Groups — what is a group? (repetition)

- a set G together with a "multiplication $\circ$" with the properties:
  - for $a, b \in G \Rightarrow c = a \circ b \in G$
  - $(a \circ b) \circ c = a \circ (b \circ c)$
  - $\forall a \in G : \exists e \in G$ with $a \circ e = e \circ a = a$
  - $\forall a \in G : \exists a^{-1} \in G$ with $a \circ a^{-1} = a^{-1} \circ a = e$
- if $a \circ b = b \circ a \ \forall a, b \in G$: abelian group, otherwise non-abelian
  - abelian: $\{\mathcal{R}, +\}$ or $\{\mathcal{R}^+, \times\}$
  - non-abelian: regular square matrices with the matrix multiplication
- continuous groups: the elements depend on a continuous parameter
  - example: rotations around an axis $R[\theta]$ with $\theta \in [0, 2\pi)$
- Lie group: a continuous group with an analytic multiplication
  - $g[\vec{x}] \circ g[\vec{y}] = g[f(\vec{x}, \vec{y})]$ with $f(\vec{x}, \vec{y})$ analytic in $\vec{x}$ and $\vec{y}$
  - the unit element is $e = g[\vec{0}]$

Lie groups and Lie algebras

- The $n \times n$ (complex) matrices form representations of Lie groups

- group multiplication is analytic $\Rightarrow$ expansion around unit element
  - unit element $e = \mathbf{1}_{n\times n}$
  - representation $T(g[\alpha]) = \exp[i\alpha_i X_i]$ $\Rightarrow$ $X_k = -i\frac{\partial T(g[\alpha])}{\partial \alpha_k}|_{\vec{\alpha}=0}$
  - generators $\{X_k\}$ span the representation of the Lie group

- the generators $\{X_k\}$ fulfill the Lie algebra $[X_j, X_k] = C_{jk}{}^{\ell} X_\ell$
  - with the antisymmetric structure constants $C_{jk}{}^{\ell} = -C_{kj}{}^{\ell}$
  - rank of the group: number of commuting generators
  - a Casimir operator commutes with all generators $\Rightarrow$ $\propto e$

- the indices i, j, k, ℓ need not indicate single numbers!
  - for the generators we will have $X_i = X_{[mn]} = -X_{[nm]}$

Representations of the Lie group

- using the Jacobi identity

$$
\begin{aligned}
0 &= [A,[B,C]] + [B,[C,A]] + [C,[A,B]] \\
&= ABC - ACB - BCA + CBA \ + \ BCA - BAC - CAB + ACB \ + \ CAB - CBA - ABC + BAC
\end{aligned}
$$

we get for the structure constants

$$
\begin{aligned}
0 &= C_{bc}{}^{d}[A,D] + C_{ca}{}^{d}[B,D] + C_{ab}{}^{d}[C,D] \\
&= C_{bc}{}^{d}C_{ad}{}^{e} + C_{ca}{}^{d}C_{bd}{}^{e} + C_{ab}{}^{d}C_{cd}{}^{e} = -(C_{ca}{}^{d}C_{db}{}^{e} - C_{cb}{}^{d}C_{da}{}^{e}) + C_{ab}{}^{d}C_{cd}{}^{e}
\end{aligned}
$$

- writing the structure constants as matrices $(M_k)_j{}^{\ell} = C_{jk}{}^{\ell}$ we have

$$
0 = -[(M_a)_c{}^{d}(M_b)_d{}^{e} - (M_b)_c{}^{d}(M_a)_d{}^{e}] + C_{ab}{}^{d}(M_d)_c{}^{e}
$$

or

$$
[M_a, M_b] = C_{ab}{}^{d} M_d
$$

$\Rightarrow$ structure constants form the adjoint representation of the Lie group

Translation and Rotation Operators

- The momentum operator $\vec{P} = -i\frac{\partial}{\partial\vec{x}} = -i\vec{\partial}$ generates translations:
  - in index notation: $P_k = -i\frac{\partial}{\partial x^k} = -i\partial_k$

$$
\begin{aligned}
e^{ia^k P_k} f(x) &= e^{a^k \partial_k} f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (a^k \partial_k)^n f(x) \\
&= f(x) + a^k \partial_k f(x) + \frac{1}{2} a^j a^k \partial_j \partial_k f(x) + \ldots
\end{aligned}
$$

