Vectors, Tensors, and **notation**

in the plane — i.e. in the 2D (Euclidean) space

- we can pick a coordinate system and describe points with coordinates
  - Cartesian coordinates (x, y)
  - Polar coordinates (r, θ)

- a vector can be understood as a difference of points

- position vector: difference between the position and the origin

- we can write the vector $\vec{v}$
  - as a **row** (v_x, v_y)
  - or as a **column** $\binom{v_x}{v_y}$
  - or in **index notation** v_i or v^i, where we identify $v_x = v_1$ and $v_y = v_2$

Vectors, Tensors, and **notation**

in more dimensional space we just have more coordinates

- In 3D space (our 3D world):
  - $\vec{v} = (v_x, v_y, v_z) = v_i$ (in cartesian coordinates)
- In **4D Minkovsky space** people do **not** write an arrow:
  - **momentum** $p = (E = p^t, p^x, p^y, p^z) = (p^0, p^1, p^2, p^3) = p^\mu$
    - and the index is usually a greek letter: μ, ν, ρ, etc.
  - position $r = (ct, x, y, z) = (x^0, x^1, x^2, x^3) = r^\mu$
    - time $ct = x^0$ is measured like spacial distances in meters
    - The constant speed of light c is used as the conversion factor between seconds and meters

For the rest of the lecture we set $c = 1$. (i.e.: $3 \cdot 10^8$m = 1s)

- so we measure time in seconds and distances in light-seconds (=300.000km)
- or distances in meters and time in "3 nanoseconds" (the time light needs to travel 1m)

What are invariant objects?

- Objects that are the same for every inertial observer
- Examples in 3D: rotations or translations
  - the distances ℓ between points: $\ell^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$
  - the angle α between directions: $\cos\alpha = (\vec{a} \cdot \vec{b})/(|\vec{a}| * |\vec{b}|)$
- In 4D Minkovsky space: $(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$
  - The time t is measured like spacial distances in meters
  - The constant speed of light c is used as the conversion factor between seconds and meters

- Any scalar product of four-vectors in Minkovsky space:

$$(p.q) = p^\mu q^\nu g_{\mu\nu} = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3 \quad (1)$$

Classification of $d^2 = (d \cdot d) = d^\mu d_\mu = d^\mu d^\nu g_{\mu\nu}$

- If $d^2 > 0$ they are **time-like** separated:
  - one event happens before the other in every frame
  - there is a frame, where A and B happen at the same position
  - in this frame $d^\mu = (\Delta t, 0, 0, 0)$ with $\Delta t = \sqrt{d^2}$
- If $d^2 = 0$ they are **light-like** related. If $A \neq B$:
  - there is no frame, where A and B happen at the same time
  - there is no frame, where A and B happen at the same position
  - there is a frame, where $d^\mu = (\eta, \eta, 0, 0)$ with η arbitrary
- If $d^2 < 0$ they are **space-like** separated:
  - there is a frame, where A and B happen at the same time
  - in this frame $d^\mu = (0, \Delta s, 0, 0)$, with $\Delta s = \sqrt{-d^2}$,
    if the x-axis is oriented in the direction $\overline{AB}$

Special scalar products

- Particles are described by their energy-momentum four-vector:

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p}) \qquad (2)$$

  - The **mass** of the particle is defined in its **rest-frame**: $\vec{p} = 0$
  - There, the energy-momentum four-vector is $p^\mu = (m, 0)$
  - Since $p^2 = (p \cdot p)$ is a scalar, it is the same in every frame
  - In the rest-frame $p^2 = m^2 - \vec{0}^2 = m^2$
  - Therefore in every frame

$$m^2 = E^2 - \vec{p}^{\,2} \ ! \qquad (3)$$

- This can be applied to collisions, too: $(p_1 + p_2)^2$ is constant
  - In the rest-frame of $(p_1 + p_2)$ we have $\vec{p}_1 + \vec{p}_2 = 0 \ \Rightarrow (p_1 + p_2)^2 = (E_1 + E_2)^2$
    - E_1 and E_2 are the energy values of p_1 and p_2 in the rest-frame of $(p_1 + p_2)$!

