket ship clock, the ship passes an interplanetary navigational station that is fixed relative to the Earth and whose clocks read Earth time. What time is it at the station? 0.6 POINTS

   (b) How far from Earth, in Earth coordinates, is the station? 0.6 POINTS

   (c) At 12:30 p.m. rocket time, the ship reports by radio back to Earth. When does Earth receive this signal (in Earth time)? 0.6 POINTS

   (d) Earth replies immediately. When does the rocket receive the response (in rocket time)? 0.6 POINTS

   (e) The spacetime diagrams 0.4+0.4 POINTS

4.7. In an interplanetary race, slow team X is travelling in their old rocket at speed 0.9c relative to the finish line. They are passed by faster team Y, observing Y to pass X at 0.9c. But team Y observes fastest team Z to pass Y's own rocket at 0.9c. What are the speeds of teams X, Y and Z relative to the finish line? 0.8 POINTS

4.8. An unstable particle at rest in the Lab frame splits into two identical pieces, which fly apart in opposite directions at Lorentz factor $\gamma = 100$ relative to the Lab frame. What is one particle's Lorentz factor relative to the other? What is its speed relative to the other? 0.6 POINTS

Homework: Particle kinematics

David Hoggs, Chapter 6, p. 34, Problems 6.7, 6.8, 6.9, and 6.10:

6.7. A particle of mass M, at rest, decays into two smaller particles of masses m_1 and m_2. What are their energies and momenta? 0.4 POINTS

6.8. Solve problem 6.7 again for the case $m_2 = 0$. Solve the equations for p and E_1 and then take the limit $m_1 \to 0$. 0.6 POINTS

6.9. If a massive particle decays into photons, explain using 4-momenta why it cannot decay into a single photon, but must decay into two or more. Does your explanation still hold if the particle is moving at high speed when it decays? 2 POINTS

6.10. A particle of rest mass M, travelling at speed v in the x-direction, decays into two photons, moving in the positive and negative x-direction relative to the original particle. What are their energies? What are the photon energies and directions if the photons are emitted in the positive and negative y-direction relative to the original particle (i.e., perpendicular to the direction of motion, in the particle's rest frame). 1+1 POINTS

Homework: FLRW universes

The Friedmann equations

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{k}{a^2} \qquad \frac{\ddot a}{a} = -\frac{4}{3}\pi G(\rho + 3\mathbf{p}) = -\frac{4}{3}\pi G(1+3w)\rho$$

can be solved explicitely for specific matter content. $\dot a := \frac{\partial a}{\partial t}$ and $\ddot a := \frac{\partial^2 a}{\partial t^2}$

Show that the parametric solutions are really solutions and determine the parameter b for each solution:

A.1 For $w = 0$ we have $\rho = m * a^{-3}$ and

$$\begin{array}{ccc} \text{for } k=-1 & \text{for } k=0 & \text{for } k=+1 \\ \begin{aligned} a &= b\,(\cosh\phi - 1) \\ t &= b\,(\sinh\phi - \phi) \end{aligned} & a = \left(\frac{9b}{2}\right)^{1/3} t^{2/3} & \begin{aligned} a &= b\,(1-\cos\phi) \\ t &= b\,(\phi - \sin\phi) \end{aligned} \end{array}$$

2+1+2 POINTS

A.2 For $w = \frac{1}{3}$ we have $\rho = E * a^{-4}$ and

$$\begin{array}{ccc} \text{for } k=-1 & \text{for } k=0 & \text{for } k=+1 \\ a = \left[(2b+t)t\right]^{1/2} & a = (4b)^{1/4}\, t^{1/2} & a = \left[(2b-t)t\right]^{1/2} \end{array}$$

1+1+1 POINTS

A.3 For $w = -1$ we have $\rho = \frac{\Lambda}{8\pi G}$.

If $\Lambda < 0$ we have $k = -1$ and $a = b^{-1}\sin bt$. 1 POINTS

If $\Lambda > 0$ we have

$$\begin{array}{ccc} \text{for } k=-1 & \text{for } k=0 & \text{for } k=+1 \\ a = b^{-1}\sinh bt & a = b^{-1}e^{bt} & a = b^{-1}\cosh bt \end{array}$$

1+1+1 POINTS

B Find Killing vectors for the Robertson-Walker metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}^2\Omega\right]$$

6 POINTS

Homework: Spin

David Griffiths, Chapter 4, p. 139, n. 4.23 and inspired by David Griffiths, Introduction to Quantum Mechanics, p. 169, n. 4.38:

4.23. The extension of everything in Section 4.4 to higher spin is relatively straightforward. For spin 1 we have three state ($m_s = +1,\ 0,\ -1$), which can we may represent as column vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively. The only problem is to construct the 3×3 matrices $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$. The latter is easy:

(a) Construct $\hat{S}_z$ for spin 1. 0.5 POINTS

To obtain $\hat{S}_x$ and $\hat{S}_y$ it is easiest to start with the "raising" and "lowering" operators, $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$, which have the property

$$\hat{S}_\pm |sm\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)}\,|s(m \pm 1)\rangle$$

(b) Construct the matrices $\hat{S}_+$ and $\hat{S}_-$ for spin 1. 0.5 POINTS

(c) Using (b) determine the spin-1 matrices $\hat{S}_x$ and $\hat{S}_y$. 0.2 POINTS

(d) Do the same construction for spin $\frac{3}{2}$. 1.3 POINTS

"4.38" Consider a "boundstate" B, made out of two spin-$\frac{1}{2}$ states, f_1 and f_2. These states f_i are described by $\hat{S}^{(i)}$. The boundstate has a spinoperator

$$\hat{S} = \hat{S}^{(1)} + \hat{S}^{(2)}$$

with $\hat{S}^{(i)}$ acting on the ith constituent f_i.

(a) What are the possible eigenvalues of $\hat{S}_z$ for B? 0.5 POINTS

(b) What are the possible eigenvalues of $\hat{S}^2$ for B? 0.5 POINTS

(c) What are the possible eigenstates of B with respect to $\hat{S}^2$ and $\hat{S}_z$ in terms of the constituent eigenstates f_i? 0.5 POINTS

(d) How does that compare to the previous exercise 4.23 ? 0.5 POINTS

(e) How can you apply that to the two $SU(2)$ subgroups of the Lorentz group? 1.0 POINTS