  - the Taylorseries of $f(x+a)$ is

$$
f(x+a) = f(x) + a^k \partial_k f(x) + \frac{1}{2} a^j a^k \partial_j \partial_k f(x) + \cdots = e^{i\vec{a}\vec{P}} f(x)
$$

$\Rightarrow$ the operator $e^{i\vec{a}\vec{P}}$ moves the function f by the amount $\vec{a}$

- The angular momentum operator $\vec{L} = \vec{X} \times \vec{P}$ generates rotations
  - in index notation: $L_j = \epsilon_{jk\ell}\, x^k P_\ell = -i\epsilon_{jk\ell}\, x^k \partial_\ell$
  - or $L_x = i(z\partial_y - y\partial_z)$, $L_y = i(x\partial_z - z\partial_x)$, $L_z = i(y\partial_x - x\partial_y)$

Translation and Rotation Operators

- The components of $\vec{L}$ do not commute:
  - if you rotate around the $\hat{x}$-axis and then around the $\hat{y}$-axis, you get a different result than rotating first around $\hat{y}$ and then $\hat{x}$.
  - mathematically:

$$\begin{aligned}
[L_y, L_x] &= i^2[(x\partial_z - z\partial_x)(z\partial_y - y\partial_z) - (z\partial_y - y\partial_z)(x\partial_z - z\partial_x)] \\
&= i^2[(x\partial_y + xz\partial_z\partial_y - xy\partial_z^2 - z^2\partial_x\partial_y + zy\partial_x\partial_z) \\
&\quad -(zx\partial_y\partial_z - z^2\partial_y\partial_x - yx\partial_z^2 + y\partial_x + yz\partial_z\partial_x)] \\
&= i^2[x\partial_y - y\partial_x] = -iL_z
\end{aligned}$$

  - or in index notation: $[L_j, L_k] = i\epsilon_{jk\ell}L_\ell \quad \Rightarrow$ Rotationgroup
- but the square $L^2 = \vec{L}\cdot\vec{L} = L_kL_k$ does commute:

$$\begin{aligned}
[L^2, L_j] &= L_k[L_k, L_j] + [L_k, L_j]L_k = L_k i\epsilon_{kj\ell}L_\ell + i\epsilon_{kj\ell}L_\ell L_k \\
&= L_h i\epsilon_{hjm}L_m + i\epsilon_{mjh}L_hL_m = i(\epsilon_{hjm} + \epsilon_{mjh})L_hL_m = 0
\end{aligned}$$

$\Rightarrow$ use L^2 and L_z to describe quantum mechanical states (particles)

Eigenstates of the Rotationgroup

- We write an eigenstate of the operators L^2 and L_z as $|\lambda, m\rangle$

$$L^2|\lambda, m\rangle = \lambda|\lambda, m\rangle \qquad \text{and} \qquad L_z|\lambda, m\rangle = m|\lambda, m\rangle$$

  – $|f\rangle$ is called a ket and used to denote a quantum mechanical state.

- We define the ladder operators $L_\pm = L_x \pm iL_y$ with

$$\begin{aligned} [L^2, L_\pm] &= [L^2, L_x] \pm i[L^2, L_y] = 0 \qquad \text{and} \\ [L_z, L_\pm] &= [L_z, L_x] \pm i[L_z, L_y] = iL_y \pm i(-iL_x) = \pm(L_x \pm iL_y) = \pm L_\pm \end{aligned}$$

$\Rightarrow$ $L_\pm|\lambda, m\rangle$ is also an eigenstate of L^2 and L_z :

$$\begin{aligned} L^2(L_\pm|\lambda, m\rangle) &= ([L^2, L_\pm] + L_\pm L^2)|\lambda, m\rangle = 0 + L_\pm L^2|\lambda, m\rangle \\ &= L_\pm \lambda|\lambda, m\rangle = \lambda(L_\pm|\lambda, m\rangle) \end{aligned}$$

and

$$\begin{aligned} L_z(L_\pm|\lambda, m\rangle) &= ([L_z, L_\pm] + L_\pm L_z)|\lambda, m\rangle = (\pm L_\pm + L_\pm L_z)|\lambda, m\rangle \\ &= (\pm L_\pm + L_\pm m)|\lambda, m\rangle = (m \pm 1)(L_\pm|\lambda, m\rangle) \end{aligned}$$

Eigenstates of the Rotationgroup

- $L_\pm$ does not change the eigenvalue λ of the state $|\lambda, m\rangle$
- $L_\pm$ changes the eigenvalue m of the state $|\lambda, m\rangle$