2. Special Relativity (SR) — Summary of SR1

Space-time objects are described by four-vectors

for better space management on the slides
I write all vectors as row vectors

- position $r^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$
  - Time by the zero component $r^0 = x^0 = ct$
  - Space by the components > 0
    $\vec{r} = r^i = x^i = (x^1, x^2, x^3) = (x, y, z)$
- momentum $p^\mu = (E/c, p_x, p_y, p_z) = (p^0, p^1, p^2, p^3)$
  - Energy (or mass) by the zero component $p^0 = E/c$
  - three momentum by the components > 0
    $\vec{p} = (p^1, p^2, p^3) = (p_x, p_y, p_z)$
- mass is defined in the rest-frame: $\vec{p} = 0$
  - There, the energy-momentum four-vector is $p^\mu = (m, 0)$

2. Special Relativity (SR) — raising and lowering indices

The metric $g^{\mu\nu}$ (or $g_{\mu\nu}$) is used to raise (or lower) indices

- in flat Minkovsky space the components of the metric tensor are

$$g_{00} = g^{00} = 1 \qquad g_{ii} = g^{ii} = -1 \qquad g_{j\neq k} = g^{j\neq k} = 0 \tag{4}$$

- that gives for the contravariant four vector $p^\mu = (E/c, p_x, p_y, p_z)$
  - the covariant four vector

$$p_\mu = g_{\mu\nu} p^\nu = g_{\mu 0} E/c + g_{\mu 1} p_x + g_{\mu 2} p_y + g_{\mu 3} p_z \tag{5}$$

  so

$$\begin{aligned}
p_0 &= g_{00} E/c + g_{01} p_x + g_{02} p_y + g_{03} p_z = 1 \times E/c + 0 \times p_x + 0 \times p_y + 0 \times p_z = E/c \\
p_1 &= g_{10} E/c + g_{11} p_x + g_{12} p_y + g_{13} p_z = 0 \times E/c + (-1) \times p_x + 0 \times p_y + 0 \times p_z = -p_x \\
p_2 &= g_{20} E/c + g_{21} p_x + g_{22} p_y + g_{23} p_z = \ldots = -p_y \\
p_3 &= g_{30} E/c + g_{31} p_x + g_{32} p_y + g_{33} p_z = \cdots = -p_z
\end{aligned}$$

  - and hence

$$p_\mu = (E/c, -p_x, -p_y, -p_z) \tag{6}$$

2. Special Relativity (SR) — raising and lowering indices

The metric $g^{\mu\nu}$ (or $g_{\mu\nu}$) is used to raise (or lower) indices

- check yourself that the invariant product, eq.(1), is really invariant

$$(p.q) = p^\mu q^\nu g_{\mu\nu} = p^\mu q_\mu = p_\nu q^\nu = p_\mu q_\nu g^{\mu\nu} \tag{7}$$

- for tensors we have to operate on each index
  - lowering both indices of a tensor $s^{\mu\nu}$

$$s_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} s^{\alpha\beta} = g_{\mu\mu'} g_{\nu\nu'} s^{\mu'\nu'} \tag{8}$$

  - what we **cannot** write:

$$\cancel{s_{\mu\nu} = g_{\mu\mu} g_{\nu\nu} s^{\mu\nu}} \tag{9}$$

  - we would make several mistakes:
    * the indices left and right do not match: $_{\mu\nu} \neq {}^{\mu\nu}$
    * the summation is not defined: we have 4 appearances of μ or ν

Lorentz transformations

... will be derived in two weeks

$$ct' = \gamma(ct - \beta\, x) \qquad x' = \gamma(x - \beta\, ct) \tag{10}$$

where

$$\beta := v/c \tag{11}$$

$$\gamma := [1 - \beta^2]^{-1/2} \tag{12}$$

- Time dilation is described in chapter 2.1 of D.Hogg's notes
  - by means of the thought experiment of a light clock
  - it can be summarized by

    moving clocks go slower by the factor γ

  - that means: the decay time of a moving particle is longer: $t = \gamma\tau$
- Length contraction is discussed in a similar way in chapter 2.3:

  a moving object seems shorter by the factor γ

  - the length of a moving volume is shorter: $\ell_{\text{measured}} = \frac{1}{\gamma}\ell_{\text{rest-frame}}$