Vectors, Tensors, and **notation**

in more dimensional space we just have more coordinates

- In 3D space (our 3D world):
  - $\vec{v} = (v_x, v_y, v_z) = v_i$ (in cartesian coordinates)

- In **4D Minkovsky space** people do **not** write an arrow:
  - **momentum** $p = (E = p^t, p^x, p^y, p^z) = (p^0, p^1, p^2, p^3) = p^\mu$
    - and the index is usually a greek letter: μ, ν, ρ, etc.
  - position $r = (ct, x, y, z) = (x^0, x^1, x^2, x^3) = r^\mu$
    - time $ct = x^0$ is measured like spacial distances in meters
    - The constant speed of light c is used as the conversion factor between seconds and meters

For the rest of the lecture we set $c = 1$. (i.e.: $3 \cdot 10^8\text{m} = 1\text{s}$)

- so we measure time in seconds and distances in light-seconds (=300.000km)
- or distances in meters and time in "3 nanoseconds" (the time light needs to travel 1m)

What are invariant objects?

- Objects that are the same for every inertial observer
- Examples in 3D: rotations or translations
  - the distances ℓ between points: $\ell^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$
  - the angle α between directions: $\cos\alpha = (\vec{a} \cdot \vec{b})/(|\vec{a}| * |\vec{b}|)$
- In 4D Minkovsky space: $(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$
  - The time t is measured like spacial distances in meters
  - The constant speed of light c is used as the conversion factor between seconds and meters
- Any scalar product of four-vectors in Minkovsky space:

$$(p.q) = p^\mu q^\nu g_{\mu\nu} = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3 \quad (1)$$

Special scalar products

- Particles are described by their energy-momentum four-vector:

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p}) \quad (2)$$

  - The **mass** of the particle is defined in its **rest-frame**: $\vec{p} = 0$
  - There, the energy-momentum four-vector is $p^\mu = (m, 0)$
  - Since $p^2 = (p \cdot p)$ is a scalar, it is the same in every frame
  - In the rest-frame $p^2 = m^2 - \vec{0}^2 = m^2$
  - Therefore in every frame

$$m^2 = E^2 - \vec{p}^{\,2} \ ! \quad (3)$$

- This can be applied to collisions, too: $(p_1 + p_2)^2$ is constant
  - In the rest-frame of $(p_1 + p_2)$ we have $\vec{p}_1 + \vec{p}_2 = 0 \ \Rightarrow (p_1 + p_2)^2 = (E_1 + E_2)^2$
    - E_1 and E_2 are the energy values of p_1 and p_2 in the rest-frame of $(p_1 + p_2)$!

The metric $g^{\mu\nu}$ (or $g_{\mu\nu}$) is used to raise (or lower) indices

- in flat Minkovsky space the components of the metric tensor are

$$g_{00} = g^{00} = 1 \qquad g_{ii} = g^{ii} = -1 \qquad g_{j\neq k} = g^{j\neq k} = 0 \tag{4}$$

- that gives for the contravariant four vector $p^\mu = (E/c, p_x, p_y, p_z)$
  - the covariant four vector

$$p_\mu = g_{\mu\nu} p^\nu = g_{\mu 0} E/c + g_{\mu 1} p_x + g_{\mu 2} p_y + g_{\mu 3} p_z \tag{5}$$

  so

$$p_0 = g_{00} E/c + g_{01} p_x + g_{02} p_y + g_{03} p_z = 1 \times E/c + 0 \times p_x + 0 \times p_y + 0 \times p_z = E/c \tag{6}$$

$$p_1 = g_{10} E/c + g_{11} p_x + g_{12} p_y + g_{13} p_z = 0 \times E/c + (-1) \times p_x + 0 \times p_y + 0 \times p_z = -p_x \tag{7}$$

$$p_2 = g_{20} E/c + g_{21} p_x + g_{22} p_y + g_{23} p_z = \ldots = -p_y \tag{8}$$

$$p_3 = g_{30} E/c + g_{31} p_x + g_{32} p_y + g_{33} p_z = \cdots = -p_z \tag{9}$$

  - and hence

$$p_\mu = (E/c, -p_x, -p_y, -p_z) \tag{10}$$

- an equation with four-vectors, eq.(5), are four equations, eqs.(6-9)
  - four-vectors give naturally energy-momentum conservation