$\Rightarrow$ the states $|\lambda, m+n\rangle$ with $n \in Z$ are related

$\Rightarrow$ for each λ there would be ∞ many states unless there is

 * $a = m_{\text{max}}$ with $L_+|\lambda, a\rangle = 0$ and
 * $b = m_{\text{min}}$ with $L_-|\lambda, b\rangle = 0$

- using

$$\begin{aligned} L_\mp L_\pm &= (L_x \mp iL_y)(L_x \pm iL_y) &&= L_x^2 \pm iL_xL_y \mp iL_yL_x + L_y^2 \\ &= (L_x^2 + L_y^2 + L_z^2) - L_z^2 \pm i[L_x, L_y] &&= L^2 - L_z^2 \pm i(iL_z) \\ &= L^2 - L_z(L_z \pm 1) \end{aligned}$$

we can relate a and b

Eigenstates of the Rotationgroup

- relating a and b:
  - $0 = L_- L_+ |\lambda, a\rangle = (\lambda - (a^2 + a))|\lambda, a\rangle \quad \Rightarrow \quad \lambda = a^2 + a$
  - $0 = L_+ L_- |\lambda, b\rangle = (\lambda - (b^2 - b))|\lambda, b\rangle \quad \Rightarrow \quad \lambda = b^2 - b$

$$a(a+1) = b(b-1) \quad \text{or} \quad a = -b$$

- Applying (L_-) n times on the state $|\lambda, a\rangle$ gives $|\lambda, a - n\rangle$
- for some n we have to reach $|\lambda, b\rangle \quad \Rightarrow \quad a - n = b$
- with $a = -b$ we get $a - n = -a$ or $m_{\text{max}} = a = \frac{n}{2}$
- The rotationgroup allows for half integer eigenstates

$$\Rightarrow \quad \text{Spinors}$$

- used to describe fermions: electron, proton, neutron, neutrino, . . .

Lie Algebra of the rotation group

- a rotation around the $\hat{z}$-axis by the angle θ is done by the matrix

$$R[\theta] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{i\theta L_z}$$

  – in index notation: $R[\theta]^j{}_k = \cos\theta(\delta^j_1\delta^1_k + \delta^j_2\delta^2_k) - \sin\theta(\delta^j_1\delta^2_k - \delta^j_2\delta^1_k) + \delta^j_3\delta^3_k$

- so the generator of the rotations, iL_z, is

$$iL_z = \left.\frac{\partial R[\theta]}{\partial\theta}\right|_{\theta=0} = \left.\begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\right|_{\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  – and similar

$$iL_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad iL_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- these rotations (incl. L_x and L_y) act on 3d column vectors $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

Lie Groups — What is a generator (in a Lie group) ?

the generator captures the ”feature” of the group element

- group elements from the same generator commute
- the generator X_k is the abstract ”direction”, as seen from
  - $X_k = -i \frac{\partial T(g[\alpha])}{\partial \alpha_k}|_{\vec{\alpha}=0}$
  - or $P_k = -i \frac{\partial}{\partial x^k} = -i\partial_k$
  - or $(L_z)^j{}_k = i(y\partial_x - x\partial_y)^j{}_k = \delta^j_2 \delta^1_k - \delta^j_1 \delta^2_k$
    * (L_z) takes the value of $y\,(-x)$ and puts it into $x\,(y)$

- X_k creates the exponential representation: $T(g[\alpha]) = \exp[i\alpha_k X_k]$
  - with the group elements $g[\alpha]$
  - parametrized by $\vec{\alpha} = \alpha_k$ ... i.e. the group parameter space
    * example: $(0, 2\pi) \otimes (0, \pi) \otimes (0, 2\pi)$ for 3D rotations with Euler axis and angle
  - representing the group element $g[\alpha]$ as the matrix $T(g[\alpha])$

Lie Algebra of the rotation group

- with simple matrix multiplication we can see:

$$[iL_x, iL_y] = -iL_z \qquad [iL_y, iL_z] = -iL_x \qquad [iL_z, iL_x] = -iL_y$$

  - or in index notation with $x = 1$, $y = 2$, and $z = 3$: $[L_j, L_k] = i\epsilon_{jk\ell} L_\ell$

- but there is a smaller dimensional realisation of the rotation group!
  - using the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  - one can define the Spin matrices $S_k = \frac{1}{2}\sigma_k$, which give

$$[S_j, S_k] = i\epsilon_{jk\ell} S_\ell$$

- these Spin matrices act on 2d complex column vectors $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$ $\Rightarrow$ Spinors

$\Rightarrow$ fundamental representation of the rotation group $SU(2)$

Rotations of Spinors

- with simple matrix multiplication we can see for the Pauli matrices:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{2\times 2}$$

- So the finite rotation of a spinor around the $\hat{y}$-axis is

$$\begin{aligned} R[\theta] &= e^{i\theta S_y} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta \tfrac{1}{2}\sigma_y)^n = \sum_{n=2m} \frac{1}{n!}(i\tfrac{\theta}{2})^n \sigma_y^n + \sum_{n=2m+1} \frac{1}{n!}(i\tfrac{\theta}{2})^n \sigma_y^n \\ &= \sum_{n}^{\infty} \frac{(-1)^n (\frac{\theta}{2})^{2n}}{(2n)!}(\sigma_y^2)^n + i\sum_{n}^{\infty} \frac{(-1)^n (\frac{\theta}{2})^{2n+1}}{(2n+1)!}(\sigma_y^2)^n \sigma_y \\ &= \cos\tfrac{\theta}{2} * 1_{2\times 2} + i \sin\tfrac{\theta}{2}\sigma_y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

  - acting on the spinor $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\Rightarrow$ spinors rotate only with half of the rotation angle θ

Lorentz transformations (like Galilean transformations)
consist of Boosts and Rotations

- a boost in $\hat{x}$ was done by

$$\Lambda(\eta)^{\mu}{}_{\nu} \;=\; \cosh\eta(\delta^{\mu}_{0}\delta^{0}_{\nu}+\delta^{\mu}_{1}\delta^{1}_{\nu})-\sinh\eta(\delta^{\mu}_{0}\delta^{1}_{\nu}+\delta^{\mu}_{1}\delta^{0}_{\nu})+\delta^{\mu}_{2}\delta^{2}_{\nu}+\delta^{\mu}_{3}\delta^{3}_{\nu}$$

- a rotation between $\hat{y}$ and $\hat{z}$ can be done by

$$\Lambda(\theta)^{\mu}{}_{\nu} = \delta^{\mu}_{0}\delta^{0}_{\nu}+\delta^{\mu}_{1}\delta^{1}_{\nu}+\cos\theta(\delta^{\mu}_{2}\delta^{2}_{\nu}+\delta^{\mu}_{3}\delta^{3}_{\nu})-\sin\theta(\delta^{\mu}_{2}\delta^{3}_{\nu}-\delta^{\mu}_{3}\delta^{2}_{\nu})$$

- we obtain the generators for boosts with $-i\frac{\partial\Lambda(\eta)^{\mu}{}_{\nu}}{\partial\eta}|_{\eta=0} =$

$$-i\sinh\eta(\delta^{\mu}_{0}\delta^{0}_{\nu}+\delta^{\mu}_{1}\delta^{1}_{\nu})+i\cosh\eta(\delta^{\mu}_{0}\delta^{1}_{\nu}+\delta^{\mu}_{1}\delta^{0}_{\nu})|_{\eta=0} = i(\delta^{\mu}_{0}\delta^{1}_{\nu}+\delta^{\mu}_{1}\delta^{0}_{\nu})$$

- we obtain the generators for rotations with $-i\frac{\partial\Lambda(\theta)^{\mu}{}_{\nu}}{\partial\theta}|_{\theta=0} =$

$$+i\sin\theta(\delta^{\mu}_{2}\delta^{2}_{\nu}+\delta^{\mu}_{3}\delta^{3}_{\nu})+i\cos\theta(\delta^{\mu}_{2}\delta^{3}_{\nu}-\delta^{\mu}_{3}\delta^{2}_{\nu})|_{\theta=0} = i(\delta^{\mu}_{2}\delta^{3}_{\nu}-\delta^{\mu}_{3}\delta^{2}_{\nu})$$