3. Special Relativity (SR) — Collisions

Classical mechanics requires energy and momentum conservation

- these are four separate equations
  - one for energy
  - three for the three components of the momentum

- in the energy-momentum four-vector $p^\mu = (p^0, \vec{p})$ we have
  - Energy (or mass) in the zero component $p^0 = E/c$
  - three momentum in the components > 0
    $\vec{p} = (p^1, p^2, p^3) = (p_x, p_y, p_z)$

- writing energy-momentum conservation for homework 3.16

$$p_A^\mu + p_B^\mu = q_{C_1}^\mu + q_{C_2}^\mu + \cdots + q_{C_n}^\mu = \sum_{i=1}^{n} q_i^\mu \tag{11}$$

  - These are four equations – the form is the same in every frame
  - specifying a single of the four-vectors fixes the frame:
    * in homework 3.16, B is at rest, meaning: $p_B^\mu = (m_B, \vec{0}\,)$

3. Special Relativity (SR) — collision frame examples

Homework problem 3.16 has two meaningful frames (coordinate systems)

- the Lab-frame with the given initial four-vectors:

$$p_A^\mu = (E_A, \vec{p}_A) \quad \text{and} \quad p_B^\mu = (m_B, \vec{0}\ ) \tag{12}$$

- the CM-frame (center-of-mass coordinate system)
  - where the sum of initial (or final) 3-momenta is zero

$$\vec{p}\,'_A + \vec{p}\,'_B = 0 = \vec{q}\,'_{C_1} + \vec{q}\,'_{C_2} + \cdots + \vec{q}\,'_{C_n} = \sum_{i=1}^{n} \vec{q}_i\,' \tag{13}$$

  - the prime $'$ on the vectors indicates the "different" frame
    - the momenta were first given in the Lab-frame, but I kept the names

- the CM-frame is better suited to discuss properties of the final state
  - but we do not know the final momenta, not even the inital ones! (in this frame)
  - eq.(11) is still valid . . . $\Rightarrow$ use invariants

Homework problem 3.16 with invariants

- introducing an initial state four-vector P^μ in eq.(11)

$$p_A^\mu + p_B^\mu = P^\mu = q_{C_1}^\mu + q_{C_2}^\mu + \cdots + q_{C_n}^\mu = \sum_{i=1}^{n} q_i^\mu \tag{11a}$$

- we know that the "norm" of P^μ is the same in every frame:

$$P_{\text{Lab}}^2 = P^\mu P_\mu = P_{\text{CM}}'^2 = P'^\mu P'_\mu = \text{invariant} \tag{14}$$

- we can use this invariant to determine E_A in the Lab-frame:

$$\begin{aligned} \text{invariant} &= (E_A + m_B, \vec{p}_A)^2 = (E_A + m_B)^2 - (\vec{p}_A)^2 \qquad (15) \\ &= E_A^2 + 2E_A m_B + m_B^2 - (E_A^2 - m_A^2) = 2E_A m_B + m_B^2 + m_A^2 \end{aligned}$$

  – or simplified:

$$E_A = \frac{\text{invariant} - m_A^2 - m_B^2}{2m_B} \tag{16}$$

$\Rightarrow$ the smaller the invariant the smaller also E_A

Homework problem 3.16 with invariants

- How can we minimize the invariant ?
  - in the CM-frame

$$\text{invariant} = P'^2_{\text{CM}} = \left(\sum_{i=1}^{n} q_i'^0\right)^2 - \left(\sum_{i=1}^{n} \vec{q}_i{}'\right)^2 = \left(\sum_{i=1}^{n} E_i'\right)^2 \tag{17}$$

  - a particle has the minimal energy if it does not move:

$$E_{\text{min}} = \lim_{|\vec{p}|\to 0} \sqrt{m^2 + \vec{p}^2} = m \tag{18}$$

$\Rightarrow$ the minimal invariant for all particles at rest in the CM-frame !

$$\text{invariant}_{\text{min}} = \left(\sum_{i=1}^{n} m_i\right)^2 := M^2 \tag{19}$$