Boosts and Rotations ... continued

- The other boosts go in $\hat{y}$ or $\hat{z}$ direction: $i(\delta^\mu_0 \delta^i_\nu + \delta^\mu_i \delta^0_\nu)$,
  or with the indices $0i$ down: $(M_{0i})^\mu{}_\nu = i(\delta^\mu_0(-g_{i\nu}) + \delta^\mu_i g_{0\nu})$
- The other rotations go in $\hat{x}\hat{y}$ or $\hat{x}\hat{z}$ direction: $i(\delta^\mu_j \delta^k_\nu - \delta^\mu_k \delta^j_\nu)$,
  or with the indices jk down: $(M_{jk})^\mu{}_\nu = i(\delta^\mu_j(-g_{k\nu}) - \delta^\mu_k(-g_{j\nu}))$
- both generators have now the same form:

$$(M_{\alpha\beta})^\mu{}_\nu = -i(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu})$$

- with $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ we get

$$\Lambda(\omega)^\mu{}_\nu = \exp[i(M_{\alpha\beta}\omega^{\alpha\beta})^\mu{}_\nu] = \exp[(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu})\omega^{\alpha\beta}]$$

- How to understand / use this formula? ... How to get a matrix?
  1. pick the indices of $\omega^{\alpha\beta}$: ω^{0i} (ω^{jk}) for a boost (rotation) in $\hat{i}$- ($\hat{j}\hat{k}$-) direction
  2. write the matrix $\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu}$ with row-(column-) number μ (ν)
     * it will only have two non-zero entries
  3. squaring the matrix gives a diagonal matrix with only two equal entries
  4. the powerseries expansion gives you the expected boost / rotation

Generators for the Lorentz transformations

- these generators fulfill the Lie algebra of the Lorentz group:

$$[M_{\alpha\beta}, M_{\gamma\delta}]^{\mu}{}_{\nu} = i(g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\delta}M_{\beta\gamma} + g_{\beta\delta}M_{\alpha\gamma})^{\mu}{}_{\nu}$$

- unifying time and spatial translations $P_\mu = (H, P_i)$
- we get the rest of the Poincaré algebra:

$$[P_\mu, P_\nu] = 0 \qquad \text{and} \qquad [M_{\alpha\beta}, P_\mu] = i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)$$

- the generators of the Poincaré group are: P_μ and $M_{\alpha\beta}$
  - all rotations, boosts, and translations are elements of the Poincaré group

Invariants of the Poincaré group

- are objects that commute with all elements of the Poincaré group
  - it is enough to check if they commute with the generators . . .

Invariants of the Poincaré group

- obviously $[ab, c] = a[b, c] + [a, c]b = abc - acb + acb - cab = abc - cab$
- so $[P_\mu, P^2] = [P_\mu, P_\nu]P^\nu + P^\nu[P_\mu, P_\nu] = 0$
- and
$$\begin{aligned}[M_{\alpha\beta}, P^2] &= g^{\mu\nu}[M_{\alpha\beta}, P_\mu]P_\nu + g^{\mu\nu}P_\mu[M_{\alpha\beta}, P_\nu] \\ &= g^{\mu\nu}i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)P_\nu + g^{\mu\nu}P_\mu i(g_{\alpha\nu}P_\beta - g_{\beta\nu}P_\alpha) \\ &= -2i[P_\alpha, P_\beta] = 0\ .\end{aligned}$$

⇒ $P^2 = m^2$ invariant is a consequence of the Poincaré algebra!

- Another invariant is W^2
  - with the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}M_{\nu\rho}P_\lambda$
$$\begin{aligned}[P_\kappa, W^\mu] &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}([P_\kappa, M_{\nu\rho}]P_\lambda + M_{\nu\rho}[P_\kappa, P_\lambda]) \\ &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}i(g_{\rho\kappa}P_\nu - g_{\nu\kappa}P_\rho)P_\lambda = 0 \quad \Rightarrow \quad [P_\kappa, W^2] = 0\end{aligned}$$
  - $0 = [M_{\alpha\beta}, W^2]$ is true, but checking is too difficult . . .

⇒ Particles can be characterised by the eigenvalues of P^2 and W^2

Eigenvalues of P^2 and W^2

- the spin vector W^μ is orthogonal to P_μ :

$$(P.W) = P^\mu \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda} M^{\nu\rho} P^\lambda = 0$$

- For a particle at rest: $P_\mu = (m, 0)$
  - $P^2 = m^2 \Rightarrow$ the eigenvalue of P^2 is m^2
  - $W_\mu = \frac{1}{2} m \epsilon_{\mu\nu\rho 0} M^{\nu\rho} = m(0, \vec{J})$
  - so $W^2 = m^2(0^2 - \vec{J}^2) = -m^2\vec{J}^2 \to -m^2 s(s+1)$