  - then

$$E_{A,\text{min}} = \frac{M^2 - m_A^2 - m_B^2}{2m_B} \tag{20}$$

3. Special Relativity (SR) — collision frame examples

Homework problem 3.19 has two (three) meaningful frames

- the rest-frames of A (= Lab-frame) or B (or C)

$$p_A^\mu = (m_A, \vec{0}\,) \quad \text{and} \quad p_B^\mu = (E_B, \vec{p}_B) \quad \text{and} \quad p_C^\mu = (E_C, \vec{p}_C) \tag{21}$$

  – taking eq.(11a) and renaming:

$$P^\mu \to p_A^\mu \quad \text{and} \quad p_A^\mu \to p_C^\mu \quad \text{and keeping} \quad p_B^\mu \tag{22}$$

  "solves" the setting of the frame

  – the conservation equations are $p_A^\mu = p_B^\mu + p_C^\mu$, or

$$m_A = E_B + E_C \quad \text{and} \quad \vec{p}_B = -\vec{p}_C := \vec{p} \tag{23}$$

- calculating the invariants in the three frames gives

$$p_A^2 = m_A^2 = (E_B + E_C)^2 - (\vec{p}_B + \vec{p}_C)^2 = (E_B + E_C)^2$$
$$p_B^2 = m_B^2 = (m_A - E_C)^2 - (\vec{p}_C)^2 = m_A^2 - 2m_A E_C + E_C^2 - (\vec{p}_C)^2 = m_A^2 - 2m_A E_C + m_C^2$$
$$p_C^2 = m_C^2 = (m_A - E_B)^2 - (\vec{p}_B)^2 = m_A^2 - 2m_A E_B + E_B^2 - (\vec{p}_B)^2 = m_A^2 - 2m_A E_B + m_B^2$$

Homework problem 3.22 has two meaningful frames

- the rest-frame of A (= Lab-frame)

$$p_A^\mu = (m_A, \vec{0}\,) \quad \text{and} \quad p_B^\mu = (E_B, \vec{p}_B) \quad \text{and} \quad p_{C_i}^\mu = (E_{C_i}, \vec{p}_{C_i}) \tag{24}$$

  – notation: C_1 means C, C_2 means D, etc. . . .

- the rest-frame of the particles C_i:

  – can be written more explicitely by

$$p_A'^\mu - p_B'^\mu = P'^\mu = q_{C_1}'^\mu + q_{C_2}'^\mu + \cdots + q_{C_n}'^\mu = \sum_{i=1}^{n} q_i'^\mu \tag{11b}$$

  – and then solved in analogy to problem 3.16

    * just note, that the minumum of E_A in 3.16 will correspond to the minimum of $-E_B$ in 3.22, which describes the maximum of E_B

3. Special Relativity (SR) — collision frame examples

Homework problem 3.25 has three meaningful frames

the energy-momentum conservation can be written in the

$$s\text{-channel} \quad S^\mu = p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu \tag{25}$$

$$t\text{-channel} \quad T^\mu = p_A^\mu - p_C^\mu = p_D^\mu - p_B^\mu \tag{26}$$

$$u\text{-channel} \quad U^\mu = p_A^\mu - p_D^\mu = p_C^\mu - p_B^\mu \tag{27}$$

- the invariants

$$s = S^2 \,, \quad t = T^2 \,, \text{ and } \quad u = U^2 \tag{28}$$

  have the same value in every frame

- calculating the invariants in terms of energies E and momenta $\vec{p}$
  - allows to compare the values of E and $\vec{p}$ between different frames
    * one frame for the initial input
    * another frame for the requested output

3. Special Relativity (SR) — collision frame examples

Homework problem 3.18 has three meaningful frames

$$p + p + (p) \to p + p + K^+ + K^- + (p) \qquad (29)$$
$$= [p + p + K^+] + K^- + p \to [p + p + K^+] + \Omega^- + K^0 + K^+$$

- the Lab-frame with

$$p^\mu = (E_p, \vec{p}) \ , \quad p^\mu = p^\mu = (m_p, \vec{0}) \ , \text{ and } \quad k^{-\mu} = (E_{K^-}, \vec{k}^{\,-}) \qquad (30)$$

- the rest-frame of $K^0 + K^+$
  - for the minimal needed value of E_{K^-}
- the rest-frame of $[p + p + K^+]$
  - for the minimum value of E_p
    - depending on the minimal needed value of E_{K^-}