  $\Rightarrow$ the eigenvalue of W^2 is $m^2 s(s+1)$

- For a massless particle $P_\mu = (\eta, \eta, 0, 0)$
  - we have $P^2 = (P.W) = W^2 = 0$

    $\Rightarrow$ the eigenvalues of P^2 and W^2 are 0
  - we can construct the operator $0 = \lambda^2 P^2 - 2\lambda(P.W) + W^2 = (\lambda P - W)^2$
    - where λ depends on the representation (i.e. the spin) of the particle
  - we get: $W^\mu = \lambda P^\mu$ with the helicity $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$

$\Rightarrow$ Particles are characterised by mass and spin !

Investigating the Lorentz group

- distinguishing again boosts and rotations

$$K_i = M_{0i} = -M^{0i} \qquad \text{and} \qquad J_i = \tfrac{1}{2}\epsilon_{ijk}M^{jk} \ ,$$

the Lorentz algebra gives

$$[J_j, J_k] = i\epsilon_{jk\ell}J_\ell \ , \quad [K_j, K_k] = -i\epsilon_{jk\ell}J_\ell \ , \quad [J_j, K_k] = i\epsilon_{jk\ell}K_\ell$$

- defining

$$L_i = N_i = \tfrac{1}{2}(J_i + iK_i) \qquad \text{and} \qquad R_i = N_i^\dagger = \tfrac{1}{2}(J_i - iK_i)$$

one gets

$$[L_j, R_k] = 0 \ , \quad [L_j, L_k] = i\epsilon_{jk\ell}L_\ell \ , \quad [R_j, R_k] = i\epsilon_{jk\ell}R_\ell$$

$\Rightarrow$ the Lorentz algebra is similar to $SU(2)_L \otimes SU(2)_R$!

- it has two invariants: $L_iL_i = n(n+1)$ and $R_iR_i = m(m+1)$
  - the angular momentum is $J_i = L_i + R_i \quad \Rightarrow \quad$ spin $j = n + m$

Investigating the Lorentz group

- Parity leaves rotations invariant $J_i \xrightarrow{\mathsf{P}} J_i$, but flips boosts $K_i \xrightarrow{\mathsf{P}} -K_i$,
  $\Rightarrow$ $L_i \xleftrightarrow{\mathsf{P}} R_i$, $(n,m) \xleftrightarrow{\mathsf{P}} (m,n)$, $SU(2)_L \xleftrightarrow{\mathsf{P}} SU(2)_R$
- Charge conjugation also interchanges $SU(2)_L \Leftrightarrow SU(2)_R$
  - like Parity

$\Rightarrow$ the combined transformation CP leaves $SU(2)_L$ and $SU(2)_R$ invariant
  - but it still includes mathematically a complex conjugation

- Time reversal T is an antiunitary transformation
  - it includes a complex conjugation

$\Rightarrow$ any quantum field theory

built from the representations of the Poincaré algebra

- that means: scalars, spinors, vectors, . . .

has to be invariant under CPT

classifying particles

according to the eigenstates (n, m) of $SU(2)_L \otimes SU(2)_R$

- $(0, 0)$ is a scalar
- $(\frac{1}{2}, 0)$ is the χ_a left-handed Weyl-spinor
  - transforms with $\Lambda(\omega)_a{}^b = [e^{i\omega_{\alpha\beta}\sigma^{\alpha\beta}}]_a{}^b$
- $(0, \frac{1}{2})$ is the $\bar\eta^{\dot a}$ right-handed Weyl-spinor
  - transforms with $\Lambda(\omega)^{\dot a}{}_{\dot b} = [e^{i\omega_{\alpha\beta}\bar\sigma^{\alpha\beta}}]^{\dot a}{}_{\dot b}$
- $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ is $\Psi = \binom{\chi_a}{\bar\eta^{\dot a}}$, the Dirac-spinor
  - transforms with $\Lambda(\omega)^a{}_b = [e^{i\omega_{\alpha\beta}(-\frac{i}{4}[\gamma^\alpha,\gamma^\beta])}]^a{}_b$, with $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$
    - a and b go from 1 to 4, (3 and 4 representing the dotted indices)
- $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$ is $(\chi\sigma^\mu\bar\eta) = \chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\eta^{\dot\alpha}$, the spin-1 four-vector

$\Rightarrow$ in that sense is the spinor the square root of the vector